Extra loops II.
On loops with identities of Bol — Moufang type

To professor Ottó Varga on his 60 th birthday

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§ 1. Introduction

The class of Bol loops and Moufang loops play an important role in the theory of quasigroups and in their applications in other branches of mathematics.

Identities characterizing the class of Bol loops and Moufang loops are well known, see for example [1], [3]. All these identities are of the following form: both sides of such an identity consist of the same three different letters taken in the same order but one of them occures twice on each side.

Identities of this form will be called *identities of Bol—Moufang type*. It is natural to ask what are all these identities and which loops satisfy them (separately).

In what follows first we give all possible nontrivial identities of Bol—Moufang type (the number of these identities is 60) and then we establish their connections to each other when we consider loops. Thus we get among others the curious fact that 30 of these identities are equivalent to the associative law. Our results make possible to decide of any two identities of Bol—Moufang type whether one of them imply the other or not.

The terminology is the same as in [2].

§ 2. Results

The list of all the 60 identities of Bol—Moufang type is the following:

(1) $xy\cdot zx=(xy\cdot z)x$

(2) $xy\cdot zx=(x\cdot yz)x$ Moufang identity

(3) $xy\cdot zx=x(y\cdot zx)$

(4) $xy\cdot zx=x(yz\cdot x)$ Moufang identity

(5) $(xy\cdot z)x=(x\cdot yz)x$

(6) $(xy\cdot z)x=x(y\cdot zx)$ extra identity

(7) $(xy\cdot z)x=x(yz\cdot x)$

(8)	$(x\cdot yz)x=x(y\cdot zx)$	
(9)	$(x\cdot yz)x=x(yz\cdot x)$	
(10)	$x(y\cdot zx)=x(yz\cdot x)$	
(11)	$xy\cdot xz=(xy\cdot x)z$	
(12)	$xy\cdot xz=(x\cdot yx)z$	
(13)	$xy\cdot xz=x(yx\cdot z)$	extra identity
(14)	$xy\cdot xz=x(y\cdot xz)$	
(15)	$(xy\cdot x)z=(x\cdot yx)z$	
(16)	$(xy\cdot x)z=x(yx\cdot z)$	
(17)	$(xy\cdot x)z=x(y\cdot xz)$	Moufang identity
(18)	$(x\cdot yx)z=x(yx\cdot z)$	
(19)	$(x\cdot yx)z=x(y\cdot xz)$	left Bol identity
(20)	$x(yx\cdot z)=x(y\cdot xz)$	
(21)	$yx\cdot zx=(yx\cdot z)x$	
(22)	$yx\cdot zx=(y\cdot xz)x$	extra identity
(23)	$yx\cdot zx=y(xz\cdot x)$	
(24)	$yx\cdot zx=y(x\cdot zx)$	
(25)	$(yx\cdot z)x=(y\cdot xz)x$	
(26)	$(yx\cdot z)x=y(xz\cdot x)$	right Bol identity
(27)	$(yx\cdot z)x=y(x\cdot zx)$	Moufang identity
(28)	$(y\cdot xz)x=y(xz\cdot x)$	
(29)	$(y\cdot xz)x=y(x\cdot zx)$	
(30)	$y(xz\cdot x)=y(x\cdot zx)$	
(31)	$yx\cdot xz=(yx\cdot x)z$	
(32)	$yx\cdot xz=(y\cdot xx)z$	
(33)	$yx\cdot xz=y(xx\cdot z)$	
(34)	$yx\cdot xz=y(x\cdot xz)$	
(35)	$(yx\cdot x)z=(y\cdot xx)z$	
(36)	$(yx\cdot x)z=y(xx\cdot z)$	
(37)	$(yx\cdot x)z=y(x\cdot xz)$	C-identity
(38)	$(y\cdot xx)z=y(xx\cdot z)$	
(39)	$(y\cdot xx)z=y(x\cdot xz)$	
(40)	$y(xx\cdot z)=y(x\cdot xz)$	
(41)	$xx\cdot yz=(x\cdot xy)z$	LC-identity

