On bi-ideals and quasi-ideals in semigroups

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The purpose of this paper is twofold: an examination of elementary properties of bi-ideals and an investigation of the relationships between bi-ideals and quasi-ideals. In the first section we define an equivalence relation $\mathscr{B}$ on a semigroup which is finer than $\mathscr{H}$ and show that any bi-ideal is the union of $\mathscr{B}$-classes. An investigation of O-minimal bi-ideals follows. We show that in a semigroup with O, a bi-ideal is O-minimal if and only if it is a non-zero $\mathscr{B}$-class union $\{O\}$.

In the second section we show that under certain conditions a O-minimal bi-ideal is also a O-minimal quasi-ideal; this is not always true. We say that a semigroup is in the class $\mathscr{B}\mathscr{Q}$ whenever its sets of bi-ideals and quasi-ideals coincide. Several sufficient conditions are found (eg. right O-simple, regular) for a semigroup to be in this class. An example is given which sheds some further light on the characterization of $\mathscr{B}\mathscr{Q}$ semigroups. We conclude with a characterization of these semigroups announced by Calais at the Semigroup Symposium in Bratislavia, Czechoslovakia in June, 1968.

We follow the notation and terminology of [3]. We will always use $\subset$ for proper containment. Equivalence relations will be denoted by script letters with the subscripted capital italic denoting the corresponding equivalence class. Thus R_a denotes the $\mathscr{R}$-class of a.

1. Bi-ideals

(1. 1) *Definition.* A (non-empty) subset B of a semigroup S is a bi-ideal if $BS^1B \subseteq B$. (Clearly a bi-ideal is a subsemigroup.)

We now define a relation on a semigroup which will be useful in our investigation of bi-ideals.

(1. 2) *Definition.* For $a, b \in S$, a given semigroup, we write $a\mathscr{B}b$ if 1) $a=b$ or 2) there exists $u, v \in S$ such that $aua=b$ and $bvb=a$.

The following two propositions can be readily verified:

(1. 3) Proposition. *The relation $\mathscr{B}$ defined in* (1. 2) *is an equivalence relation; indeed* $\mathscr{B} \subseteq \mathscr{H}$.

(1. 4) Proposition. *If A is a bi-ideal of a semigroup then* $A = \bigcup_{a \in A} B_a$, i.e., *any bi-ideal is the union of its $\mathscr{B}$-classes.*

In what follows we will be concerned mainly with semigroups with O though we could procede as in [8] for semigroups with proper Suschkewitsch kernel and obtain more general resuits. With this in mind we recall the following:

(1. 5) *Definition.* A non-zero bi-ideal B of a semigroup S with O is said to be O-minimal if there is no bi-ideal B' of S with $\{O\} \subset B' \subset B$.

The following result follows immediately from (1. 4):

(1. 6) *Corollary.* Let S be a semigroup with O. If a bi-ideal, B, is a non-zero $\mathscr{B}$-class union $\{O\}$ then it is a O-minimal bi-ideal.

The converse of this corollary is also true as we show in the following:

(1. 7) **Theorem.** *Let S be a semigroup with O. A bi-ideal is O-minimal if and only if it is a non-zero $\mathscr{B}$-class union $\{O\}$.*

PROOF. Let B be a O-minimal bi-ideal of S. Let $a, b \in B \setminus \{O\}$. Since $\{b, b^2\} \cup bSb$ and $\{a, a^2\} \cup aSa$ are clearly non-zero bi-ideals contained in B we must have $B = \{b, b^2\} \cup bSb = \{a, a^2\} \cup aSa$.

Now assume $a \neq b$. We can procede from the last equality by cases.

Suppose $a = b^2$. We have two sub-cases to consider.

1) If also $b = a^2$ then $a = b^2 = aa^2a = b(ba^2b)b$ and also $b = a^2 = a(ab^2a)a$. It follows that $a\mathscr{B}b$.

2) If $b \neq a^2$ we must have $b \in aSa$ and $b = aua$ for some $u \in S$. Then $a = b^2 = auaaua = b(buaaub)b$. Again it follows that $a\mathscr{B}b$.

