On nonlinear tensorial connexions

To professor Ottó Varga on his 60th birthday

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1. *Introduction.* A tensorial connexion of type (r, s) is a direct linear connexion between tensors of type (r, s). Such a direct connexion is more general, than that which is induced by the usual linear connexion between the vectors of the tangent spaces of a manifold. In the case of tensors of type (2, 0) this tensorial connexion is given by

(1) $$Dt^{ij} \equiv dt^{ij} + \gamma_{kl}{}^{ij}{}_r t^{kl}\, dx^r = 0.$$

It reduces to the induced connexion

$$Dt^{ij} \equiv dt^{ij} + \Gamma_k{}^i{}_r t^{kj}\, dx^r + \Gamma_k{}^j{}_r t^{ik}\, dx^r = 0$$

if and only if

$$\gamma_{kl}{}^{ij}{}_r = \Gamma_k{}^i{}_r \delta_l^j + \Gamma_l{}^j{}_r \delta_k^i.$$

Tensorial connexions have been studied in a number of papers.[1]) A nonlinear tensorial connexion is a direct connexion between the tensors, not necessarily linear in the components of the tensor. Such a connexion occurs at H. FRIESECKE [4] and W. BARTHEL [1].

In this paper the curvature and equivalence theories of the nonlinear homogeneous tensorial connexions are discussed. Curvature quantities will be obtained through the investigation of the equivalence. The results will show certain analogy to but also some differences against the theory of linear tensorial connexions. For the sake of a better lucidity we will confine ourselves to the case of tensors of type (2, 0).

2. *Nonlinear tensorial connexion.* Let $t^{ij}(x)$ be a tensor of type (2, 0) attached to the point x of a differentiable manifold V_n. A nonlinear connexion between these tensors is defined by writing an absolute differential in the form

(2) $$Dt^{ij} \overset{\text{def}}{=} dt^{ij} + B^{ij}{}_r(x, t)\, dx^r,$$

where $B^{ij}{}_r(x, t)$ is continuously differentiable in all its variables. In order that Dt^{ij} be a tensor, $B^{ij}{}_r$ has to have the following law of transformation:

(3) $$B^{rs}{}_k(x, t)\, A_r^{i'} A_s^{j'} = -(A_{mk}^{i'} A_p^{j'} + A_{pk}^{j'} A_m^{i'})\, t^{mp} + B^{i'j'}{}_{l'}(x', t')\, A_k^{l'},$$

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[1]) See for example [3], [5], [8].

where x^i and $x^{i'}$ are coordinates in two coordinate systems connected by

$$x^{i'} = x^{i'}(x)$$

and

$$A^{i'}_r \equiv \frac{\partial x^{i'}}{\partial x^r}, \quad A^{i'}_{mk} \equiv \frac{\partial^2 x^{i'}}{\partial x^m \partial x^k}; \quad A^r_{i'} A^{i'}_s = \delta^r_s.$$

A tensor t^{ij} is parallel displaced, if $Dt^{ij}=0$. (3) makes it obvious that the linearity of $B^{ij}{}_r(x, t)$ in t^{kl} is invariant under transformations of coordinates. Thus (2) reduces to (1) if and only if B is linear and homogeneous in t^{kl}.

Now we want to assume, that $B^{ij}{}_r(x, t)$ is homogeneous of degree 1 in the t^{kl}. This means, that

$$\frac{\partial B^{ij}{}_r}{\partial t^{pm}} t^{pm} = B^{ij}{}_r \tag{4}$$

and this is equivalent to

$$D(\lambda t^{ij}) = \lambda D t^{ij}$$

accodring to (2). A nonlinear tensorial connexion satisfying (4) will be called homogeneous. Nonlinear connexions will mean homogeneous ones throughout this paper.