(42) $$xx\cdot yz=(xx\cdot y)z$$

(43) $$xx\cdot yz=x(x\cdot yz)$$

(44) $$xx\cdot yz=x(xy\cdot z)$$

(45) $$(x\cdot xy)z=(xx\cdot y)z$$

(46) $$(x\cdot xy)z=x(x\cdot yz)$$ LC-identity

(47) $$(x\cdot xy)z=x(xy\cdot z)$$

(48) $$(xx\cdot y)z=x(x\cdot yz)$$ LC-identity

(49) $$(xx\cdot y)z=x(xy\cdot z)$$

(50) $$x(x\cdot yz)=x(xy\cdot z)$$

(51) $$yz\cdot xx=(yz\cdot x)x$$

(52) $$yz\cdot xx=(y\cdot zx)x$$

(53) $$yz\cdot xx=y(zx\cdot x)$$ RC-identity

(54) $$yz\cdot xx=y(z\cdot xx)$$

(55) $$(yz\cdot x)x=(y\cdot zx)x$$

(56) $$(yz\cdot x)x=y(zx\cdot x)$$ RC-identity

(57) $$(yz\cdot x)x=y(z\cdot xx)$$ RC-identity

(58) $$(y\cdot zx)x=y(zx\cdot x)$$

(59) $$(y\cdot zx)x=y(z\cdot xx)$$

(60) $$y(zx\cdot x)=y(z\cdot xx)$$

It is clear that any of these identities is implied by the associative law.

The following facts are immediately obvious for any loop:

(i) *Any one of the identities* (1), (3), (5), (7), (8), (10), (11), (12), (14), (16), (18), (20), (21), (23), (24), (25), (28), (29), (31), (32), (33), (34), (44), (47), (49), (50), (52), (55), (58), (59) *is equivalent to the associative law.*

(ii) *Any one of the identities* (9), (15), (30) *is equivalent to the flexible identity* $xy\cdot x=x\cdot yx$.

(iii) *Any one of the identities* (40), (43), (45) *is equivalent to the left alternative identity* $x^2y=x\cdot xy$.

(iv) *Any one of the identities* (35), (51), (60) *is equivalent to the right alternative identity* $yx^2=yx\cdot x$.

Moreover also hold the following:

(v) *A loop* $(G,\cdot)$ *satisfies the identity* (42) *(or* (38) *or* (54)*) if and only if* x^2 *lies in the left (middle, right) nucleus of* $(G,\cdot)$ *for all* x *in* G.

(vi) *A loop* $(G,\cdot)$ *satisfies the identity* (36) *(or* (39)*) if and only if it is right (left) alternative and* x^2 *lies in the middle nucleus of* $(G,\cdot)$ *for all* x *in* G.

(vii) *The three LC-identities* (41), (46), (48) *(RC-identities* (53), (56), (57)*) are equivalent for any loop.*

(viii) *The four Moufang identities* (2), (4), (17), (27) *are equivalent for any loop* (cf. [1]).

(ix) *The three extra identities* (6), (13), (22) *are equivalent for any loop* (cf. [2]).

Theorem 1. *The following statements are equivalent:*

(a) $(G, \cdot)$ *is an extra loop,*

(b) $(G, \cdot)$ *is a Moufang loop and* $N(x)=\langle I, R_xR_x, R_xR_x\rangle$ *is an autotopism (i.e.* x^2 *lies in the nucleus) of* $(G, \cdot)$ *for every* $x \in G$.

Proof. Assume (a). Then $A_1(x)=\langle L_x, R_x^{-1}, L_xR_x^{-1}\rangle$ and, by Theorem 3 of [2], $M(x)=\langle L_x, R_x, L_xR_x\rangle$ are autotopisms of $(G, \cdot)$ for all x in G, so $A_1^{-1}(x)M(x)=$ $=\langle I, R_xR_x, R_xR_x\rangle$ is an autotopism. Therefore, $(G, \cdot)$ satisfies (b).

