Convergent sequences in semimetric spaces

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The ordering of the real numbers is often used to obtain certain refinements of theorems concerning convergent sequences of real numbers. It may seem that such refinements are available only for real number's. The semimetric spaces of Nachbin (a particular case of the same authors more general semiuniform spaces, see [1], [2]) are appropriate for extensions to more general situations. In this note we will prove two such results and then use one of them for sequences with a certain subadditive property.

For theorem 1. and 2. let X be a space with a semimetric ϱ, that is a function $\varrho: X \times X \to [0, \infty)$ so that:

$$\varrho(x, y) + \varrho(y, z) \geqq \varrho(x, z), \tag{i}$$

$$\varrho(x, y) = \varrho(y, x) = 0 \Leftrightarrow x = y. \tag{ii}$$

The topology of X will not be that of the semimetric ϱ, but that of the metric space (X, d), where d is the metric defined by:

$$d(x, y) = \max \{\varrho(x, y), \varrho(y, x)\}.$$

Since for sequences of real numbers one of our principal tools is Cesáro's lemma, we shall put the additional requirement that X should have the Montel property, this means that every closed bounded subset of X be compact.

Theorem 1. *Let $\{x_n\}_{n \in N}$ be a bounded sequence in X and C a positive integer so that for every $\varepsilon > 0$ there is a rank $N(\varepsilon)$ so that:*

$$m - C > n \geqq N(\varepsilon) \Rightarrow \varrho(x_n, x_m) < \varepsilon.$$

The sequence $\{x_n\}_{n \in N}$ is then convergent.

Proof. Consider first the case $C = 0$. By the Montel property of X it is required that $\{x_n\}_{n \in N}$ should have a convergent subsequence $\{x_{p_k}\}_{k \in N}$; let a denote its limit. We shall prove that $\{x_n\}_{n \in N}$ has the limit a. Let $\varepsilon > 0$ be given, we shall determine a number M with the property:

$$N \geqq M \Rightarrow d(x_n, a) < \varepsilon. \tag{1}$$

Take for this first N_1, N_2 so that:

$$m > n \geqq N_1 \Rightarrow \varrho(x_n, x_m) < \frac{\varepsilon}{2},$$

$$k \geqq N_2 \Rightarrow d(x_k, a) < \frac{\varepsilon}{2}$$

and

$$p_k \geqq N_1.$$

$M = \max\{p_{N_2+1}, N_1\}$ will satisfy (1). If $n \geqq M$ then there exists a number $k \geqq N_2$ for which: $p_{k+2} > n > p_k$. Then the four numbers

$$\varrho(x_n, x_{p_{k+2}}), \varrho(x_{p_{k+2}}, a), \varrho(x_{p_k}, x_n), \varrho(a, x_{p_k})$$

are less than $\varepsilon/2$. By the "triangle" inequality for semimetrics it follows:

$$\varrho(a, x_n) \leqq \varrho(a)\, x_{p_k}) + \varrho(x_{p_k}, x_n) < \varepsilon,$$

$$\varrho(x_n, a) \leqq \varrho(x_n, x_{p_{k+2}}) + \varrho(x_{p_{k+2}}, a) < \varepsilon,$$

wherefrom (1) and by this also q.e.d. are easily obtained.

It remains now to extend this result for $C > 0$. Consider for this the following sequences:

$$\{x_{(k-1)(C+1)+t}\}_{k \in N} \quad \text{and} \quad \{x_{(k-1)(C+2)+1}\}_{k \in N} \qquad (t = 1, 2, \dots, C+1).$$

They all satisfy the hypotheses of our theorem in the above proved case and are hence convergent. As it is easily observed, the last of these sequences has a common subsequence with every one of the other $C+1$ sequences, so that they all have the same limit. By this it is also proved that $\{x_n\}_{n \in N}$ is convergent.

Theorem 2. *Let $\{x_n\}_{n \in N}$ be a bounded sequence in X with the following property: for every pair (ε, k), where $\varepsilon > 0$ and k is a natural number there is a rank $N(\varepsilon, k)$ for which:*

$$n \geqq N(\varepsilon, k) \Rightarrow \varrho(a_k, a_n) < \varepsilon.$$

The sequence $\{x_n\}_{n \in N}$ is then convergent.

PROOF. The proof is similar to that of theorem 1., though the obtained results are of different nature.

