Publ. Math. Debrecen
49 / 1-2 (1996), 69–76

Stability of the Cauchy equation on a restricted domain

By JACEK TABOR (Cracow)

Abstract. Let G be a commutative semigroup, E a Banach space and $D \subset G \times G$. Let $\varepsilon > 0$ be given and let $f : G \to E$ satisfy the inequality

$$\|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \le \varepsilon \text{ for } (\alpha, \beta) \in D.$$

We prove that under certain assumptions on D and G there exists a unique additive function $A : G \to E$ such that

$$\|f(\alpha) - A(\alpha)\| \le \varepsilon \text{ for } \alpha \in G.$$

By $\mathbb{R}_+$ we understand the set $[0, \infty)$, by $\mathbb{N}$ the set of all positive integers and by $\mathcal{P}(X)$ the family of all subsets of X. By a vector space we mean a real vector space. Let G be a semigroup. According to the tradition, we will use the additive notation (even in the case when G is noncommutative). When G is a semigroup without zero $G \setminus \{0\}$ denotes G. If $D \subset G \times G$ then we write

$$D_X = \{x | (x, y) \in D\},\ D_Y = \{y | (x, y) \in D\},\ D_{X+Y} = \{x+y \mid (x, y) \in D\}.$$

The set D is called a Pexider domain of stability if there exists a constant $K > 0$ such that whenever three functions $f : D_X \to \mathbb{R}$, $g : D_Y \to \mathbb{R}$, $h : D_{X+Y} \to \mathbb{R}$ satisfy the inequality

$$|f(x) + g(y) - h(x + y)| \le \varepsilon \text{ for } (x, y) \in D,$$

Mathematics Subject Classification: Primary 39B72.

then there exists an additive function $A : G \to \mathbb{R}$ and two constants $a, b \in \mathbb{R}$ such that

$$|f(x) - A(x) - a| \le K\varepsilon \text{ for } x \in D_X,$$
$$|g(y) - A(y) - b| \le K\varepsilon \text{ for } y \in D_Y,$$
$$|h(z) - A(z) - a - b| \le K\varepsilon \text{ for } z \in D_{X+Y}.$$

To our knowledge this definition was introduced by Zs. Páles.

We say that a set D is a Cauchy domain of stability if there exists a constant $K > 0$ such that whenever $f : D_X \cup D_Y \cup D_{X+Y} \to \mathbb{R}$ satisfies the inequality

$$|f(x+y) - f(x) - f(y)| \le \varepsilon \text{ for } (x,y) \in D,$$

then there exists an additive function $A : G \to \mathbb{R}$ such that

$$|f(x) - A(x)| \le K\varepsilon \text{ for all } x \in D_X \cup D_Y \cup D_{X+Y}.$$

On the 32-nd International Symposium on Functional Equations Zs. Páles posed the following problem [4]: Is the set $D = \{(x,y) : y \ge x^2\}$ a Pexider domain of stability? The answer to this question is negative (c.f. [6]). We take $f(x) = 0,\ g(x) = h(x) = \ln(1 + |x|)$.

One may ask a closely related question. Is the same set a Cauchy domain of stability? We will show (see Example 1) that the answer is positive. This paper was inspired by this problem.

Definition 1. Let G be a commutative semigroup and let $B \subset G \times G$. We say that $W \subset G$ is B-bounded if

(i) $\forall\, \alpha \in G\backslash\{0\}\ \exists k_0 \in \mathbb{N}\ \forall k \ge k_0 : k\alpha \notin W$,

(ii) $\forall \alpha, \beta \in G\backslash\{0\},\ (\alpha,\beta) \in B\ \exists k_0 \in \mathbb{N}\ \forall k_1 \ge k_0,\ k_2 \ge 1 :$ $k_1\alpha + k_2\beta \notin W$.

One can easily notice that the family of B-bounded subsets of a semigroup forms an ideal of sets.

Definition 2. Let G be a semigroup. We say that $B \subset G \times G$ is full in G if for every group H, and every function $A : G \to H$ such that

$$A(\alpha + \beta) = A(\alpha) + A(\beta) \text{ for } (\alpha,\beta) \in B, \tag{1}$$

A is additive.

