# Some remarks on the cosine functional equation

To professor Ottó Varga on his 60th birthday

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**1.** Let us consider the cosine functional equation

$$f(xy)+f(xy^{-1}) = 2f(x)f(y), \qquad x, y \in G, \tag{1}$$

where $(G, .)$ is a group (not necessarily abelian) and the range of $f$ is a commutative ring $(R, +, .)$. For $(R, .)$ we suppose that

$$2a \neq 0 \quad \text{for} \quad a \neq 0 \qquad a \in R \tag{I}$$

holds, i.e., every equation $2a=2b$ in $R$ can be cancellated by 2.

If we put $y=1$, the unity of $G$, in (1) then we have

$$2f(x)[1-f(1)] = 0.$$

Thus it is a natural supposition to assume

$$f(1)=1. \tag{2}$$

This assumption and (1) imply

$$f(y^{-1})=f(y), \qquad y \in G \tag{3}$$

by substituting $x=1$ in (1).

Note that the commutability conditions

$$f(xy)=f(yx), \qquad x, y \in G \tag{4}$$

(inside) and

$$f(x)f(y)=f(y)f(x), \qquad x, y \in G \tag{5}$$

(outside) are equivalent each to another since we have

$$f(xy) = 2f(x)f(y) - f(xy^{-1}),$$
$$f(yx) = 2f(y)f(x) - f(yx^{-1}) =$$
$$= 2f(y)f(x) - f(xy^{-1})$$

by (1) and (3).

P. L. KANNAPPAN has treated (1) in the case where $(R, +\ .,)$ is the complex field. He used the assumption

$$(6) \qquad f(xyz)=f(yxz), \qquad x, y, z \in G$$

in order to obtain the solutions of (1) in the form

$$f(x)=\frac{1}{2}\,[a(x)+a(x)^{-1}] = \text{ch}\,\alpha(x),$$

where $a(x)$ is an arbitrary homomorphism of $(G, .)$ into $(R, .)$. Clearly, the condition (6) is much more restrictive than (4). Indeed, (4) is the special case of (6) when $z=1$.

**2.** In what follows we investigate (1) under assumption (2) and (6) in the general case where $(R, +, .)$ is an *arbitrary* commutative ring restricted only by I). The commutativity of $(R, .)$ is a natural supposition since now we have (5) as a consequence of (6).

The technic of the solution of a functional equation depends very greatly on the following characteristic numbers:

1. the number $n$ of the independent variables;

2. the number $m$ of the given binary operations figuring inside of the required function $f$.

E.g., in our equation (1) we have $n=m=2$ as $x, y$ are independent variables and $xy$, $xy^{-1}$ are given binary operations. It would be useful to find another equation instead of (1), where the respective numbers $n_1$, $m_1$ are changed such that

$$n_1>n, \quad m_1<m.$$

Here we show such a reduction of (1). We prove that

$$(7) \qquad F(x, y) = f(xy) - f(x)f(y)$$

satisfies

$$(8) \qquad F(x, y)F(u, v)=F(x, u)F(y, v), \qquad x, y, u, v \in G$$

for every solution $f$ of (1) with subsidiary conditions (2) and (6).

For this purpose let us consider

$$\begin{aligned} f(xyz) &= 2f(xy)f(z)-f(xyz^{-1}) = \\ &= 2f(xy)f(z)-2f(x)f(yz^{-1})+f(xzy^{-1}) = \\ &= 2f(xy)f(z)-2f(x)[2f(y)f(z)-f(yz)]+f(xzy^{-1}) = \\ &= 2f(xy)f(z)+2f(x)[f(yz)-f(y)f(z)]+ \\ &+ 2f(xz)f(y)-2f(x)f(y)f(z)-f(xzy), \end{aligned}$$

hence

$$f(xyz)-f(xy)f(z) = f(x)[f(yz)-f(y)f(z)]+[f(xz)-f(x)f(z)]f(y),$$

i.e.,

$$(9) \qquad F(xy, z) = f(x)F(y, z)+F(x, z)f(y), \qquad x, y, z \in G.$$

Similarly, considering

$$\begin{aligned} F(xyu, v) &= f(xy)F(u, v) + F(xy, v)f(u) = \\ &= f(xy)F(u, v) + [f(x)F(y, v) + F(x, v)f(y)]f(u) = \\ &= f(x)F(yu, v) + F(x, v)f(yu) = \\ &= f(x)[f(y)F(u, v) + F(y, v)f(u)] + F(x, v)f(yu), \end{aligned}$$

we conclude to

$$[f(xy) - f(x)f(y)]\,F(u, v) = F(x, v)\,[f(yu) - f(y)f(u)]$$

and this is equivalent to (8).

**3.** The Kannappan's condition (6) seems to be very restrictive. E.g., (6) implies that $f$ maps every commutator $xyx^{-1}y^{-1}$ onto

$$f(xyx^{-1}y^{-1}) = f(yxx^{-1}y^{-1}) = f(1) = 1.$$

But what can we state without supposing (6)? Observe that (6) was used in the reduction of (1) to (9) only in a unique step. Therefore, (6) is necessary and sufficient for the equivalence of (1) and (9). This raises the question weather is (6) necessary also for the equivalence of (1) and (8) or not? (Assuming naturally (2).)

