Generalized functions defined by fields which are isomorphic to the field of Mikusiński operators

To Professor O. Varga on his 60th birthday

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Introduction

There are several equivalent definitions of distributions. These definitions have as a common characteristic the following steps:

(I) As a basic material, a linear space $\mathcal{S}$ is constructed and the elements of $\mathcal{S}$ are called distributions.

(II) The linear space $\mathcal{C}$ of continuous functions is embedded in $\mathcal{S}$.

(III) The notion of the derivative is defined in $\mathcal{S}$ such that if a distribution is a function with a continuous derivative, then its derivative in the distributional sense coincides with its derivative in the ordinary sense.

(IV) There are put other notions, as much as possible, into distributions. For example there are defined the multiplication of distributions by infinitely derivable functions, the definite integral of a distribution, the limit of a distribution at a point, the convolution of two distribution, etc.

In the theory of distributions of L. Schwartz $\mathcal{S}$ is the dual space of certain space formed of infinitely differentiable functions with compact carriers. In the Mikusiński—Sikorski sequential theory of distributions $\mathcal{S}$ is the space of abstraction classes of certain equivalent fundamental sequences ([1]).

The equivalence of the different notions of distributions is based on the fact that (in a case of one variable) for every distribution and for any closed subinterval there exists an integer k such that the distribution is the k-th derivative of a continuous function in this interval.

The concept of Mikusiński's convolution quotients provides a further possibility to obtain a generalization of functions. Indeed, if $\mathcal{S}=\mathcal{M}$ is the field of Mikusiński operators, the properties (I), (II) and (III) are fulfilled. According to the book of Erdélyi ([2] p. 25): "The embedding of continuous and locally integrable functions, and of the delta function, in $\mathcal{M}$ suggests that convolution quotients be regarded as generalized functions." It is known that the notion of generalized functions in the above sense is a generalization of distributions with left-sided bounded carriers (see e.g. Wloka [3]). We have recently introduced some notions for generalized functions enumerated in (IV). ([4]).

In the paper [5] is introduced the notion of generalized convolution quotients.

The starting point of this theory is a linear space $\mathscr{C}^+(\alpha, \beta)$ formed of continuous functions in a fixed interval $\alpha < x < \beta$ $(-\infty \leqq \alpha < \beta \leqq \infty)$. Every function $f(x)$ of $\mathscr{C}_+(\alpha, \beta)$ vanishes identically in a right-sided neighbourhood of α. The interval of vanishing depends on f. A continuous increasing function $\mu(x)$ is referred to as a base function on (α, β) if $\lim_{x \to \alpha+0} \mu(x) = -\infty$ and $\lim_{x \to \beta-0} \mu(x) = \infty$. By means of $\mu(x)$ is defined the generalized convolution in $\mathscr{C}^+(\alpha, \beta)$ in the following manner:

$$f * g = \int_{\alpha}^{\beta} f[\mu^{-1}(\mu(x) - \mu(t))] g(t) \, d\mu(t) \tag{1}$$

where $\mu^{-1}(t)$ is the inverse function of $\mu(x)$ and the integral is understood in the Stieltjes sense.

The space $\mathscr{C}^+(\alpha, \beta)$ endowed with the multiplication (1) is a commutative ring without divisors of zero, and it is denoted by $\mathscr{C}_\mu(\alpha, \beta)$. The quotient field of $\mathscr{C}_\mu(\alpha, \beta)$ is denoted by $\mathscr{M}_\mu$ and $\mathscr{M}_\mu$ is called the field of generalized convolution quotients. $\mathscr{M}_{\{x\}}$ is the field $\mathscr{M}$ of Mikusiński operators. In [5] is proved, that every $\mathscr{M}_\mu$ is isomorphic to $\mathscr{M}$.