For a broader study and the literature concerning full sets see [1], [2] or [3].

Proposition 1. *Let G be a semigroup and let $B \subset G \times G$ satisfy the following condition:*

$$\forall (\alpha,\beta) \in (G \times G)\backslash B \ \exists \gamma \in G : (\beta,\gamma), (\alpha, \beta+\gamma), (\alpha+\beta,\gamma) \in B. \tag{2}$$

Then B is full in G.

Proof. Consider an arbitrary group H and an arbitrary mapping $A : G \to H$ satisfying (1). We have to show that A is additive. Let $(\alpha,\beta) \in (G \times G)\backslash B$. Then by (2) there exists a $\gamma \in G$ such that

$$(\beta,\gamma), (\alpha,\beta+\gamma), (\alpha+\beta,\gamma) \in B,$$

so

$$\begin{aligned} A(\beta+\gamma) &= A(\beta) + A(\gamma), \\ A(\alpha+\beta+\gamma) &= A(\alpha) + A(\beta+\gamma), \\ A(\alpha+\beta+\gamma) &= A(\alpha+\beta) + A(\gamma). \end{aligned}$$

This implies that

$$A(\alpha+\beta)+A(\gamma) = A(\alpha+\beta+\gamma) = A(\alpha)+A(\beta+\gamma) = A(\alpha)+A(\beta)+A(\gamma).$$

As H is a group we obtain that $A(\alpha+\beta) = A(\alpha) + A(\beta)$. □

Definition 3. Let E be a vector space, and let $S \subset E$. We define

$$B(S) := \{(x,y) \in S \times S : y \neq rx \text{ for all } r \in \mathbb{R}_-\}.$$

Proposition 2. *Let E be a vector space such that $\dim E \geq 2$. Then $B(E)$ is full.*

Proof. We are going to show that (2) holds. Let $(x,y) \in E \times E\backslash B(E)$. Then $y = rx$ for a certain $r \in \mathbb{R}_-$. Since $\dim E \geq 2$ we can find a $z \in E$, such that $z \neq rx$ for every $r \in \mathbb{R}$. Then obviously $(y,z) \in B(E)$, $(x, y+z) \in B(E)$, $(x+y, z) \in B(E)$. Proposition 1 completes the proof. □

Let E be a vector space, and let $W \subset E$. One can easily notice that if the intersection of W with any two-dimensional subspace P of E is bounded in P then W is $B(E)$-bounded. This implies that every bounded subset of a topological vector space is $B(E)$-bounded.

Suppose that any three elements of W are linearly independent over the field $\mathbb{Q}$. Then W is $B(E)$-bounded. Condition (i) of the Definition 1 is obvious. Suppose that condition (ii) does not hold. Then we can find $x, y \in$

$E\backslash\{0\}$, $(x,y) \in B(E)$ and $\{k_i\},\{l_i\} \subset \mathbb{N}$, $\{k_i\}$ an increasing sequence, such that

$$z_i = k_i x + l_i y \in W \text{ for } i \in \mathbb{N}.$$

Since $x \neq 0$ and $\{k_i\}$ is increasing, we can find n_1, n_2, n_3 such that $z_{n_1} \neq z_{n_2}$, $z_{n_2} \neq z_{n_3}$, $z_{n_1} \neq z_{n_3}$. But obviously z_{n_1}, z_{n_2}, z_{n_3} are linearly dependent over $\mathbb{Q}$. We obtain a contradiction.

The above observations and the fact that the finite union of $B(E)$-bounded sets is still a $B(E)$-bounded set shows that the family of $B(E)$-bounded subsets of a vector space is quite large.