We answer this question negative and show that (1)—(2) and (8) may hold without supposing (6), i.e., they do not imply (6) in general.

In fact, let us suppose (1)—(2). Then we have (3). By (5), also (4) is true. Thereafter we define $F(x, y)$ by (7). This $F$ is symmetric with accordance to (4). However, (9) can not be proved without using (6). We have only (10) $F(xy, z) + F(yx, z) = 2f(x)F(y, z) + 2f(y)F(x, z)$, $x, y, z \in G$ instead of (9). Now let us consider the following equations equivalent to (8):

$$[f(xy) - f(x)f(y)]F(u, v) = F(x, v)[f(yu) - f(y)f(u)],$$

$$2f(x)[2f(y)F(u, v) + 2f(u)F(y, v)] + 2F(x, v)[f(yu) + f(uy)] =$$
$$= 2f(u)[2f(x)F(y, v) + 2f(y)F(x, v)] + 4f(xy)F(u, v),$$

$$2f(xy)F(u, v) + 2f(u)F(xy, v) + 2f(yx)F(u, v) + 2f(u)F(yx, v) =$$
$$= 2f(x)F(yu, v) + 2f(yu)F(x, v) + 2f(x)F(uy, v) + 2f(uy)F(x, v),$$

$$F(xyu, v) + F(uxy, v) + F(yxu, v) + F(uyx, v) =$$
$$= F(xyu, v) + F(yux, v) + F(xuy, v) + F(uyx, v),$$

$$F(uxy, v) + F(yxu, v) = F(yux, v) + F(xuy, v),$$

$$f(uxyv) + f(yxuv) + f(yuxv) + f(xuyv).$$

This last equation holds since we have

$$f(uxy)f(v) + f(yxu)f(v) = f(yux)f(v) + f(xuy)f(v)$$

as

$$f(uxy) = f(yux), \quad f(yxu)) = f(xuy)$$

are true by (4).

Further equivalent form equations of (8) are the followings:

$$f(xyuv)+f(yxuv) = f(xuyv)+f(xvyu),$$

(11) $$f(uxyv)+f(xuvy) = f(xuyv)+f(uxvy), \qquad x, y, u, v \in G.$$

Summing up, (11) is a necessary and sufficient condition for the solutions $f$ of (1)—(2) to satisfy (8). Clearly, if (6) is supposed, then (11) holds evidently but, conversely, (11) does not imply (6).

Note that also (11) seems to be a rather restrictive condition since it implies that every commutator $xyx^{-1}y^{-1}$ is mapped by $f$ to the unity 1. It remains an open question to find useful reduction instead of (8) without supposing such restrictive conditions as (6) or (11).

**4.** Why is, however, interesting to reduce (1) to (8)?

We show the useful of this reduction in the special case where

(II) $(R, +, .)$ is a field.

Then (8) characterize the separable functions $F$ for which

$$F(x, y) = cg(x)g(y), \qquad x, y \in G$$

holds with a suitable constant $c$ in $R$. Here it is allowed $c=0$ or $g \equiv 0$ too.

Thus (1) leads to a system of functional equations of the form

(8′), (9′)
$$\left.\begin{aligned} f(xy) &= f(x)f(y)+cg(x)g(y), \\ g(xy) &= f(x)g(y)+g(x)f(y), \end{aligned}\right\} \qquad x, y \in G,$$

where both $f$ and $g$ are unknown functions.

Thereafter, having this addition theorem form functional equations, it can be built up the matrix

(12) $$A(x) = \begin{bmatrix} f(x) & g(x) \\ cg(x) & f(x) \end{bmatrix}$$

satisfying

(13) $$A(xy) = A(x)A(y), \qquad x, y \in G.$$

Therefore, the most general form of $f$ and $g$ can be obtained by a special matrix representation $A(x)$ of the group $(G, .)$. In this way the solution of (1) is reduced to the determination of certain homomorphism $x \to A(x)$ of the group $(G, .)$ into a special matrix semigroup over $(R, +, .)$.

Note here that calculating $af(xy \cdot z) = af(x \cdot yz)$ in detail, whithout using (II)—(6), we obtain only

(8″), (9″)
$$\left.\begin{aligned} af(xy) &= af(x)f(y)+g(x)g(y), \\ ag(xy) &= af(x)g(y)+ag(x)f(y), \end{aligned}\right\} \quad x, y \in G,$$

with

(14) $$\begin{cases} a = F(u, u) = f(u^2)-f(u)^2 = f(u)^2-1, \\ g(x) = F(x, u) = f(xu)-f(x)f(u) = f(x)f(u)-f(xu^{-1}). \end{cases}$$

This yields an other matrix representation

$$B(x)=\begin{bmatrix} af(x) & ag(x) \\ g(x) & af(x) \end{bmatrix} \tag{12'}$$

instead of the above $A(x)$ by which we have only

$$aB(xy)=B(x)B(y), \qquad x, y\in G \tag{13'}$$

instead of (13).