Now if $a \neq b$ and $a \neq b^2$ we must have $a \in bSb$ so that $a = bvb$ for some $v \in S$. Again we examine b by cases as above. If $b = a^2$ we have simply case 2) with the roies of a and b interchanged. If $b \in aSa$ then $b = aua$ for some $u \in S$. In either case it follows that $a\mathscr{B}b$.

By (1. 4) we may conclude that $B = B_b \cup \{O\}$.

The converse is just (1. 6).

We now remark, with thanks to Professor Otto Steinfeld and Mr. Bruce Mielke, that the above proof suffices to show that $a\mathscr{B}b$ if and only if $B(a) = \{a, a^2\} \cup aSa = \{b, b^2\} \cup bSb = B(b)$, ie., two elements are $\mathscr{B}$ related precisely when their principal bi-ideals coincide. Thus $\mathscr{B}$ has the same relation to bi-ideals as $\mathscr{L}$ to left ideals, $\mathscr{R}$ to right ideals, $\mathscr{H}$ to quasi-ideals [4] and $\mathscr{J}$ to two-sided ideals.

We now procede to investigate the structure of O-minimal bi-ideals. The reader will recall that a null subsemigroup N is a subsemigroup with O in which $ab = O$ for any $a, b \in N$.

(1. 8) **Theorem.** *Let S be a semigroup with O. A O-minimal bi-ideal, B, of S is either a null subsemigroup or a group with $\{O\}$.*

PROOF. From (1. 7) we have $B = B_b \cup \{O\}$ for any $b \in B \setminus \{O\}$. We recall from (1. 3) that $\mathscr{B} \subseteq \mathscr{H}$. Thus if $b^2 \neq 0$ we conclude $b^2\mathscr{B}b$ and $b^2\mathscr{H}b$. It now follows that H_b is a group ([3] Theorem 2. 16). Now if $a \in H_b$ we would also have $a\mathscr{B}b$ since the equations $a = bvb$ and $aua = b$ can be solved for u and v in the group H_b. Thus B is the group H_b union $\{O\}$.

On the other hand if $b^2 = 0$ for each $b \in B \setminus \{O\}$ and if $a \in B \setminus \{O\}$ we have $a\mathscr{B}b$, $a\mathscr{H}b$ and then $ab\mathscr{D}b^2 = 0$ by [3] Theorem 2. 4. Since it is clear that $D_0 = \{O\}$ we have $ab = 0$ and B is a null subsemigroup.

We now procede to investigate the relationship between 0-minimal left and right ideals and 0-minimal bi-ideals.

(1. 9) Proposition. *Let S be a semigroup with 0. If R is a 0-minimal right dieal and L is a 0-minimal left ideal then either $RL=\{O\}$ or RL is a 0-minimal bi-ideal of S.*

Proof. Suppose $RL\neq\{O\}$ and that there is a bi-ideal B with $\{O\}\subset B\subset RL$. Since $RL\subseteqq R\cap L$ we have $BS^1\subseteqq RS^1\subseteqq R$. It follows that $RS^1=R$. Now $B\subseteqq L$ so that we have $B\subset RL=BS^1B\subseteqq B$, a contradiction. Since RL is a bi-ideal ([3] p 85 ex 18 c) the proof is complete.

Unfortunately not every 0-minimal bi-ideal in a semigroup with 0 can be obtained as the product of a 0-minimal right ideal and a 0-minimal left ideal as the following simple example shows.

(1. 10) *Example.* Let S be the semigroup $\{a, a^2, a^3, 0\}$ where $a^4=0$. It is easy to check that $\{a^2, 0\}$ is a 0-minimal bi-ideal which is not the product of a 0-minimal right ideal and 0-minimal left ideal.

This also fails even when the 0-minimal bi-ideal is a group with $\{O\}$ as in the following:

(1. 11) *Example.* Let S be the semigroup consisting of those 2×2 matrices of the form $\begin{pmatrix} i & 0 \\ j & a\end{pmatrix}$ where i, j are non-negative integers and $a=0, 1$. One readily checks that $B=\left\{0, \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\right\}$ is a 0-minimal bi-ideal; indeed B is even a 0-minimal left ideal. $\left(0=\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}.\right)$ Moreover, B is a group with $\{O\}$. Since there are no 0-minimal right ideals in S we can not obtain B as the product of a 0-minimal right ideal and 0-minimal left ideal. However we do have the following:

(1. 12) Proposition. *Let S be a semigroup with 0. If B is a 0-minimal bi-ideal of S then for any right ideal R contained in BS^1 and any left ideal L contained in S^1B we have either $RL=\{O\}$ or $RL=B$.*

Proof. Let $R\subseteqq BS^1$ and $L\subseteqq S^1B$. Then $RL\subseteqq BS^1S^1B\subseteqq BS^1B\subseteqq B$. Since RL is a bi-ideal and B is 0-minimal it follows that $RL=\{O\}$ or $RL=B$.

