# The topological group of the $p$-adic integers

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**Introduction.** In this paper *) we characterize the topological group $E_p$ of $p$-adic integers (for some prime number $p$) (see Theorem 1). It is also shown that $E_p$ does not allow of any locally compact group topology other than the natural compact topology and the discrete topology (Theorem 2.). Finally all compact abelian groups with this property are characterized (Theorem 3).

The author wishes to express his best thanks to Professor M. VENKATARAMAN for all his encouragement and guidance in the preparation of this paper.

**§ 1.** In this section we show that four properties characterize the $E_p$'s. Our proof depends strongly on the duality theory of locally compact groups.

**Theorem 1.** *Let $G$ be an infinite compact topological group. Then the following conditions are equivalent:*

(1) *$G$ is abelian and every infinite collection of closed subgroups has a trivial intersection.*

(2) *$G$ is abelian and every nontrivial closed subgroup of $G$ is open.*

(3) *$G$ is topologically isomorphic to the group $E_p$ for some prime $p$.*

(4) *The closed subgroups of $G$ form a chain ordered by set-inclusion.*

(5) *$G$ is abelian and every closed subgroup is $nG$ for some integer $n$.*

PROOF. We will give a cyclic proof.

(1)$\Rightarrow$(2): Let $S$ be any nontrivial closed subgroup of $G$. If $G/S$ had an infinity of nontrivial closed subgroups correspondingly $G$ will have infinity of closed subgroups whose intersection contains $S$ contradicting (1). Hence $G/S$ has only a finite number of closed subgroups and so its dual is a discrete abelian group with only a finite number of subgroups since each subgroup in the dual is the annihilator of a closed subgroup of $G/S$. But it is easy to see that an abelian group with just a finite number of subgroups is finite and so self-dual. Thus $G/S$ is finite and so $S$ is open.

(2)$\Rightarrow$(3). Let $H$ be the character group of $G$ and $S'a$ proper subgroup of $H$. If $S$ is the annihilator in $G$ of $S'$, then by hypothesis $S$ is open and so $G/S$ is finite. But then $S'$ is the dual of $G/S$ and so $S'$ is finite. Thus every proper subgroup of $H$ is finite and so $H \simeq c(p^\infty)$ for some prime [1, p. 67]. Hence $G$ (dual of $H$) is topologically isomorphic to $E_p$ (the dual of $c(p^\infty)$).

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*) This is a substantially improved version of the paper presented to the 31st conference of the Indian Mathematical Society in Dec. '65 under the same title. Part of the paper was prepared while the author was a N. I. S. I. Research Fellow.

(3)$\Rightarrow$(4). Since $G$ is topologically isomorphic to $E_p$, the character group $H$ of $G$ is $\simeq c(p^\infty)$. But in $c(p^\infty)$ the subgroups from a chain. Hence their annihilators in $G$ namely the closed subgroups of $G$ form a chain, since whenever $S' \supset T'$, $S \subset T$ $S$, $T$ being the annihilators of $S'$, $T'$ respectively [7].

(4)$\Rightarrow$(5). If $a$ and $b$ are elements of $G$, the closures of the abelian groups $\{a\}$ and $\{b\}$ are abelian subgroups of $G$ [5] and are comparable so that $a$ and $b$ commute. Hence $G$ is abelian. If $H$, the character group of $G$, contains an element $a$ of infinite order the subgroups $\{2a\}$ and $\{3a\}$ are not comparable and so their annihilators cannot be comparable. Thus $H$ has to be a torsion group. The annihilator $H(n)$ of the group $nG$ consists of all elements $x'$ in $H$ $s.\ t \cdot nx' = 0$ [4]. Since $H$ is torsion each of its elements belongs to some $H(n)$. The closed subgroup $\bigcap_n nG$ will be annihilated by each element of $H$ and so $\bigcap_n nG = 0$.