The purpose of the present paper is to show that every $\mathscr{M}_\mu$ provides a generalization of the concept of a function. Since $\mathscr{M}_\mu$ is a linear space, and the continuous functions of $\mathscr{C}_\mu(\alpha, \beta)$ are embedded in $\mathscr{M}_\mu$, the properties (I) and (II) are fulfilled for $\mathscr{S} = \mathscr{M}_\mu$. In the present work we shall introduce the concept of the derivative in $\mathscr{M}_\mu$ such that also the property (III) will be fulfilled. Thus we may regard the elements of $\mathscr{M}_\mu$ as generalized functions. In general it is impossible the elements of $\mathscr{M}_\mu$ identify with distributions. Therefore, we make a convention, that the elements of $\mathscr{M}_\mu$ and only these will be referred to as generalized functions.

Moreover, we shall define the multiplication of generalized functions by some kind of infinitely derivable functions. We will introduce the notion of the limit of a generalized function $f(x)$ as $x \to \beta$. We define the indefinite and the definite integral of a generalized function of $\mathscr{M}_\mu$. It will be showed that the notion of the definite integral may be extended for functions too, which are not elements of $\mathscr{M}_\mu$. It will be proved that every Lebesgue integrable function is integrable in the generalized sense given here, and the generalized integral is equal to the Lebesgue integral. However, it may happen, that an in the usual sense divergent integral is convergent in the generalized sense.

Though every $\mathscr{M}_\mu$ provides a generalization of the concept of a function, these different generalizations are not equivalent. We shall show namely, that, for example, the integral $\int_b^\infty t^m dt$ $(b \geqq 0)$ does not exist in $\mathscr{M}$, for the integer $m \geqq 1$.

However, if $\mu(x) = \log x$, the integral exists in $\mathscr{M}_\mu$ and

$$\int_b^\infty t^\lambda \, dt = -\frac{b^{\lambda+1}}{\lambda+1} \qquad (b \geqq 0)$$

for all complex numbers $\lambda \neq -1$. This result coincides with the result obtained in the theory of distributions ([6]). Furthermore, not only the existence but also the value of the integral may depend on the choice of the base function $\mu(x)$. Namely, it will be showed that

$$\int_0^\infty t^m\,dt = \frac{(-1)^{m+1}}{m+1} \qquad (m>0),$$

provided $\mu(x)=\log(x+1)$.

In the quantum theory of radiation is raised the following divergent integral: $\int_1^\infty \frac{1}{t}\,dt$. This integral does not exist in $\mathcal{M}$, neither in $\mathcal{M}_\mu$ if $\mu(x)=\log x$, nor if $\mu(x)=\log(x+1)$. However it will be proved that

$$\int_1^\infty \frac{1}{t}\,dt = 0,$$

provided $\mu(x)=\log\log x$.

Summarized our results, it may be established, that every regularization of a divergent integral is connected with a base function $\mu(x)$. If there are raised divergent integrals in some kind of a theory, as there are raised divergent integrals for instance in the quantum theory of radiations, then it may happen that there exists a base function $\mu(x)$, such that certain, in the usual sense divergent integrals are convergent in the generalized sense in $\mathcal{M}_\mu$. However, in order to elimeinate the contradictions, it is not allowed to make a change in μ, once for all throughout the theory.

§ 0. Preliminary notions

In this section we shall summarize the results which are proved in the paper [5].

Let $\mathscr{C}_\mu(\alpha,\beta)$ be the ring, in which the multiplication is defined by means of a base function $\mu(x)$ by the formula (1). The symbol $\{f(x)\}_\mu$ will denote that the function $f(x)\in\mathscr{C}^+(\alpha,\beta)$ is regarded as an element of the ring $\mathscr{C}_\mu(\alpha,\beta)$. Thus we can preserve the usual notations of the algebraic operations in $\mathscr{C}_\mu(\alpha,\beta)$ without misunderstandings:

(0. 1) $$\{f(x)\}_\mu+\{g(x)\}_\mu=\{f(x)+g(x)\}_\mu,$$

(0. 2) $$\{f(x)\}_\mu\{g(x)\}_\mu=\Big\{\int_\alpha^\beta f[\mu^{-1}(\mu(x)-\mu(t))]g(t)\,d\mu(t)\Big\}_\mu.$$

In the case of $\mu(x)=x$ we shall observe the original notations of Mikusiński and we shall write simply