Assume (b). Then $M(x)$ and $N(x)$ are autotopisms of $(G, \cdot)$, and so $M(x)N(x^{-1})=A_1(x)$ is also an autotopism. Thus (b) implies (a).

Corollary 1. *If the mapping* $x \to x^2$ *is a permutation of the extra loop* $(G, \cdot)$, *then* $(G, \cdot)$ *is a group.*

Corollary 2. *Every finite extra loop of odd order is a group.*

Definition. If a loop $(G, \cdot)$ satisfies a C-, LC-, RC-identity for all x, y, z in G, then it is called a C-, LC-, RC-loop respectively.

Theorem 2. *If* $(G, \cdot)$ *is an LC-loop, then*

(a) $(G, \cdot)$ *has the left inverse property,*

(b) $(G, \cdot)$ *is left alternative,*

(c) x^2 *is in the left nucleus of* $(G, \cdot)$ *for all* x *in* G.

(Analogous results hold for RC-loops.)

Proof. (a) Let $(G, \cdot)$ be an LC-loop, and define x^{-1} by $xx^{-1}=1$ for all x in G. Substituting $y=x^{-1}$ in

$$(x\cdot xy)z=x(x\cdot yz) \tag{46}$$

we get $xz=x(x\cdot x^{-1}z)$, whence $z=x\cdot x^{-1}z$ i.e. (a) holds. Moreover we see from this (by $z=x$) that $(x^{-1})^{-1}=x$ also holds.

(b) (46) reduces with $y=1$ to the left alternative identity $x^2z=x\cdot xz$.

(c) Applying (b) to (46) we obtain $x^2y\cdot z=x^2\cdot yz$. This completes the proof.

Now we shall use the following notation. Let $(G, \cdot)$ be an LC-loop and let $x \in G$. We define the powers of x as follows: $x^0=1$; if n is a positive integer, then define $x^n=x\cdot x^{n-1}$ and $x^{-n}=(x^{-1})^n$.

Lemma 1. *If* $(G, \cdot)$ *is an LC-loop, then every pair* x, y *of elements of* G *satisfies the law*

$$x^{m+1}y = x\cdot x^m y = x^m \cdot xy \tag{61}$$

where m *is an arbitrary integer.*

Note. The statements of Lemma 1 and Theorem 3 are proved for right Bol loops by D. A. Robinson in [3], and his method applicable for LC-loops.

Proof. The lemma is obvious for $m=-1$, $m=0$ and $m=1$. Now let $k>2$ be an integer and suppose that (61) holds for each i where $k>i>1$. Then

$$x\cdot x^k y = x(x\cdot x^{k-1}y) = x[x(x^{k-2}\cdot xy)] = (x\cdot xx^{k-2})\cdot xy = x^k\cdot xy,$$

and

$$x\cdot x^k y = x(x\cdot x^{k-1}y) = (x\cdot xx^{k-1})y = x^{k+1}y.$$

So

(62) $$x^m y = x \cdot x^{m-1} y = x^{m-1} \cdot xy$$

is satisfied by all x, y in G and each non-negative integer m. Taking $x = x^{-1}$ in (62) we have

$$(x^{-1})^m y = x^{-1}.(x^{-1})^{m-1} y \quad \text{whence} \quad x \cdot x^{-m} y = x^{-m+1} y,$$

and taking $x = x^{-1}$, $y = xy$ in (62) we get

$$(x^{-1})^m \cdot xy = (x^{-1})^{m-1}(x^{-1} \cdot xy) \quad \text{whence} \quad x^{-m} \cdot xy = x^{-m+1} y.$$

Thus

(63) $$x^{-m+1} y = x \cdot x^{-m} y = x^{-m} \cdot xy$$

holds. The lemma is proved by (62) and (63).

We omit the proof of the next theorem (cf. the proof of Theorem 2. 2 in [3]).

Theorem 3. *If $(G, \cdot)$ is an LC-loop, then every pair x, y of elements of G satisfies the law*

(64) $$x^{m+n} y = x^n \cdot x^m y$$

for all integers m and n.