Let now $S$ be any non zero closed subgroup of $G$. Then $S$ contains some $nG$ since $S \not\subset \bigcap_n nG = 0$, and the closed subgroups form a chain. Consider the group $\bar{G} = G/nG$. This is a torsion group of bounded order any two finite subgroups are comparable. Hence if $\bar{x}$ is an element of maximum order in $\bar{G}$ then $\bar{G} = \{\bar{x}\}$. Since $\bar{G}$ is cyclic, any of its subgroups is $m\bar{G}$ for some $m$. If $x$ is a preimage of $\bar{x}$ then $mx \to m\bar{x}$ and so $mx \notin nG$. Hence $mG \supset nG$ and $mG \to m\bar{G}$. Hence if the closed subgroup $S$ maps upon $m\bar{G}$ we should have $S = mG$ since there is a 1—1 correspondence between closed subgroups of $G$ containing $nG$ and closed subgroups of $\bar{G}$.

(5)$\Rightarrow$(1). Let $\{S_\alpha\}$ be any collection of closed subgroups of $G$ and let $\cap S_\alpha \neq 0$. Then $\cap S_\alpha = nG$ for some $n$. Since each closed subgroup of $G$ is of the form $mG$, the closed subgroups of $\bar{G} = G/nG$ are $\bar{G}, 2\bar{G}, \ldots, (n-1)\bar{G}$ only. Hence the closed subgroups of $G$ containing $nG$ are $G, 2G, \ldots, (n-1)G$ only. Thus the collection $\{S_\alpha\}$ is a finite collection and so our result follows. This establishes Theorem 1.

Corollary 1. *In $E_p$, any proper closed subgroup is of the form $p^k E_p$.*

PROOF. By Theorem 1, any proper closed subgroup is of the form $nG$, $G = E_p$. Now $G/nG$ is the dual of a subgroup of $c(p^\infty)$ and hence is cyclic of order $p^k$ say. Now every element $x$ of $G/nG$ satisfies $nx = 0$ so that $n = p^k \cdot m$. Now $p^k G \subseteq nG$ since $G/nG$ is of order $p^k$. $nG = m \cdot p^k G \subseteq p^k G$.

Hence $nG = p^k G$.

Corollary 2. *In $E_p$, the $p^k E_p$'s, $k = 0, 1, 2, \ldots$ form a basis of neighbourhoods at the identity.*

PROOF. $E_p$, being the dual of the (infinite) discrete torsion group $c(p^\infty)$, is a (nondiscrete) compact totally disconnected group and hence [5] has a basis at the identity consisting of compact open subgroups. Then corollary 1 completes the proof.

§ 2. A. HULANICKI [2] has characterized all abelian groups which allow of a unique compact group topology, as a consequence of which it follows that for any $E_p$ the natural topology is the only compact group topology on it. Another well known fact which we need in this section is that no $E_p$ allows of a homomorphism into $E_p$ for any $q \neq p$. This could be easily proved using the facts that $E_p$ is a reduced group $\left(\bigcap_n nE_p = 0\right)$, and is divisible by all integers prime to $p$; but not by $p$ [as a matter of fact no non-zero subgroup of $E_p$ is divisible by $p$].

**Theorem 2.** The only locally compact group topologies on the group $E_p$ of the $p$-adic integers are the natural compact topology and the discrete topology.

PROOF. Let $(E_p, \tau)$ be a locally compact topological group. If $\tau$ is discrete there is nothing to prove. Let $\tau$ be nondiscrete. Then by [7] $E_p = R^n + A$, where $A$ contains a compact open subgroup $S$. Since $E_p$ is a reduced group, $n=0$. Also $S$ cannot have a connected component since a compact connected abelian group is divisible and $E_p$ is reduced. Hence $S$ is totally disconnected. Also $S$ is torsion free since $E_p$ is torsion free. Hence $S$ is algebraically isomorphic to a complete product of some $E_p$'s. If one of these $q$'s is $\neq p$ we will have a isomorphism of some $E_q$ into $S \subset E_p$. Since this is not possible, we have each $q=p$; so $S$ contains a $E_p$.