Theorem 1. *Let G be a nontrivial commutative semigroup and let $B \subset G \times G$ be full in G. Let $W : G \to \mathcal{P}(G)$ be a mapping such that $G\backslash W(\alpha)$ is a B-bounded set for every $\alpha \in G$. Let E be a Banach space and let $\varepsilon > 0$. Suppose that $f : G \to E$ satisfies the following inequality:*

$$\|f(\alpha+\beta) - f(\alpha) - f(\beta)\| \leq \varepsilon \text{ for } \alpha \in G,\ \beta \in W(\alpha).$$

Then there exists a unique additive function $A : G \to E$ such that

$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in G.$$

Proof. Let $\alpha \in G\backslash\{0\}$. Since $G\backslash W(\alpha)$ is B-bounded, there exists $n \in \mathbb{N}$ such that $i\alpha \in W(\alpha)$ for $i \geq n$. Then we have for $k > n$

$$\begin{aligned} \left\|\frac{f(k\alpha)}{k} - f(\alpha)\right\| &\leq \left\|\frac{f(k\alpha) - (k-n)f(\alpha)}{k}\right\| + \left\|\frac{nf(\alpha)}{k}\right\| \\ &\leq \sum_{i=n}^{k-1} \frac{\|f((i+1)\alpha) - f(i\alpha) - f(\alpha)\|}{k} + \frac{1}{k}\|f(n\alpha)\| + \frac{1}{k}\|nf(\alpha)\| \\ &\leq \frac{k-n}{k}\varepsilon + \frac{1}{k}\|f(n\alpha)\| + \frac{1}{k}\|mnf(\alpha)\|. \end{aligned}$$

Thus

$$\limsup_{k\to\infty} \left\|\frac{f(k\alpha)}{k} - f(\alpha)\right\| \leq \varepsilon \text{ for } \alpha \in G\backslash\{0\}. \tag{3}$$

Replacing in this inequality α by $n\alpha$ and dividing both sides by n we obtain

$$\limsup_{k\to\infty} \left\|\frac{f(kn\alpha)}{kn} - \frac{f(n\alpha)}{n}\right\| \leq \frac{\varepsilon}{n},$$

and inserting l instead of k and m instead of n

$$\limsup_{l\to\infty}\left\|\frac{f(ml\alpha)}{ml}-\frac{f(m\alpha)}{m}\right\|\le\frac{\varepsilon}{m}.$$

Thus

$$\left\|\frac{f(n\alpha)}{n}-\frac{f(m\alpha)}{m}\right\|\le\frac{\varepsilon}{n}+\frac{\varepsilon}{m}.$$

This proves that $\left\{\frac{f(n\alpha)}{n}\right\}$ is a Cauchy sequence for $\alpha\in G\backslash\{0\}$. If G is a semigroup with zero then obviously $\left\{\frac{f(n0)}{n}\right\}$ is convergent. We define

$$A(\alpha):=\lim_{n\to\infty}\frac{f(n\alpha)}{n}\text{ for }\alpha\in G.$$

Due to (3) and the definition of A we have $\|f(\alpha)-A(\alpha)\|\le\varepsilon$ for $\alpha\in G\backslash\{0\}$. In the case when G is a semigroup with zero there exists a $\beta\notin W(0)$. Then we have $\|f(0+\beta)-f(0)-f(\beta)\|\le\varepsilon$, so $\|f(0)\|\le\varepsilon$. Hence

$$\|f(\alpha)-A(\alpha)\|\le\varepsilon\text{ for }\alpha\in G,$$

and replacing α by $n\alpha$ and dividing by n we obtain

$$\|\frac{f(n\alpha)}{n}-A(\alpha)\|\le\frac{\varepsilon}{n}\text{ for }\alpha\in G.$$

We prove the additivity of A. Let $\alpha,\beta\in G$. Suppose that $(\alpha,\beta)\in B$. As $A(0)=0$ we may assume that $\alpha,\beta\in G\backslash\{0\}$. Consider an arbitrary $\delta>0$ and take an $n\in\mathbb{N}$ such that $\frac{\varepsilon}{n}\le\delta$. Then, as $G\backslash W(n\beta)$ is B-bounded, there exists $k\in\mathbb{N}\backslash\{0\}$ such that

$$i\alpha+j(n\beta)\in W(n\beta)\text{ for }i\ge nk,\ j\ge 1.$$

Hence

$$\begin{aligned}\|A(\alpha+\beta)-A(\alpha)-A(\beta)\|&\le\|\frac{f(nk(\alpha+\beta))}{nk}-\frac{f(nk\alpha)}{nk}-\frac{f(n\beta)}{n}\|+3\delta\\ &\le\sum_{i=0}^{k-1}\frac{1}{nk}\|f((i+1)n\beta+nk\alpha)-f(in\beta+nk\alpha)-f(n\beta)\|+3\delta\\ &\le\frac{k}{nk}\varepsilon+3\delta\le 4\delta.\end{aligned}$$