Let again $\{x_{p_k}\}_{k \in N}$ be a convergent subsequence of $\{x_n\}_{n \in N}$ and a its limit. For a given $\varepsilon > 0$ let M be a number so that

$$k \geqq M \Rightarrow d(x_{p_k}, a) < \frac{\varepsilon}{2}.$$

We shall prove that

$$n \geqq N\left(\frac{\varepsilon}{2}, p_M\right) \Rightarrow d(x_n, a) < \varepsilon.$$

As a matter of fact, if $n \geqq N\left(\frac{\varepsilon}{2}, p_M\right)$, then:

$$\varrho(x_{p_M}, x_n) < \frac{\varepsilon}{2}, \ \varrho(a, x_{p_M}) < \frac{\varepsilon}{2},$$

$$\varrho(a, x_n) \leqq \varrho(a, x_{p_M}) + \varrho(x_{p_M}, x_n) < \varepsilon.$$

On the other hand take $p_j \geqq \max\left\{p_M, N\left(\frac{\varepsilon}{2}, n\right)\right\}$ and we have:

$$\varrho(x_{p_j}, a) < \frac{\varepsilon}{2}, \quad \varrho(x_n, x_{p_j}) < \frac{\varepsilon}{2},$$

$$\varrho(x_n, a) \leqq \varrho(x_n, x_{p_j}) + \varrho(x_{p_j}, a) < \varepsilon.$$

Hence $d(x_n, a) = \max\{\varrho(x_n, a), \varrho(a, x_n)\} < \max\{\varepsilon, \varepsilon\} = \varepsilon$, q.e.d.

If we translate these two theorems in the language of real numbers, the semimetric ϱ will be:

$$\varrho(x, y) = \max\{y - x, 0\}$$

and we have:

Theorem 1'. *Let $\{x_n\}_{n\in N}$ be a bounded sequence of real numbers and C a positive integer so that for every $\varepsilon > 0$ there is a rank $N(\varepsilon)$ so that:*

$$m - C > n \geqq N(\varepsilon) \Rightarrow x_m - x_n < \varepsilon.$$

The sequence $\{x_n\}_{n\in N}$ is then convergent.

Theorem 2'. *Let $\{x_n\}_{n\in N}$ be a bounded sequence of real numbers with the following property: for every pair (ε, k), where $\varepsilon > 0$ and k is a natural number, there is a rank $N(\varepsilon, k)$ for which:*

$$n > N(\varepsilon, k) \Rightarrow a_n - a_k < \varepsilon.$$

The sequence $\{x_n\}_{n\in N}$ is then convergent.

The reader has perhaps found out by now which are the theorems of which theorems 1'. and 2'. are refinements: the general Cauchy criterion and the theorem concerning the convergence of decreasing sequences.

Now the theorem which in its proof uses theorem 2'.

Theorem 3. *Let $\{x_n\}_{n\in N}$ be a sequence of positive real numbers and $\{p_n\}_{n\in N}$ a strictly increasing sequence of natural numbers. For every way of representing a number p_m as a sum of not-necessarily different numbers of the sequence $\{p_n\}_{n\in N}$:*

$$p_m = \sum_{j=1}^{k} p_{m_j} \ (k \text{ is arbitrary in every such sum})$$

we have:

$$x_m \leqq \sum_{j=1}^{k} x_{m_j}.$$

Under these requirements the sequence $\left\{\frac{x_n}{p_n}\right\}_{n\in N}$ is convergent.

PROOF. We use theorem 2' and prove first the existence of the numbers $N(\varepsilon, 1)$.

An auxiliar result necessary to fulfill our task will be now proved.

There exist two natural numbers M, N and a constant C so that for any $m \geqq M$ we may find non-negative integers $\alpha_{k,m}$ $(k=1, \ldots, N)$ satisfying the following relations:

$$\sum_{k=1}^{N} \alpha_{k,m} \cdot p_k = p_m,$$

$$\sum_{k=2}^{N} \alpha_{k,m} \cdot p_k \leqq C.$$

Let D denote the highest common factor of the numbers $\{p_n\}_{n \in N}$. We may determine then a rank N and natural numbers $h_1, \ldots, h_N$ so that:

$$h_1 \cdot p_1 + \ldots + h_N \cdot p_N \equiv D \,(\mathrm{mod}\, p_1).$$

The proof can be now easily completed, which we leave to the reader.

We return now to the proof of the existence of the numbers $N(\varepsilon, 1)$. Let $m \geqq M$ and $A = \max\left\{\frac{x_1}{p_1}, \ldots, \frac{\varkappa_N}{p_N}\right\}$, we have:

$$\frac{x_m}{p_m} \leqq \frac{\sum_{k=1}^{N} \alpha_{k,m} \cdot \frac{x_k}{p_k} \cdot p_k}{\sum_{k=1}^{N} \alpha_{k,m} \cdot p_k} \leqq \frac{A \cdot \sum_{k=1}^{N} \alpha_{k,m} \cdot p_k}{\sum_{k=1}^{N} \alpha_{k,m} \cdot p_k} = A$$

which shows that the sequence $\left\{\frac{x_n}{p_n}\right\}_{n \in N}$ is bounded (all its terms are positive). Further on, for $m \geqq M$ we have:

$$\frac{x_m}{p_m} - \frac{x_1}{p_1} \leqq \frac{\alpha_{1,m} \cdot x_1 + A \cdot C}{\alpha_{1,m} \cdot p_1} - \frac{x_1}{p_1} = \frac{A \cdot C}{\alpha_{1,m} \cdot p_1}.$$

A and C being constant and $\alpha_{1,m}$ growing indefinitely, the existence of the numbers $N(\varepsilon, 1)$ is proved.

The existence of $N((\varepsilon, k)$ can be proved similarly.

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References

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