(0. 3) $$\{f(t)\}\{g(t)\}=\Big\{\int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau\Big\}$$

and the ring $\mathscr{C}_{\{x\}}(-\infty,\infty)$ will be denoted simply by $\mathscr{C}^+(-\infty,\infty)$. $\mathscr{C}_\mu(\alpha,\beta)$ is a commutative ring without zero divisors. Thus $\mathscr{C}_\mu(\alpha,\beta)$ can be extended to a quotient field $\mathcal{M}_\mu$. $\mathcal{M}_{\{x\}}$ is the field $\mathcal{M}$ of Mikusiński operators. The elements of $\mathcal{M}_\mu$ will be denoted by the symbols

$$\frac{\{f(x)\}_\mu}{\{g(x)\}_\mu}, \ \dots \qquad (f, g \in \mathcal{C}_\mu(\alpha, \beta)).$$

In the case of $\mu(x)=x$, the elements of $\mathcal{M}$ are the Mikusiński operators, and these will be denoted usually by

$$\frac{\{f(t)\}}{\{g(t)\}}, \ \dots \qquad (f, g \in \mathcal{C}^+(-\infty, \infty)).$$

Every $\mathcal{M}_\mu$ is isomorphic to $\mathcal{M}$. This isomorphism is given by the mapping L_μ defined as follows:

$$L_\mu\{f(t)\} = \{f[\mu(x)]\}_\mu, \quad \text{if} \quad f(t) \in \mathcal{C}^+(-\infty, \infty),$$

$$L_\mu\left(\frac{\{f(t)\}}{\{g(t)\}}\right) = \frac{\{f[\mu(x)]\}_\mu}{\{g[\mu(x)]\}_\mu} = \frac{L_\mu(f)}{L_\mu(g)}, \quad \text{if} \quad \frac{\{f(t)\}}{\{g(t)\}} \in \mathcal{M}. \tag{0. 4}$$

The field $\mathcal{K}$ of complex numbers is embedded in $\mathcal{M}_\mu$ and the following equation holds:

$$\lambda = \frac{\{\lambda f(x)\}_\mu}{\{f(x)\}_\mu} \tag{0. 5}$$

for all $\lambda \in \mathcal{K}$ and $f \in \mathcal{C}_\mu(\alpha, \beta)$, $f \neq 0$.

It may be easily shown that

$$L_\mu(\lambda) = \lambda. \tag{0. 6}$$

Let $\mathcal{M}_\mu(\alpha, \beta)$ be the class of functions $f(x)$ defined in $\alpha < x < \beta$ such that (i) $f(x) = 0$ almost everywhere with respect to the measure $\mu(x)$ in a right-sided neighbourhood of α. This neighborhood depends on f, (ii) $f(x)$ is integrable with respect to $\mu(x)$ in every subinterval (α_1, β_1) $(\alpha \leqq \alpha_1 < \beta_1 < \beta)$ in the sense of Lebesgue — Stieltjes. The functions $\mathcal{L}_\mu(\alpha, \beta)$ are embedded in $\mathcal{M}_\mu$ and holds

$$\{f(x)\}_\mu = L_\mu\{f[\mu^{-1}(t)]\} \tag{0. 7}$$

for $f(x) \in \mathcal{L}_\mu(\alpha, \beta)$. Thus, the function

$$H_\lambda(x) = \begin{cases} 0 & \text{if} \quad x < \lambda \\ 1 & \text{if} \quad \lambda \leqq x \end{cases} \tag{0. 8}$$

is an element of $\mathcal{M}_\mu$ for every fixed $\lambda \in (\alpha, \beta)$. For the zero x_0 of $\mu(x)$, the function $l = \{H_{x_0}(x)\}_\mu$ is called the operator of integration with respect to $\mu(x)$, namely we have

$$l\{f(x)\}_\mu = \Big\{\int_\alpha^x f(t)\, d\mu(t)\Big\}_\mu \tag{0. 9}$$

for all $f \in \mathcal{L}_\mu(\alpha, \beta)$.

