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A sharp inequality for Bergman-Nevanlinna functions

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Abstract. In this note we prove inequalities, one for harmonic functions in the unit disc Δ which are representable as the Poisson integral of a finite measure on $\partial\Delta$, and another one for Bergman-Nevanlinna functions. These are used to characterize those entire functions f whose associated autonomous nonlinear superposition operator transforms Nevanlinna functions into Bergman-Nevanlinna functions.

1. Introduction

Let f be an entire function and $H(\Delta)$ the space of analytic functions in the unit disc Δ. By h^1 we mean the Banach space of harmonic functions in Δ which are equal to the difference of two positive harmonic functions. The nonlinear superposition operator F_f is defined by $F_f(u) = f \circ u$, whenever $u \in H(\Delta)$. General information about this operator may be found in [1]. About the action of F_f between Bergman spaces the reader may consult [5]. We shall denote by N the well known Nevanlinna space of functions u in $H(\Delta)$ such that $\log^+|u(z)|$ has a harmonic majorant. The symbol BN will denote the Bergman-Nevanlinna space of functions u in $H(\Delta)$ such that

$$\iint_\Delta \log^+|u(z)|dxdy < \infty.$$

It is natural to ask what are the entire functions f such that F_f transforms N into BN. In other words, characterize those entire functions f for which $f \circ u \in BN$ whenever $u \in N$. In this paper we solve this problem. To do this we need an inequality for functions in h^1 which may

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be of independent interest. We shall also need an inequality for functions in BN which, in a certain way, sharpens the usual one.

We shall use the following notation: for $0 < p < \infty$

$$M_p(r,h) = \left(\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}},$$

$$\|h\|_p = \sup_{r<1} M_p(r,h).$$

When $p = \infty$ we shall denote by $M_\infty(r,h)$ the usual maximum modulus of h on the circle of radius r.

In [4] I proved the following inequality

Theorem A. *If $h \in h^1$ then*

$$\int_0^1 M_{2-\varepsilon}^{2-\varepsilon}(r,h)dr \leq C\|h\|_1^{2-\varepsilon}, \quad 0 < \varepsilon < 1.$$

For $\varepsilon = 0$ the inequality is no longer true as showln by the function $h(z) = \frac{1-|z|^2}{|1-z|^2}$.

We shall improve this inequality by changing the function $t^{2-\varepsilon}$ to a convex nondecreasing function $\phi(t)$ which grows more slowly than t^2. More precisely we have

Theorem 1. *If $h = h_1 - h_2$, where h_1 and h_2 are two positive harmonic functions in Δ, and $\phi : [1,\infty) \to \mathbb{R}^+$ is a non-decreasing convex function such that*

$$\int_1^\infty \frac{\phi(t)}{t^3} dt < \infty,$$

then

$$\iint_\Delta \phi(|h(z)|)dxdy < \infty.$$

This theorem sharpens the result $h^1 \subset b_q$, $\forall q < 2$, where b_q is the Banach space of Bergman harmonic functions ([2], page 167).

The subharmonicity of $\log^+|u(z)|$ for $u \in BN$ easily implies the well known inequality

$$\log^+|u(z)| \leq \frac{C}{(1-|z|)^2}.$$

We shall improve this inequality in the following manner.

Theorem 2. *Let $u \in BN$. Then there exists a non-decreasing convex function $\phi : [1, \infty) \to \mathbb{R}^+$ with*

$$\int_1^\infty \frac{\phi(t)}{t^3} dt < \infty,$$

such that

$$\log^+ |u(z)| \le \phi\left(\frac{1}{1-|z|}\right).$$

As an application of Theorems 1 and 2 we characterize those entire function f for which the nonlinear operator F_f acts from N to BN.

Theorem 3. *Let f be an entire function. Then F_f acts from N to BN if and only if there exists a non-decreasing convex function $\phi : [0, +\infty) \to \mathbb{R}^+$ with*

$$\int_1^\infty \frac{\phi(t)}{t^3} dt < \infty$$

and $\log^+ M_\infty(r, f) \le \phi(\log^+ r)$.

The way we have followed to prove Theorem 3 has forced us to state it in terms of a convex function ϕ. Actually we can get rid of ϕ and state Theorem 3 in the following equivalent form.

Theorem 3'. *Let f be an entire function. Then F_f acts from N to BN if and only if*

$$\int_1^\infty \frac{\log M_\infty(e^t, f)}{t^3} dt < \infty.$$

This result follows easily from Theorem 3 by the convexity (as a function of t) of the function $\log M_\infty(e^t, f)$ (Hadamard's Three Circles Theorem).