After (9″) $ag(xy)=ag(yx)$ holds. Thus we have

$$\begin{aligned} a^2f(xyz) &= a\, f(xy)f(z)+ag(xy)g(z) = \\ &= a^2f(yx)f(z)+ag(yx)g(z) = a^2f(yxz) \end{aligned}$$

for every choosing of $u$ in $G$, i.e.,

$$[f(u)^2-1]^2[f(xyz)-f(yxz)]=0, \qquad x, y, z, u\in G \tag{15}$$

holds identically. This means that there is not a great distance between conditions (6) and (11).

Remark that this reduction method here showen for obtaining (9) resp. (9″) can be applied also for more general type

$$f(xy)+g(xy^{-1}) = \sum_{i=1}^{n} f_i(x)g_i(y)$$

instead of (1).

If $(R, +, .)$ is the complex field, then (8′) −(9′) is equivalent with

$$\varphi(xy)=\varphi(x)\varphi(y), \qquad \psi(xy)=\psi(x)\psi(y), \qquad x, y\in G,$$

where

$$\varphi(x)=f(x)+\sqrt{-c}\,g(x), \qquad \psi(x)=f(x)-\sqrt{-c}\,g(x).$$

Finally, remark that the importance of (8) can be seen also by putting suitable special values of the independent variables $x, y, u, v$. E.g., if we substitute $y=x^{-1}$, $v=u^{-1}$, then it becomes

$$[1-f(x)^2][1-f(u)^2] = [f(xu)-f(x)f(u)]^2, \qquad x, u\in G,$$

which is an interesting equation also in itself.

## References

[1] P. L. Kanappan, On cosine and sine functional equations, *Ann. Polon. Math.* **20** (1968), 245—249.

[2] P. L. Kannappan, The functional equation $f(xy)+f(xy^{-1})=2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc.* **19** (1968), 69.

[3] E. Vincze, Eine allgemeinere Methode in der Theorie der Funktionalgleichungen, I. *Publ. Math. Debrecen* **9** (1962), 149—163.

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Summary

The cosine functional equation

$$f(xy)+f(xy^{-1}) = 2f(x)f(y), \qquad x, y \in G \tag{1}$$

is considered, where $f$ maps a group $(G,.)$ into a commutative ring $(R, +,.)$. Assuming $f(1)=1$ and $2a\neq 0$ for $a\neq 0$ in $R$, the following results are established:

1. the relations $f(x^{-1})=f(x)$, $f(xy)=f(yx)$ are true;
2. the Kannappan's condition $f(xyz)=f(yxz)$ implies that

$$F(x, y)F(u, v)=F(x, u)F(y, v), \qquad x, y, u, v \in G, \tag{8}$$

$$F(xy, z) = f(x)F(y, z)+F(x, z)f(y), \qquad x, y, z \in G \tag{9}$$

hold, where $F$ is defined by

$$F(x, y) = f(xy)-f(x)f(y), \qquad x, y \in G; \tag{7}$$

3. a necessary and sufficien condition for (8) is

$$f(uxyv)+f(xuvy) = f(xuyv)+f(uxvy), \qquad x, y, u, v \in G; \tag{11}$$

4. (8) implies

$$f(xy) = f(x)f(y)+c\,g(x)g(y), \qquad x, y \in G, \tag{8'}$$

$$g(xy) = f(x)g(y)+g(x)f(y), \tag{9'}$$

i. e.

$$A(xy)=A(x)A(y), \tag{13}$$

$$A(x)=\begin{bmatrix} f(x) & g(x) \\ c\,g(x) & f(x) \end{bmatrix}, \tag{12}$$

provided that $(R, +,.)$ is a field;

5. (8) implies

$$[1-f(x)^2][1-f(u)^2] = [f(xu)-f(x)f(u)]^2, \qquad x, u \in G, \tag{16}$$

$$a^2[f(xyz)-f(yxz)] = 0, \qquad x, y, z \in G, \tag{15}$$

and

$$a f(xy) = a f(x)f(y)+g(x)g(y), \qquad x, y \in G, \tag{8''}$$

$$a\,g(xy) = a f(x)g(y)+a\,g(x)f(y), \tag{9''}$$

$$a\,B(xy)=B(x)B(y), \tag{13'}$$

$$B(x) = \begin{bmatrix} af(x) & a\,g(x) \\ g(x) & a f(x), \end{bmatrix}, \tag{12'}$$

where $g$, $a$ are given by

$$\begin{cases} a = F(u, u) = f(u^2)-f(u)^2 = f(u)^2-1, \\ g(x) = F(x, u) = f(xu)-f(x)f(u) = f(x)f(u)-f(xu^{-1}) \end{cases} \tag{14}$$

with arbitrarily fixed $u \in G$.

The method of reduction showen here for (1) seems to be applicable also for the more general type

$$f(xy)+g(xy^{-1}) = \sum_{i=1}^{n} f_i(x)g_i(y).$$