The derivative of $f(x)$ with respect to $\mu(x)$ is defined by the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{\mu(x+h) - \mu(x)}$$

if it exists and it will be denoted by $\frac{df(x)}{d\mu(x)}$. Since $\mu(x)$ is a monotonic function, it follows that

$$\frac{df(x)}{d\mu(x)} = \frac{f'(x)}{\mu'(x)}$$

almost everywhere, if $\frac{df(x)}{d\mu(x)}$ exists everywhere, provided $\mu'(x) \neq 0$. The element $\bar{s} = \frac{1}{l} \in \mathcal{M}_\mu$ is the operator of differentiation with respect to $\mu(x)$. In [5] is proved that

$$\bar{s}\{f(x)\}_\mu = \left\{\frac{df(x)}{d\mu(x)}\right\}_\mu \tag{0. 10}$$

provided both f and $\frac{df}{d\mu}$ are functions of $\mathscr{C}_\mu(\alpha, \beta)$.

Every value of the function

$$h(\lambda) = L_\mu(e^{-\mu(\lambda)s}) \tag{0. 11}$$

is a shift operator in $\mathcal{M}_\mu$ and the following equation holds:

$$h(\lambda)\{f(x)\}_\mu = \{f[\mu^{-1}(\mu(x) - \mu(\lambda))]\}_\mu \tag{0. 12}$$

for all $\lambda \in (\alpha, \beta)$ and $f \in \mathscr{L}_\mu(\alpha, \beta)$.

It follows from $\mu(x_0) = 0$ by (0. 6) and (0. 11) that

$$h(x_0) = 1. \tag{0. 13}$$

Let $f(x)$ be a function with a continuous derivative in respect to $\mu(x)$. As a generalization of (0. 10), the following formula holds:

$$\bar{s}\{f(x)H_\lambda(x)\}_\mu = \left\{\frac{df(x)}{d\mu(x)}H_\lambda(x)\right\}_\mu + f(\lambda)h(\lambda) \qquad (\lambda \in (\alpha, \beta)) \tag{0. 14}$$

A sequence of functions $f_n \in \mathscr{C}^+(\alpha, \beta)$ is said to be convergent in $\mathscr{C}^+(\alpha, \beta)$ to the function $f \in \mathscr{C}^+(\alpha, \beta)$, if the following properties (i) and (ii) are fulfilled.

(i) There exists an interval $\alpha < x < \xi < \beta$ in which the functions $f_1, f_2, \ldots$ and f vanish identically.

(ii) The sequence f_n is convergent to the limit f uniformly in any closed subinterval $[\xi, \eta]$ of $[\xi, \beta)$. We write in such a case

$$f_n \rightrightarrows f \quad \text{in} \quad \mathscr{C}^+(\alpha, \beta) \quad \text{as} \quad n \to \infty. \tag{0. 15}$$

A sequence of elements $A_n \in \mathcal{M}_\mu$ is said to be convergent in $\mathcal{M}_\mu$ to the limit $A \in \mathcal{M}_\mu$, if there exist representatives

$$\frac{\{f_n(x)\}_\mu}{\{g_n(x)\}_\mu} = A_n \qquad (f_n, g_n \in \mathscr{C}_\mu(\alpha, \beta),\ n = 1, 2, \ldots)$$

and

$$\frac{\{f(x)\}_\mu}{\{g(x)\}_\mu} = A \qquad (f, g \in \mathscr{C}_\mu(\alpha, \beta))$$

such that

$$f_n \rightrightarrows f, \quad g_n \rightrightarrows g$$

in $\mathscr{C}^+(\alpha, \beta)$ as $n \to \infty$.

In [5] is proved that the mapping L_μ is continuous, i.e. if a sequence of Mikusiński operators $a_n \in \mathscr{M}$ is convergent to the limit $a \in \mathscr{M}$, then the sequence $L_\mu(a_n)$ is convergent to the limit $L_\mu(a)$ in $\mathscr{M}_\mu$.

Let $\mathbf{F}$ be an operator transformation of $\mathscr{M}$. Then

$$\bar{\mathbf{F}} = L_\mu \mathbf{F} L_\mu^{-1} \tag{0.16}$$

is a transformation of $\mathscr{M}_\mu$. The map $\bar{\mathbf{F}}$ is called the equivalent of $\mathbf{F}$ in $\mathscr{M}_\mu$. It follows from (0. 16) that

$$\overline{\mathbf{F}+\mathbf{G}} = \bar{\mathbf{F}} + \bar{\mathbf{G}}, \quad \overline{\mathbf{FG}} = \bar{\mathbf{F}}\,\bar{\mathbf{G}}. \tag{0.17}$$

§ 1. Linear transformations of $\mathscr{M}_\mu$

Let $\mathscr{M}_\mu$ be the field of generalized convolution quotients determined by a base function $\mu(x)$ on $\alpha < x < \beta$.