Corollary 3. *Every LC-loop is power-associative.*

Theorem 4. *A loop is a C-loop if and only if it is both an LC- and RC-loop.*

Proof. Suppose that $(G, \cdot)$ is a C-loop. This means that

(37) $$(yx \cdot x)z = y(x \cdot xz)$$

holds for all x, y, z in G. First of all, we prove that $(G, \cdot)$ has the inverse property. Define ${}^{-1}x$ and x^{-1} by ${}^{-1}xx = 1$ and by $xx^{-1} = 1$ respectively for all x in G. Then for any given elements x, y of G, using the solutions u, v of the equations $xu = y$ and $vx = y$, and (37), we obtain

$${}^{-1}x \cdot xy = {}^{-1}x(x \cdot xu) = ({}^{-1}xx \cdot x)u = xu = y$$

and

$$yx \cdot x^{-1} = (vx \cdot x)x^{-1} = v(x \cdot xx^{-1}) = vx = y.$$

Therefore $(G, \cdot)$ is an inverse property loop, and ${}^{-1}x = x^{-1}$ also holds for each x in G. So (37) implies that $\langle R_x R_x, L_x^{-1} L_x^{-1}, I \rangle$ is an autotopism of $(G, \cdot)$ for all x in G. Thus, by Lemma A of [2],

$$\langle I, JL_x^{-1} L_x^{-1} J, R_x R_x \rangle = \langle I, R_x R_x, R_x R_x \rangle$$

and

$$\langle JR_x R_x J, I, L_x^{-1} L_x^{-1} \rangle^{-1} = \langle L_x L_x, I, L_x L_x \rangle$$

are also autotopisms of $(G, \cdot)$. And therefore the RC-identity (56) and the LC-identity (46) holds for $(G, \cdot)$.

Conversely, if $(G, \cdot)$ is both an LC- and RC-loop, then, by Theorem 2, $(G, \cdot)$ is an inverse property loop. It follows from (46) that $\langle L_x L_x, I, L_x L_x \rangle$ is an autotopism of $(G, \cdot)$ for all x in G, and so, by the inverse property of $(G, \cdot)$, $\langle JL_x L_x J, L_x L_x, I \rangle^{-1} = \langle R_x R_x, L_x^{-1} L_x^{-1}, I \rangle$ is also an autotopism. This shows that $(G, \cdot)$ satisfies the C-identity (37). The theorem is proved.

Summarizing our results *we have the following implications:*

group → *extra loop* —
- → *Moufang loop* —
  - → *left Bol loop.*
  - → *right Bol loop.*
- → *C-loop* —
  - → *LC-loop.*
  - → *RC-loop.*

It can be shown by examples that *all these implications are irreversible.*

The loop given by the multiplication Table 1 is a *C*-loop but it is not a Moufang loop.

The multiplication Table 2 defines an *LC*-loop but this is neither a *C*-loop nor a left Bol loop.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	1	5	10	3	8	9	6	7	4
3	3	5	1	7	2	10	4	9	8	6
4	4	10	7	1	8	9	3	5	6	2
5	5	3	2	8	1	7	6	4	10	9
6	6	8	10	9	7	1	5	2	4	3
7	7	9	4	3	6	5	1	10	2	8
8	8	6	9	5	4	2	10	1	3	7
9	9	7	8	6	10	4	2	3	1	5
10	10	4	6	2	9	3	8	7	5	1

Table 1.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	4	3
3	3	5	1	6	2	4
4	4	3	2	1	6	5
5	5	6	4	3	1	2
6	6	4	5	2	3	1

Table 2.

References

[1] R. H. Bruck, A survey of binary systems, *Berlin—Göttingen—Heidelberg,* 1958.
[2] F. Fenyves, Extra loops I., *Publ. Math. Debrecen,* **15** (1968), 235—238.
[3] D. A. Robinson, Bol loops, *Trans. Amer. Math. Soc.* **123** (1966), 341—354.

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