We note that the semigroup in example (1. 11) shows that RL may be $\{O\}$ even if B is a group with $\{O\}$ if we take $R=\left\{\begin{pmatrix} 0 & 0 \\ j & 0\end{pmatrix} \middle| j\geqq 0\right\}$ and $L=B$. (However $SB=B$.)

We now conclude this section with the following proposition which we are informed can also be found in [5]:

(1. 13) Proposition. *The product of two bi-ideals in a semigroup is always a bi-ideal.*

Proof. Let A and B be two bi-ideals of a given semigroup S. Then $(AB)S^1(AB)=A(B(S^1A)B)\subseteqq A(BSB)\subseteqq AB$. It follows that AB is a bi-ideal.

2. Bi-ideals and quasi-ideals

In this section we examine the relationship between bi-ideals and quasi-ideals. The desirable property of bi-ideals proven in (1. 13) and Steinfeld's question [7] as to whether quasi-ideals also had this property motivated much of the following investigation. The reader will recall the following definition and proposition.

(2. 1) *Definition.* (A non-empty) subset Q of a semigroup S is called a quasi-ideal if $QS \cap SQ \subseteq Q$.

(2. 2) Proposition. ([3] p 85 ex 18). *Every quasi-ideal of a semigroup is a bi-ideal. In a regular semigroup every bi-ideal is also a quasi-ideal.*

It is now natural to define the following class of semigroups. It is clearly non-vacuous by (2. 2):

(2. 3) *Definition.* The class $\mathscr{B}\mathscr{Q}$ of semigroups will consist precisely of those semigroups whose sets of bi-ideals and quasi-ideals coincide.

Because of (1. 13) the following corollary is immediate:

(2. 4) *Corollary.* The product of two quasi-ideals of a semigroup $S \in \mathscr{B}\mathscr{Q}$ is always a quasi-ideal.

We now procede in two directions in an attempt to find the extent of the class $\mathscr{B}\mathscr{Q}$. First we try to determine large subclasses of $\mathscr{B}\mathscr{Q}$, then we will try to determine the relationship between the minimal bi-ideals and quasi-ideals in a semigroup and to see whether this has any determinabie effect on the total sets of bi-ideals and quasi-ideals of the semigroup. Preliminary to this we quote the following theorem of Steinfeld:

(2. 5) **Theorem** ([7] Theorem 1). The intersection of a left and right ideal of a semigroup is a quasi-ideal. Conversely, every quasi-ideal of S can be obtained as the intersection of a left and right ideal.

(2. 6) Proposition. *If S is a left [right] simple semigroup then each bi-ideal of S is a right [left] ideal.*

Proof. Suppose S is left simple and let B be a bi-ideal of S. Since S^1B is a left ideal of S and S is left simple we must have $S^1B = S$. Thus $B \supseteq BS^1B = B(S^1B) = BS$ and it follows that B is a right ideal. The proof for a right simple semigroup is dual.

The following corollary is now an immediate result of the above and (2. 5):

(2. 7) *Corollary.* If S is a left [right] simple semigroup then $S \in \mathscr{B}\mathscr{Q}$.

We can modify the above argument to prove:

(2. 8) Proposition. *Let S be a semigroup with* 0. *If S is a left [right]* 0*-simple semigroup then $S \in \mathscr{B}\mathscr{Q}$.*

Proof. If B is a non-zero bi-ideal of S then $O \in BSB \subseteq B$. Thus if $SB = \{O\}$ we would be done since in this case B is a left ideal; otherwise $SB = S$, since S is by hypothesis left 0-simple. We can now procede as in (2. 6) and (2. 7).