Definition 1. 1. A map $\mathbf{F}_\mu$ of $\mathscr{M}_\mu$ into $\mathscr{M}_\mu$ is called a linear transformation of $\mathscr{M}_\mu$ if

(i)
$$\mathbf{F}_\mu(f+g) = \mathbf{F}_\mu(f) + \mathbf{F}_\mu(g)$$

(ii)
$$\mathbf{F}_\mu(\lambda f) = \lambda \mathbf{F}_\mu(f)$$

for all $f, g \in \mathscr{M}_\mu$ and $\lambda \in \mathscr{K}$.

Let $\mathscr{T}_\mu$ be the set of all linear transformations $\mathbf{F}_\mu$ of $\mathscr{M}_\mu$. Obviously, $\mathscr{T}_\mu$ forms a ring with respect to the usual operations of addition and multiplication of transformations. We denote by $\mathscr{T}$ the ring of all linear transformations of $\mathscr{M}$ (i.e. $\mathscr{T} = \mathscr{T}_{\{x\}}$). Let $\bar{\mathscr{T}}$ be the set of all equivalent transformations of $\mathscr{T}$ in $\mathscr{M}_\mu$, i.e. $\bar{\mathscr{T}}$ is the set of all transformations

$$\bar{\mathbf{F}} = L_\mu \mathbf{F} L_\mu^{-1} \tag{1.1}$$

where $\mathbf{F} \in \mathscr{T}$.

Theorem 1. 1. $\mathscr{T}_\mu = \bar{\mathscr{T}}$.

Proof. Let f and g be elements of $\mathscr{M}_\mu$, and $\mathbf{F} \in \mathscr{T}$. Then

$$\bar{\mathbf{F}}(f+g) = L_\mu \mathbf{F} L_\mu^{-1}(f+g) = L_\mu \mathbf{F}[L_\mu^{-1}(f) + L_\mu^{-1}(g)] =$$

$$= L_\mu[\mathbf{F}L_\mu^{-1}(f) + \mathbf{F}L_\mu^{-1}(g)] = L_\mu[\mathbf{F}L_\mu^{-1}(f)] + L_\mu[\mathbf{F}L_\mu^{-1}(g)] =$$

$$= (L_\mu \mathbf{F} L_\mu^{-1})(f) + (L_\mu \mathbf{F} L_\mu^{-1})(g) = \bar{\mathbf{F}}(f) + \bar{\mathbf{F}}(g).$$

Furthermore, by (0. 6), we get

$$\bar{\mathbf{F}}(\lambda f) = L_\mu \mathbf{F} L_\mu^{-1}(\lambda f) = L_\mu \mathbf{F}[L_\mu^{-1}(\lambda) L_\mu^{-1}(f)] = L_\mu \mathbf{F}[\lambda L_\mu^{-1}(f)] =$$
$$= L_\mu [\lambda \mathbf{F}(L_\mu^{-1}(f))] = L_\mu(\lambda) L_\mu [\mathbf{F}(L_\mu^{-1}(f))] = \lambda (L_\mu \mathbf{F} L_\mu^{-1})(f) =$$
$$= \lambda \bar{\mathbf{F}}(f).$$

Thus, we have proved that $\bar{\mathcal{T}} \subseteqq \mathcal{T}_\mu$. It may be similarly seen that $\bar{\mathcal{T}} \supseteqq \mathcal{T}_\mu$, if we put $\bar{\mathbf{F}}_\mu = L_\mu^{-1} \mathbf{F}_\mu L_\mu$, and the theorem is proved.

