Another definition of invertibility

By P. H. DOYLE and J. G. HOCKING (East Lansing, Mich.)

A separable metric space M *has only small closed sets* if for each proper closed set $C \subset M$ and each $\varepsilon > 0$, there exists a homeomorphism h of M onto itself such that $h(C)$ has diameter $\delta(h(C)) < \varepsilon$.

Invertible spaces were introduced in [1].

Theorem 1. *Necessary and sufficient conditions that a compact metric space M be invertible are that M have only small closed sets and that each point of M may be moved by some homeomorphism of M onto itself (M non-degenerate).*

PROOF. The necessity of these two conditions is obvious. Assuming these two conditions let C be a proper closed subset of M. For each positive integer n there exists a homeomorphism h_n of M onto itself with $\delta(h_n(C)) < 1/n$. Since M is compact, some subsequence of $\{h_n(C)\}$ converges to a point $p \in M$. Let g be a homeomorphism of M onto itself with $g(p) \neq p$. There is an open neighborhood U of p such that $g(U) \cap U = \emptyset$. Select n sufficiently large so that $h_n(C) \subset U$. Then $gh_n(C) \subset M - h_n(C)$ whence $h_n^{-1}gh_n(C) \subset M - C$. This shows that M is invertible by applying Theorem 6 of [1].

Let M be a metric continuum having only small closed sets. Assume that M is locally n-euclidean at some point. Then the set C of points at which M is not locally n-euclidean is a proper closed subset of M. For each positive integer n, there exists a homeomorphism h_n of M onto itself such that $\delta(h_n(C)) < 1/n$. Some subsequence of $\{h_n(C)\}$ then converges to a point p. Furthermore, for each positive integer n, $M - U_n$, where U_n is the $1/n$-spherical neighborhood $U_n = \{x | d(p, x) < 1/n\}$, is locally n-euclidean. It follows that $M - p$ is locally n-euclidean and hence $C = p$.

Let M_1 be a component of $M - p$. Then M_1 is an open connected n-manifold and $M - M_1$ is a proper closed subset of M. By the same reasoning as above, $M - M_1$ has diameter zero and hence is a point. Thus $M = \overline{M}_1 = M \cup p$.

Now let C be any compact subset of M_1. Since C can be made arbitrarily small by homeomorphisms of M onto itself, we consider two possibilities, (1) $h(C) \subset M_1$ for each homeomorphism h of M onto itself and (2) $p \in h(C)$ for some h. In the later case, it follows that p has an open n-cell neighborhood in M whence M is an n-manifold with only small closed subsets. By Theorem 1, M is invertible and hence by [2] M is an n-sphere. If $h(C) \subset M_1$ for each homeomorphism h, then clearly C lies in an open n-cell in M_1 whence by [2], M_1 is homeomorphic to E^n. Then $M = M_1 \cup p$ is the unique 1-point compactification of E^n and again M is homeomorphic to S^n.

Theorem 2. *A metric continuum which is locally n-euclidean somewhere and which has only small closed sets is a sphere.*

Corollary. Let P be a polyhedron with a triangulation K such that each proper closed subset of P can be carried into the open star of a vertex of K by some homeomorphism of P onto itself. Then P is a sphere, or is degenerate.

This also suggests the following topological characterization of E^n.

Theorem 3. *Let M^n be a non-compact topological n-manifold. If for each proper closed subset $D \subset M^n$ and each compact set $C \subset M^n$ there is a homeomorphism h of M^n onto itself such that $h(D) \subset M^n - C$, then M^n is homeomorphic to E^n.*

PROOF. Let $\overline{M}^n = M^n \cup p$ be the 1-point compactification of M^n. Let L be a proper closed subset of $\overline{M}^n$ and let U be an open neighborhood of p in $\overline{M}^n$. Then $L-p$ is closed in M^n and $\overline{M}^n - U$ is compact in M^n whence there exists a homeomorphism h of M^n onto itself such that $h(L-p) \subset M^n - (\overline{M}^n - U)$. Then h can be extended to $\overline{M}^n$ be setting $h(p) = p$. Then we have $h(L) \subset U$. It follows that $\overline{M}^n$ has only small closed sets whence Theorem 2 concludes that $\overline{M}^n$ is an n-sphere. Thus M^n is homeomorphic to $S^n - p = E^n$.

The research reported here was partially supported by NSF Contract GP-7126.

References

[1] P. H. DOYLE and J. G. HOCKING, Invertible Spaces, *Amer. Math. Monthly* **68** (1961), 959—965.
[2] P. H. DOYLE and J. G. HOCKING, A characterization of Euclidean n-space *Mich. Math. Jour.* **7** (1960), 199—200.

(Received December 11, 1967.)