3. *Equivalence of two nonlinear tensorial connexions.* We consider two nonlinear tensorial connexions. One of them is given on the $V_n(x)$ by the $B^{ij}{}_r(x, t)$, and the other is given on the $\overline{V}_n(x')$ by the $\overline{B}^{i'j'}{}_{r'}(x', t')$. They are equivalent, if the system of partial differential equations

$$B^{rs}{}_k(x, t) A^{i'}_r A^{j'}_s = -(A^{i'}_{mk} A^{j'}_p + A^{j'}_{pk} A^{i'}_m) t^{mp} + \overline{B}^{i'j'}{}_{l'}(x', t') A^{l'}_k \tag{5}$$

has an invertible solution for the $x^{i'}(x)$.

First we want to bring this system to a form, more suitable for our investigation. Differentiating (5) with respect to t^{mp} we have

$$-(A^{i'}_{mk} A^r_{i'} \delta^s_p + A^{i'}_{pk} A^s_{i'} \delta^r_m) + \overline{G}^{i'j'}{}_{u'v'l'} A^{u'}_m A^{v'}_p A^{l'}_k A^r_{i'} A^s_{j'} - G^{rs}{}_{mpk} = 0, \tag{6}$$

where

$$\frac{\partial B^{ij}{}_r}{\partial t^{pm}}(x, t) \equiv G^{ij}{}_{pmr}(x, t). \tag{7}$$

Conversely, contracting (6) by t^{mp} we obtain (5) with respect to the homogenity of degree 1 in t^{ij} of B.

Now we can express $A^{i'}{}_{mk}$ explicitly from (6) by contracting (6) with δ^p_r:

$$A^{i'}_{mk} = \tfrac{1}{2}(\overline{G}^{v'i'}{}_{u'v'l'} A^{u'}_m A^{l'}_k - G^{rs}{}_{mrk} A^{i'}_s). \tag{8}$$

Using the notation

$$\tfrac{1}{2} G^{rs}{}_{mrk} \equiv M_m{}^s{}_k \tag{9}$$

(8) gets the form

$$A^{i'}_{mk} = \overline{M}_{u'}{}^{i'}{}_{l'} A^{u'}_m A^{l'}_k - M_m{}^s{}_k A^{i'}_s.$$

Substituting this in (6) we obtain

$$\overline{F}^{a'b'}{}_{u'v'l'} A^{u'}_m A^{v'}_p A^{l'}_k = F^{rs}{}_{mpk} A^{a'}_r A^{b'}_s,$$

where

$$F^{rs}{}_{mpk} \equiv G^{rs}{}_{mpk} - M_m{}^r{}_k \delta^s_p - M_p{}^s{}_k \delta^r_m. \tag{10}$$

Now the partial differential equation system (5) gets the form

$$\text{a)}\ \frac{\partial x^{i'}}{\partial x^{j}} = A^{i'}_{j}$$

$$(11)\quad \text{b)}\ \frac{\partial A^{i'}_{m}}{\partial x^{k}} = \overline{M}_{u'}{}^{i'}{}_{l'} A^{u'}_{m} A^{l'}_{k} - M_{m}{}^{s}{}_{k} A^{i'}_{s}$$

$$\text{c)}\ \overline{F}^{a'b'}{}_{u'v'l'} A^{u'}_{m} A^{v'}_{p} A^{l'}_{k} = F^{rs}{}_{mpk} A^{a'}_{r} A^{b'}_{s}.$$

The conditions of the integrability of (11, a, b) with respect to (11, b) are

$$(12)\quad \overline{S}_{u'}{}^{i'}{}_{l'} A^{u'}_{m} A^{l'}_{k} - S_{m}{}^{s}{}_{k} A^{i'}_{s} = 0 \qquad \text{and} \qquad (13)\quad \overline{R}^{i'}{}_{u'l'p'} A^{u'}_{m} A^{l'}_{k} A^{p'}_{j} = R^{r}{}_{mkj} A^{i'}_{r},$$

where

$$(14)\qquad S_{m}{}^{s}{}_{k} \equiv M_{m}{}^{s}{}_{k} - M_{k}{}^{s}{}_{m}$$

and

$$(15)\qquad R^{r}{}_{mkj} \equiv \frac{\partial M_{m}{}^{r}{}_{k}}{\partial x^{j}} - \frac{\partial M_{m}{}^{r}{}_{j}}{\partial x^{k}} + 2 M_{s}{}^{r}{}_{[k} M_{|m|}{}^{s}{}_{j]}.$$