Thus we have an endomorphism of $E_p$. But $E_p$ is a compact topological integral domain and any endomorphism of the group $E_p$ is represented by $x \to rx$, $r \in E_p$ [1, p. 212]. Under the natural compact topology this is a continuous map and hence the image is a closed subgroup of $E_p$ and so is open by Th. 1 and hence is of finite index. Since $S$ contains this image, $S$ is also of finite index in $E_p$. Now $S$ is $\tau$-compact and $E_p$ is a finite union of $\tau$-compact sets $S, x_1 S, \ldots, x_n S$ and so $E_p$ is $\tau$-compact, i.e. $\tau$ is compact. But the only compact group topology on $E_p$ is the natural topology and hence our result follows.

**Theorem 3.** *Let $(G, T)$ be a compact abelian topological group. Then the following are equivalent:*

(1) *$G$ has no locally compact group topology other than $T$ and the discrete topology*

(2) *$G = A + B$, a topological direct sum, where $B$ is discrete finite and $A=0$ or a compact p-adic integer group $E_p$.*

PROOF.

(2)$\Rightarrow$(1). Let $G = E_p + F$. Suppose $\tau$ is a locally compact group topology on $G$. Then $\tau$ is either the 'natural' product topology or is discrete. Now $G = R^n + A$, with $A$ having a compact open subgroup $S$. If $F$ is finite of order $m$, then $mR^n = R^n \subset E_p$ and is a divisible subgroup of $E_p$. Since $E_p$ is reduced we have $n=0$. Thus $G=A$ and $S$ is a compact open subgroup of $G$. Since a compact connected group is divisible, by a similar argument, $S$ has no connected component and hence is totally disconnected. Now $mS$ is a compact group $\subset E_p$. If $mS=0$, it means that $S \subset F$ and hence is finite so that $\tau$ is discrete since it has a finite open subgroup. If $mS \neq 0$, by declaring this to be a open subgroup we get a locally compact group topology on $E_p$. This topology is not discrete because $mS$ is a compact and infinite group and a compact discrete space is finite. So this is the natural topology on $E_p$. Hence $E_p/mS$ is finite since $mS$ is open in $E_p$. Since $mS$ is $\tau$-compact, so is $E_p$. But $G/E_p$ is finite and so $E_p$ is $\tau$-open and on $E_p$ the induced topology is the natural topology of $E_p$. But any group topology having a open subgroup is determined by the topology on the subgroup [3]. Hence $\tau$ is the product topology of $E_p$ and $F$. The proof is obvious when $A=0$ since then $G$ is finite.

(1)$\Rightarrow$(2). Let $G$ have $T$ and the discrete topology as the only locally compact group topologies. Since $G$ is compact, this is the only compact group topology on it. Hence by HULANICKI [2]. $G = \prod_p D_p$, where each $D_p =$ product of a finite $p$-group and a finite number of $p$-adic integer groups with the product topology. If $E_p$, $E_q$, $q \neq p$ occur then by giving compact group topology to $E_p$ and discrete to the rest we get a new nondiscrete locally compact group topology. Hence there is at most one $p$ for which $D_p$ contains a $E_p$. Also in this $D_p$, $E_p$ cannot occur more than once since by picking out one of them and declaring it to be open with its natural compact topology we get a new nondiscrete locally compact group topology. Now each $D_q$, $q \neq p$ is finite and if there are an infinity of $q$'s for which $D_q \neq 0$, then by picking a proper infinite subset $J$ of these $q$'s and declaring $\prod_{q \in J} D_q$ to be open with its product topology we get again a new non discrete locally compact group topology. Hence we have $\Pi D_q$, $q \neq p$ is a finite group and $D_p$ is either finite or finite group + a $E_p$ so that the assertion follows.

## References

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*(Received November 27, 1966.)*