We modify below the semigroup in example (1. 11) to obtain a semigroup which is not in the class $\mathscr{B}\mathscr{Q}$. Indeed this is a rather elementary semigroup in comparison to the construction in [1] and we also find a right ideal, R, and left ideal L, whose product, RL, is not a quasi-ideal. Since right and left ideals are quasi-ideals the product of two quasi-ideals is not always a quasi-ideal.

(2. 9) *Example.* Let S be the multiplicative semigroup of 2×2 matrices, of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ where a and b are positive real numbers. Let $R = \left\{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\middle| a<b\right\}$ and $L = \left\{\begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix}\middle| q>5\right\}$. One can readily check that R is a right ideal and L a left ideal. Moreover, we can directly verify from the defining conditions of R and L that $\begin{pmatrix} 5 & 0 \\ 10 & 1 \end{pmatrix} \notin RL$. But we have

$$\begin{pmatrix} 5 & 0 \\ 10 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix}\left[\begin{pmatrix} 1/2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix}\right]\begin{pmatrix} 1/4 & 0 \\ 1 & 1 \end{pmatrix}$$

so that directly $S[RL]\cap[RL]S \subsetneqq RL$ and RL is not a quasi-ideal.

Now in example (1. 10) we can see that $B=\{a^2, 0\}$ is a 0-minimal bi-ideal which is not a quasi-ideal since $a^3 \in BS\cap SB$ but $a^3 \notin B$. However, as the following proposition shows, it is not necessary for a semigroup, S, to be in the class $\mathcal{B}\mathcal{Q}$ in order for the product of two quasi-ideals of S to be a quasi-ideal.

(2. 10) Proposition. *In a commutative semigroup the product of two quasi-ideals is a quasi-ideal.*

Proof. In commutative semigroups the set of quasi-ideals and ideals coincide. The result is thus immediate.

Thus we have seen that if a semigroup is either regular, right 0-simple, or left 0-simple it belongs to the class $\mathcal{B}\mathcal{Q}$.

We now give one further result which is a partial characterization for the class of quasi-regular semigroups. This class properly contains the class of regular semigroups. From Calais [1] we have the following definition and result:

(2. 11) *Definition.* A semigroup S is said to be quasi-regular if each right ideal and each left ideal is idempotent.

(2. 12) **Theorem.** ([1] Proposition 2. 2). A semigroup S is quasi-regular if and only if $Q=(QS)^2\cap(SQ)^2$ for each quasi-ideal Q of S.

We can now characterize those quasi-regular semigroups which belong to $\mathcal{B}\mathcal{Q}$. We note that Calais' example shows that quasi-regular semi-groups do not in general belong to the class $\mathcal{B}\mathcal{Q}$.

(2. 13) **Theorem.** *A semigroup S belongs to the class $\mathcal{B}\mathcal{Q}$ and is quasi-regular if and only if $B=(BS)^2\cap(SB)^2$ for each bi-ideal B of S.*

Proof. If we have $B=(BS)^2\cap(SB)^2$ for each bi-ideal B of S then S is clearly quasi-regular by (2. 12) since by (2. 2) every quasi-ideal of S is also a bi-ideal. But if S is quasi-regular then we have $(BS)^2=BS$ and $(SB)^2=SB$ since BS and SB are right and left ideals respectively. Whence for any bi-ideal B we have $B=(BS)^2\cap(SB)^2=BS\cap SB$ so that B is also a quasi-ideal. It is now immediate that $S\in\mathcal{B}\mathcal{Q}$.

Conversely if S is quasi-regular and $S \in \mathcal{B}\mathcal{Q}$ then each bi-ideal is a quasi-ideal. The result follows immediately upon application of (2. 12).