Differentiating (16, c), (12) and (13) with respect to x^i and eliminating $\frac{\partial x^{i'}}{\partial x^{i}}$ and $\frac{\partial A^{i'}_{m}}{\partial x^{i}}$ by means of (11, a, b) we obtain the equations E_1:

$$\nabla_{c'} \overline{S}_{u'}{}^{i'}{}_{l'} A^{u'}_{m} A^{l'}_{k} A^{c'}_{d} = \nabla_{d} S_{m}{}^{s}{}_{k} A^{i'}_{s}$$

$$\nabla_{c'} \overline{R}^{i'}{}_{u'l'p'} A^{u'}_{m} A^{l'}_{k} A^{p'}_{j} A^{c'}_{d} = \nabla_{d} R^{s}{}_{mkj} A^{i'}_{s}$$

$$\nabla_{c'} \overline{F}^{a'b'}{}_{u'v'l'} A^{u'}_{m} A^{v'}_{p} A^{l'}_{k} A^{c'}_{d} = \nabla_{d} F^{rs}{}_{mpk} A^{a'}_{r} A^{b'}_{s},$$

where ∇_{α} denotes a covariant derivative with respect to the $M_{m}{}^{s}{}_{k}$. Further derivations with respect to the x^j lead to relations similar to E_1, but involving higher covariant derivatives. They are denoted by $E_2, E_3, \ldots$. Now by making use of a well known result of J. M. Thomas and O. Veblen [2]) we obtain the following

Theorem 1. *A necessary and sufficient condition for the equivalence of two nonlinear tensorial connexions $B^{ij}{}_{k}(x, t)$ and $\overline{B}^{i'j'}{}_{k'}(x', t')$ is the existence of an integer N, such that* (11, c), (12), (13) *and* $E_1, E_2, \ldots, E_N$ *are compatible considered as equations for the $x^{i'}$ and $A^{i'}_{j}$, and that their solutions satisfy E_{N+1} and Det $|A^{i'}_{j}| \neq 0$.*

4. *Curvature quantities and their geometrical meaning.* We call S torsion tensor and R and F curvature tensors. They have certain geometrical meaning.

Let us consider the nonlinear connexion between the vectors of the tangent spaces of a V_n, given by

$$(16)\qquad D^{*}\xi^{i} = d\xi^{i} + H_{u}{}^{i}{}_{l}(x, \xi)\, \xi^{u}\, dx^{l} = 0,$$

where $H_{u}{}^{i}{}_{l}$ is homogeneous of degree zero in the ξ^j. [3])

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[2]) J. M. Thomas—O. Veblen [9], or T. Y. Thomas [10], p. 203.

[3]) See e. g. H. Friesecke [4], W. Barthel [2], or H. Rund [7] p. 83. Connexions not linear neither in the components of the vector nor in the differentials of the coordinates were studied by A. Moór [6].

We call the tensor t^{ij} square-decomposable, if t^{ij} has a representation in the form

$$t^{ij} = \xi^i \xi^j \tag{17}$$

where ξ is a vector. γ will denote the set of the square-decomposable tensors.

We can extend (16) to the elements of γ by means of the definition

$$D^*(\xi^i \xi^j) \overset{\text{def}}{=} D^* \xi^i \cdot \xi^j + \xi^i \cdot D^* \xi^j.$$

We get

$$D^*(\xi^i \xi^j) = d(\xi^i \xi^j) + [H_u{}^i{}_l(x, \xi) \xi^u \xi^j + H_u{}^j{}_l(x, \xi) \xi^i \xi^u] dx^l = 0. \tag{18}$$

Let

$$M_u{}^i{}_l(x, t) = H_u{}^i{}_l(x, \xi), \tag{19}$$

where t and ξ are related by (17). Then (18) gets the form

$$dt^{ij} + [M_u{}^i{}_l(x, t) t^{uj} + M_u{}^j{}_l(x, t) t^{iu}] dx^l = 0. \tag{20}$$

(20) is called the nonlinear tensorial connexion induced by (16) over the γ. Conversely, a nonlinear tensorial connexion reduces over γ to a nonlinear vector-connexion of type (16), if it has the form (20) over γ, and (19) and (17) hold. In this case (20) is induced by (16).