Since δ was chosen arbitrarily, we obtain that $A(\alpha+\beta)=A(\alpha)+A(\beta)$. As B is full, this proves that A is additive. Because $A(\alpha)=\lim\limits_{n\to\infty}\frac{f(n\alpha)}{n}$, A is a unique additive approximation of f. □

Corollary 1. *Let G be a semigroup, and let H be a nontrivial subsemigroup of the centre of G. Let $B \subset H \times H$ be full in H. Let $W : G \to \mathcal{P}(H)$ be a mapping such that $H \backslash W(\alpha)$ is B-bounded for $\alpha \in H$. We assume the following condition:*

(4) *for every $\alpha \in G$ there exists a $\beta \in W(\alpha)$ such that $\alpha + \beta \in H$.*

Let F be a Banach space, and let $\varepsilon > 0$. Suppose that $f : G \to F$ satisfies the inequality

$$\|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \leq \varepsilon \text{ for } \alpha \in G,\ \beta \in W(\alpha).$$

Then there exists a unique additive function $A : G \to F$ such that

$$\|f(\alpha) - A(\alpha)\| \leq 3\varepsilon \text{ for } \alpha \in G,$$
$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in H.$$

Proof. Making use of Theorem 1 for the function $f|H$ we obtain that there exists a unique additive function $A : H \to F$ such that

$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in H.$$

We will show that A has a unique additive extension onto G. Let $\alpha \in G$. Then by (4) there exists a $\beta \in W(\alpha) \subset H$ such that $\alpha + \beta \in H$. We define $\widetilde{A}(\alpha) := A(\alpha + \beta) - A(\beta)$. Now we prove that $\widetilde{A}$ is well defined. Suppose that $\alpha + \beta_1, \alpha + \beta_2 \in H$ for certain $\beta_1, \beta_2 \in H$. Then

$$(A(\alpha + \beta_1) - A(\beta_1)) - (A(\alpha + \beta_2) - A(\beta_2)) = A(\alpha + \beta_1 + \beta_2) - A(\alpha + \beta_2 + \beta_1) = 0.$$

Making use of the fact that H is contained in the center of G one can easily prove that $\widetilde{A}$ is additive. The way of defining $\widetilde{A}$ shows that it is a unique additive extension of A.

Let $\alpha \in G$. Then by (4) there exists a $\beta \in W(\alpha)$ such that $\alpha + \beta \in H$. Then

$$\|f(\alpha) - \widetilde{A}(\alpha)\| \leq \|f(\alpha + \beta) - A(\alpha + \beta) + A(\beta) - f(\beta)\| + \|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \leq 3\varepsilon.$$

□

Corollary 2. *Let E be a vector space, let $C \subset E$ be a convex cone such that $C \cap -C = \{0\}$, $C - C = E$. Let $V : E \to \mathcal{P}(E)$ be a mapping such that $C \backslash V(x)$ is a $B(E)$-bounded set for $x \in E$. Let F be a Banach space and let $\varepsilon > 0$. Suppose that $f : E \to F$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \text{ for } x \in E,\ y \in W(x).$$

Then there exists a unique additive function $A : E \to F$ such that

$$\|f(x) - A(x)\| \leq 3\varepsilon \text{ for } x \in E,$$
$$\|f(x) - A(x)\| \leq \varepsilon \text{ for } x \in C.$$

Proof. We are going to show that the assumptions of Corollary 1 are satisfied. Let $W(x) := C \cap V(x)$. Because C is a cone such that $C \cap -C = \{0\}$, $B := B(E) \cap (C \times C) = C \times (C \backslash \{0\})$. Hence B is full in C and $C \backslash W(x)$ is B-bounded for $x \in C$. Let $x \in E$. Because $C - C = E$, there exists an $a \in C$ such that $a + x \in C$. Since $C \backslash W(x)$ is $B(E)$-bounded there exists a $b \in C$ such that $a + b \in W(x)$. Then obviously $(a+b)+x \in C$. Corollary 1 completes the proof. $\square$