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1. Proof of Theorem 1

Set $A = \iint_{B(0,\frac{1}{2})} \phi(|h(z)|)dxdy$. By the Harnack inequality A is bounded above by $C_1 = \frac{\pi}{4}\phi(C(h_1(0)+h_2(0)))$, where C is an absolute constant. Therefore

$$\iint_{\Delta} \phi(|h(z)|)dxdy \leq \iint_{\Delta\setminus B(0,\frac{1}{2})} \phi(|h(z)|)dxdy + C_1.$$

By the Riesz-Herglotz Theorem ([6], Theorem 1.1) there is a function $\mu(t)$ of bounded total variation $\|\mu\| = \|h\|_1$ shuch that

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} d\mu(t), \quad z = re^{i\theta} \in \Delta.$$

Using Jensen's inequality at the appropriate step we can write

$$\begin{aligned}
&\iint_{\Delta\setminus B(0,\frac{1}{2})} \phi(|h(z)|)dxdy \\
&\leq \iint_{\Delta\setminus B(0,\frac{1}{2})} \phi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\|\mu\|(1-r^2)}{|e^{it}-re^{i\theta}|^2}\frac{d|\mu|(t)}{\|\mu\|}\right) dxdy \\
&\leq \iint_{\Delta\setminus B(0,\frac{1}{2})} \left(\int_{-\pi}^{\pi} \phi\left(\frac{\|\mu\|(1-r^2)}{2\pi|e^{it}-re^{i\theta}|^2}\right)\frac{d|\mu|(t)}{\|\mu\|}\right) dxdy \\
&= \iint_{\Delta\setminus B(0,\frac{1}{2})} \phi\left(\frac{\|\mu\|(1-r^2)}{2\pi|1-re^{i\theta}|^2}\right) dxdy.
\end{aligned}$$

If we set

$$I := \iint_{\Delta} \phi\left(\frac{\|\mu\|(1-|z|^2)}{|1-z|^2}\right) dxdy$$

it suffices to prove that

$$I \leq C(\phi)\|\mu\|^2.$$

Let $z = \frac{w-1}{w+1}$ with $w = t+is$. Then

$$\begin{aligned}
I &= 4\int_{\operatorname{Re} w>0} \phi(\|\mu\| \operatorname{Re} w)\frac{dtds}{|1+w|^4} \\
&= 4\int_0^{\infty} \phi(\|\mu\|t)dt \int_{-\infty}^{\infty} \frac{ds}{[(1+t)^2+s^2]^2} = 2\pi\int_0^{\infty} \frac{\phi(\|\mu\|t)}{(1+t)^3}dt \\
&\leq 2\pi\|\mu\|^2\int_0^{\infty} \frac{\phi(t)}{(\|\mu\|+t)^3}dt < \infty,
\end{aligned}$$

as required.

Remark. This results is sharp in the following sense. If $\theta : [0,\infty) \to \mathbb{R}^+$ is a non-decreasing function such that

$$\int_1^\infty \frac{\phi(t)}{t^3} dt = +\infty$$

then

$$\iint_\Delta \phi\left(\frac{1-|z|^2}{|1-z|^2}\right) dxdy = \infty.$$

The exponent 2 on the right hand side of the inequality cannot be improved.

In [4] I proved the following lemma.

Lemma 2. *If $u \in N$ then*

$$\int_0^1 dr \int_0^{2\pi} \left(\log^+ |u(re^{i\theta})|\right)^{2-\varepsilon} d\theta < \infty, \quad 0 < \varepsilon < 1.$$

As an application of Theorem 1 we have the following corollary which sharpens this result.

Corollary. *If $u \in N$ and $\phi : [0,\infty) \to \mathbb{R}^+$ is a non-decreasing convex function which satisfies*

$$\int_1^\infty \frac{\phi(s)}{s^3} ds < \infty \tag{1.1}$$

then

$$\int_0^1 rdr \int_0^{2\pi} \phi\left(\log^+ |u(re^{i\theta})|\right) d\theta < \infty. \tag{1.2}$$

Proof. We assume first that $u \neq 0$ in Δ. Then $\log|u| \in h^1$. Therefore

$$\int_0^1 rdr \int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|) d\theta \le \int_0^1 rdr \int_0^{2\pi} \phi(|\log|u||) d\theta < \infty$$

in view of Theorem 1. For a general $u \in N$ we take $v = \frac{u}{B_u}$, where is the Blaschke product associated to the zeroes of u. Then $v \in N$, $v \neq 0$ in Δ and $|u| \le |v|$. Thus

$$\int_0^1 rdr \int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|) d\theta \le \int_0^1 rdr \int_0^{2\pi} \phi(\log^+ |v(re^{i\theta})|) d\theta < \infty$$

by means of what was proved first.

Remark. This corollary is sharp in the sense that if $\phi : [0,\infty) \to \mathbb{R}^+$ is a non-decreasing function which satisfies (1.1) then (1.2) diverges when $u(z) = e^{\frac{1+z}{1-z}} \in N$.

3. Proof of Theorem 2

We shall need the following fundamental lemma.

Lemma 3. *If $u(z)$ is analytic in $\{|z| \leq R\}$ then*

$$\log^+ M_\infty(s,u) \leq \frac{R+s}{R-s}\,\frac{1}{2\pi}\int_0^{2\pi} \log^+ |u(Re^{i\theta})|d\theta, \quad 0 \leq s < R. \tag{3.1}$$

A proof of this can be found in [7, pag. 18].