Theorem 2. *A nonlinear tensorial connexion is induced over γ by a nonlinear connexion of vectors, if and only if F vanishes over γ.*

If the tensor F vanishes, then according to (2), (7) and (10)

$$Dt^{ij} = dt^{ij} + B^{ij}{}_l dx^l = dt^{ij} + G^{ij}{}_{uvl} t^{uv} dx^l =$$

$$= dt^{ij} + (M_u{}^i{}_l \delta^j_v + M_v{}^j{}_l \delta^i_u) t^{uv} dx^l = dt^{ij} + [M_u{}^i{}_l(x, t) t^{uj} + M_v{}^j{}_l(x, t) t^{iv}] dx^l.$$

By (19) this determines a vector connexion of type (16) and then $Dt\,(t \in \gamma)$ is iduced by this nonlinear vector connexion.

Conversely, if our nonlinear tensorial connexion is induced over γ by a nonlinear vector connexion, that is if

$$B^{ij}{}_l = H_u{}^i{}_l t^{uj} + H_v{}^j{}_l t^{iv},$$

then

$$G^{ij}{}_{uvl} = \frac{\partial B^{ij}{}_l}{\partial t^{uv}} = H_u{}^i{}_l \delta^j_v + H_v{}^j{}_l \delta^i_u \tag{21}$$

and

$$M_u{}^j{}_l = \tfrac{1}{2} G^{ij}{}_{uil} = \tfrac{1}{2}(H_u{}^j{}_l + H_u{}^j{}_l) = H_u{}^j{}_l.$$

In this case F vanishes over γ according to (10) and (21).

As a corollary of the preceding theorem we have also

Theorem 3. *If F vanishes, then R and S reduce to the curvature and torsion tensor respectively, of the nonlinear vector connexion* (16).

We call the connexion (2) affine, if a coordinate system $x^{i'}$ exists, where the $B^{i'j'}{}_{l'}(x', t')$ identically vanish.

Theorem 4. *The connexion* (2), (4) *is affine if and only if* $F=R=S=0$.

Let $F=R=S=0$. In this case the nonlinear tensorial connexion (2), (4), given by $B^{ij}{}_l(x, t)$ and the connexion given by $\bar{B}^{i'j'}{}_{l'}(x', t')\equiv 0$ are equivalent. Namely (11) reduces in this case to

$$\frac{\partial x^{i'}}{\partial x^j} = A^{i'}_j \tag{22}$$

$$\frac{\partial A^{i'}_j}{\partial x^k} = -M_j{}^s{}_k A^{i'}_s,$$

for $\bar{M}$ and $\bar{F}$ vanish because of $\bar{B}$ does so, and $F=0$ according to our condition. Thus (22) is completely integrable since $R=S=0$.

Conversely, if $\bar{B}^{i'j'}{}_l\equiv 0$, then we have from (7), (9), (10), (14) and (15) $\bar{F}=\bar{R}=\bar{S}=0$ and this is true in every coordinate system because of their tensor character.

Finally we make two remarks: *A)* If $R=S=0$ then a coordinate system x' exists, in which

$$M_{u'}{}^{i'}{}_{l'}(x', t') \equiv 0, \tag{23}$$

for in this case (22) has a solution. In this coordinate system $F^{a'b'}{}_{u'v'l'}\equiv G^{a'b'}{}_{u'v'l'}$. — Conversely, if (23) holds, then obviously $R=S=0$.

B) If $F=0$ and M does not depend from t, then $B^{ij}{}_t$ is induced by a linear vectorial connexion on γ, and conversely.

(20) means an expansion of (16) over γ. But other expansions of (16) are also conceivable, extending (16) to broader subsets of the tensors t^{ij} than γ. To these problems we wish to return at another occasion.

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