We now turn to the problem of the relationship of 0-minimal bi-ideals and 0-minimal quasi-ideals. We first give a theorem due to Steinfeld:

(2. 14) **Theorem.** ([8] Theorem 2). *Let S be a semigroup with O and let R and L be 0-minimal right and left ideals of S respectively. Then either RL is a null semigroup or it is a group with O. In the latter case we have $RL = R \cap L$ and thus RL is a 0-minimal quasi-ideal of S.*

From this we may deduce the following:

(2. 15) **Theorem.** *Let S be a semigroup with 0. If B is 0-minimal bi-ideal which is a group with $\{O\}$ and can be written as the product of a 0-minimal right ideal and 0-minimal left ideal then B is also a 0-minimal quasi-ideal.*

Now if we return to the semigroup in example (1. 11) we see that B, a 0-minimal bi-ideal, is a group with $\{O\}$ which is also a 0-minimal quasi-ideal eventhough it cannot be written as a product of a 0-minimal right ideal and a 0-minimal left ideal. Ttat 0-minimal bi-ideals which are groups with $\{O\}$ are 0-minimal quasi-ideals is shown in a note by Mr. Bruce Mielke submitted to thies Journal.

When a semigroup S does not have a 0 or if we restrict outselves to minimal bi-ideals and quasi-ideals the problem is much simpler. We conclude the paper with a discussion of this case. First from (1. 12) we can deduce:

(2. 16) Proposition. *Let S be a semigroup. If B is a minimal bi-ideal of S then BS and SB are minimal right and left ideals of S respectively and we have $B = (BS)(SB)$, i.e. B is the product of a minimal right ideal and a minimal left ideal.*

Proof. Let R be a right ideal and L a left ideal of S with $R \subseteqq BS$, $L \subseteqq SB$. Since RL is a bi-ideal with $RL \subseteqq BSSB \subseteqq BS^1B \subseteqq B$ we must have $B = RL$. But $RL \subseteqq R \cap L$ and thus $B \subseteqq R$. Hence we have $BS \subseteqq RS \subseteqq R$ so that $R \subset BS$ would ley dia contradiction. It follows that BS is a minimal right ideal. Similarly SB is a minimal left ideal. Since B is a minimal bi-ideal which contains $(BS)(SB)$, itself a bi-ideal, we must have $B = (BS)(SB)$.

(2. 17) **Lemma** ([2] Lemma 3. 4). *If R is a minimal right ideal and L a minimal left ideal of a semigroup then $RL = R \cap L$.*

(2. 18) **Lemma** ([7] Theorem 4a). *Each minimal quasi-ideal of a semigroup is a group.*

(2. 19) **Theorem.** *For any semigroup S the set of minimal bi-ideals and minimal quasi-ideals coincide.*

Proof. If B is a minimal bi-ideal then $B = RL$ for some minimal right ideal R and minimal left ideal L by (2. 16). Now $RL = R \cap L$ by (2. 17). Thus B is also a quasi-ideal by (2. 5) and it follows from (2. 2) that B is a minimal quasi-ideal.

Conversely let Q be a minimal quasi-ideal. By (2. 18) Q is a group. Now Q is also a bi-ideal and any bi-ideal is the union of $\mathcal{B}$-classes by (1. 4). Since a group is clearly contained in but one $\mathcal{B}$-class it follows immediately that Q is a minimal bi-ideal.

Finally we have the characterization given by Calais:

(2. 20) **Theorem** [Calais; Reims, France]. *Let S be a semigroup. Let $B(x, y)$ denote the smallest bi-ideal of S containing $x, y \in S$ and let $Q(x, y)$ denote the smallest quasi-ideal of S containing x, y. Then $S \in \mathcal{B}\mathcal{Q}$ if and only if $B(x, y) = Q(x, y)$ for every $x, y \in S$.*

PROOF. The condition $B(x, y) = Q(x, y)$ is clearly necessary.

Now let B be an arbitrary bi-ideal of S and let $z \in BS \cap SB$. Then we have $x, y \in B$ such that $z \in xS \cap Sy$. But $xS \cap Sy \subseteqq Q(x, y) = (\{x, y\} \cup \{x, y\}S\}) \cap \cap (\{x, y\} \cup S\{x, y\})$. Since $Q(x, y) = B(x, y)$ we have $z \in B(x, y)$. Since $B(x, y) = = \{x, y) \cup \{x, y\}^2 \cup \{x, y\}S\{x, y\}$ is the smallest bi-ideal of S containing x, y we can conclude $z \in B(x, y) \subseteqq B$. Thus $BS \cap SB \subseteqq B$ and B is quasi-ideal. It now follows from (2. 2) that $S \in \mathcal{B}\mathcal{Q}$.

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