Example 1. Let $g : \mathbb{R} \to \mathbb{R}$. We put $E = F = \mathbb{R}$, $C = \mathbb{R}_+$, $V(x) = (g(x), +\infty)$. Now by Corollary 2 we obtain that the set

$$D = \{(x, y) : y > g(x)\}$$

is a Cauchy domain of stability.

Corollary 3. *Let E be a vector space, let F be a Banach space and let $\varepsilon > 0$. Suppose that $W : E \to \mathcal{P}(E)$ is a mapping, such that $E \backslash W(x)$ is a $B(E)$-bounded set for $x \in E$. Suppose that $f : E \to F$ satisfies the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \text{ for } x \in E,\ y \in W(x).$$

Then there exists a unique additive function $A : E \to F$ such that

$$\|f(x) - A(x)\| \leq \varepsilon \text{ for } x \in E.$$

Proof. Suppose that $E = \mathbb{R}$. Let $g(x) = -f(-x)$. Then

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \text{ for } y \in W(x),$$
$$\|g(x+y) - g(x) - g(y)\| \leq \varepsilon \text{ for } y \in -W(-x).$$

Obviously the set $-W(-x)$ is $B(\mathbb{R})$-bounded for every $x \in R$. Now due to Corollary 2 we can find additive functions $A_1, A_2 : E \to F$ such that

$$\|f(x) - A_1(x)\| \le 3\varepsilon \text{ for } x \in \mathbb{R},$$
$$\|f(x) - A_1(x)\| \le \varepsilon \text{ for } x \in \mathbb{R}_+$$
$$\|g(x) - A_2(x)\| \le 3\varepsilon \text{ for } x \in \mathbb{R},$$
$$\|g(x) - A_2(x)\| \le \varepsilon \text{ for } \ x \in \mathbb{R}_+.$$

One can easily notice that then $A_1 = A_2 =: A$. Hence

$$\|f(x) - A(x)\| \le \varepsilon \quad \text{for} \quad x \in \mathbb{R}_+,$$
$$\| - f(-x) - A(x)\| \le \varepsilon \quad \text{for} \quad x \in \mathbb{R}_+.$$

The last two inequalities mean that

$$\|f(x) - A(x)\| \le \varepsilon \quad \text{for} \quad x \in \mathbb{R}.$$

Suppose that $\dim E \ge 2$. Then by Proposition 2 $B(E)$ is full, so Theorem 1 completes the proof. □

Putting $E = \mathbb{R}$, $W(x) := [-1, 1]$ we obtain a generalization of Theorem 3 [5]. Moreover we get the best possible constant K (instead of $K = 9$ we have $K = 1$).

Acknowledgements. I would like to thank the referee for his valuable remarks.

References

[1] R. Ger, On some functional equations with a restricted domain, *Fund. Math.* **LXXXIX** (1975), 131–149.

[2] R. Ger, On some functional equations with a restricted domain, II, *Fund. Math.* **XCVIII** (1978), 249–272.

[3] L. Paganoni and S. Paganoni Marzegalli, Cauchy functional equation on semigroups, *Fund. Math.* **CX** (1980), 63–74.

[4] Zs. Páles, 32 Remark and Problem, The Thirty-second International Symposium on Functional Equations, Gargnano, Italy, *Aequationes Math.* **49** (1995), 197.

[5] F. Skof, Sull'approssimazione delle applicazioni localmente δ-additive, *Atti Acad. Sc. Torino* **17** (1983), 377–389.

[6] Jacek Tabor, 20 Remark, The Thirty-third International Symposium on Functional Equations, Caldes de Malavella, Spain, *Aequationes Math.* (*to appear*).

JACEK TABOR
SŁOMIANA 20/31
30–316 CRACOW
POLAND

(Received July 11, 1995; revised November 15, 1995)