Proof of Theorem 2. Let $\phi_1(r) = \log^+ M_\infty(1-\frac{1}{r}, u)$. Clearly

$$\log^+ |u(z)| \leq \phi_1\left(\frac{1}{1-|z|}\right).$$

Now we define

$$\phi(r) = 4\int_1^{2r} T\left(1-\frac{1}{2t}, u\right) dt,$$

where

$$T(s,u) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |u(se^{i\theta})|d\theta$$

is the Nevanlinna characteristic function of u. Since $T(s,u)$ is a positive non-decreasing function of s then $\phi(r)$ is an increasing convex function of r. We also have that $\phi_1(r) \leq \phi(r)$. In fact, if we set $R = 1-\frac{1}{2r}$ and $s = 1-\frac{1}{r}$ in (3.1) we obtain

$$\log^+ M_\infty\left(1-\frac{1}{r}, u\right) \leq (4r-3)\, T\left(1-\frac{1}{2r}, u\right).$$

Hence

$$\begin{aligned}\phi(t) &\geq 4\int_r^{2r} T\left(1-\frac{1}{2t}, u\right) dt \geq 4rT\left(1-\frac{1}{2r}, u\right)\\ &\geq \frac{4r}{4r-3}\log^+ M_\infty\left(1-\frac{1}{r}, u\right) \geq \phi_1(r).\end{aligned}$$

Finally,

$$\begin{aligned}\int_1^\infty \frac{\phi(r)}{r^3}dr &= 4\int_1^\infty \frac{dr}{r^3}\int_1^{2r} T\left(1-\frac{1}{2t}, u\right) dt\\ &= 2\int_1^\infty T\left(1-\frac{1}{2t}, u\right) dt \int_{\frac{t}{2}}^\infty \frac{dr}{r^3} = 8\int_1^\infty \frac{T(1-\frac{1}{2t}, u)}{t^2}dt\end{aligned}$$

$$= 16 \int_{\frac{1}{2}}^{1} T(s,u)ds < \infty,$$

since $u \in BN$.

4. Proof of Theorem 3

First of all let us assume that $u \in N$ and

$$\log^+ M_\infty(r,f) \leq \phi(\log^+ r) + C, \quad r \geq 0,$$

for a function ϕ satisfying the conditions stated in the theorem. Then

$$\iint_\Delta \log^+ |F_f(u)(z)|dxdy = \int_0^1 rdr \left(\int_0^{2\pi} |f(u(re^{i\theta}))|d\theta \right)$$

$$\leq \int_0^1 rdr \left(\int_0^{2\pi} \log^+ M_\infty(|u(re^{i\theta})|, f)d\theta \right)$$

$$\leq \int_0^1 rdr \left(\int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|)d\theta \right) + O(1)$$

$$= \iint_\Delta \phi(\log^+ |u(z)|)dxdy + O(1) < \infty,$$

in view of the corollary to Theorem 1. Therefore, F_f acts from N to BN. Next, let us suppose thet F_f acts from N to BN. The function $w = u(z) = \exp\left\{\frac{1+z}{1-z}\right\}$ belongs to N. Thus $f \circ u$ belongs to BN. As a consequence of Theorem 2 we have a function $\phi : [1,\infty) \to \mathbb{R}^+$, convex and non-decreasing which satisfies

$$\int_1^\infty \frac{\phi(r)}{r^3} dr < \infty$$

and

$$\log^+ |f(w)| \leq \phi\left(\frac{1}{1-|z|}\right). \tag{4.1}$$

We shall confine ourselves to those z in the Stolz angle $S = \{z : |1-z| \leq C_1(1-|z|), \operatorname{Re} z \geq c_0\} \cap \Delta$. If S is big enough then $u(S) \supseteq \{w : |w| > R\}$, for some $R > 1$. Since $|w| = \exp\left\{\frac{1-|z|^2}{|1-z|^2}\right\}$ we can write

$$\frac{1}{1-|z|} = \frac{|1-z|^2}{(1-|z|)^2(1+|z|)} \log|w| \leq C_1^2 \log|w|, \quad z \in S. \tag{4.2}$$

Since ϕ_1 is non-decreasing we obtain from (4.1) and (4.2) that

$$\log^+ |f(w)| \leq \phi_1(C_1^2 \log |w|) \quad |w| > R.$$

The desired result is obtained by choosing $\phi(r) = \phi_1(C_1^2 r)$.

References

[1] J. Appel and P. P. Zabrejko, Nonlinear Superposition Operators, *Cambridge University Press*, 1990.

[2] S. Axler, P. Bourdon and W. Ramey, Harmonic Function Theory (Springer-Verlag GTM 137, ed.), 1992.

[3] A. Baernstein II, Private Communication.

[4] G. A. Cámera, The Nonlinear Superposition Operator Acting on N and BN, *Acta Cientifica Venezolana* **44** No. 1 (1993), 3–5.

[5] G. A. Cámera and J. Giménez, The Nonlinear Superposition Operator Acting on Bergman Spaces, *Compositio Mathematica* **93** (1994), 23–25.

[6] P. Duren, Theory of H^p Spaces, *Academic Press*, *New York*, 1970.

[7] W. K. Hayman, Meromorphic Functions, *Oxford Mathematical Monographs*, 1975.

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