Definition 1. 2. A transformation $\mathbf{F}_\mu$ of $\mathcal{M}_\mu$ is said to be continuous if the transformation $\mathbf{F} = L_\mu^{-1} \mathbf{F}_\mu L_\mu$ of $\mathcal{M}$ is continuous in the sense of the paper [7], (i.e. $\mathbf{F}$ is continuous, if for every interval $\mathcal{I}$ and every continuous operator function $f(\lambda)$ on $\mathcal{I}$, the operator function $F[f(\lambda)]$ is continuous on $\mathcal{I}$).

It follows from a theorem of [7], by the continuity of L_μ, that every continuous transformation $\mathbf{F}_\mu$ of $\mathcal{M}_\mu$ has the following property: if $a_n \to a$ in $\mathcal{M}_\mu$ as $n \to \infty$, then $\mathbf{F}_\mu(a_n) \to \mathbf{F}_\mu(a)$ in $\mathcal{M}_\mu$ as $n \to \infty$.

We denote by $\mathcal{CT}$ the set of all continuous linear transformations of $\mathcal{M}$. Let $\mathcal{C}\bar{\mathcal{T}}$ be the set of all continuous linear transformations of $\mathcal{M}_\mu$. Then,

$$\mathcal{C}\bar{\mathcal{T}} = \overline{\mathcal{CT}} \tag{1. 2}$$

where $\overline{\mathcal{CT}}$ is the set of equivalent transformations of all $\mathbf{F} \in \mathcal{CT}$ in $\mathcal{M}_\mu$.

We shall now give a few examples for equivalent transformations in $\mathcal{M}_\mu$ of some important continuous linear transformations of $\mathcal{M}$.

1°. The algebraic derivation $\mathbf{D}$ is defined in $\mathcal{M}$ as follows: ([1]).

$$\mathbf{D}(f) = \{-tf(t)\} \quad \text{if} \quad f \in \mathcal{C}^+(-\infty, \infty),$$
$$\mathbf{D}\left(\frac{f}{g}\right) = \frac{\mathbf{D}(f)g - \mathbf{D}(g)f}{g^2} \quad \text{if} \quad \frac{f}{g} \in \mathcal{M}. \tag{1. 3}$$

The equivalent of $\mathbf{D}$ in $\mathcal{M}_\mu$ is $\bar{\mathbf{D}} = L_\mu \mathbf{D} L_\mu^{-1}$, consequently, if $F \in \mathcal{C}_\mu(\alpha, \beta)$, then

$$\bar{\mathbf{D}}(F) = L_\mu \mathbf{D} L_\mu^{-1}(F) = L_\mu \mathbf{D}\{F[\mu^{-1}(t)]\} = L_\mu\{-tF[\mu^{-1}(t)]\} = \{-\mu(x)F(x)\}_\mu.$$

Let $\frac{F}{G} \in \mathcal{M}_\mu$ $(F, G \in \mathcal{C}_\mu(\alpha, \beta))$, then

$$\bar{\mathbf{D}}\left(\frac{F}{G}\right) = L_\mu \mathbf{D} L_\mu^{-1}\left(\frac{F}{G}\right) = L_\mu \mathbf{D}\left[\frac{L_\mu^{-1}(F)}{L_\mu^{-1}(G)}\right] =$$
$$= L_\mu \frac{\mathbf{D}[L_\mu^{-1}(F)] L_\mu^{-1}(G) - \mathbf{D}[L_\mu^{-1}(G)] L_\mu^{-1}(F)}{[L_\mu^{-1}(G)]^2} =$$
$$= \frac{L_\mu \mathbf{D} L_\mu^{-1}(F) L_\mu L_\mu^{-1}(G) - L_\mu \mathbf{D} L_\mu^{-1}(G) L_\mu L_\mu^{-1}(F)}{[L_\mu L_\mu^{-1}(G)]^2} = \frac{\bar{\mathbf{D}}(F)G - \bar{\mathbf{D}}(G)F}{G^2}.$$

It can be easily shown the following basic properties of $\overline{\mathbf{D}}$:

$$\bar{\mathbf{D}}(A+B) = \bar{\mathbf{D}}(A) + \bar{\mathbf{D}}(B) \qquad (A, B \in \mathscr{M}_\mu), \tag{1.4}$$

$$\bar{\mathbf{D}}(AB) = \bar{\mathbf{D}}(A)B + A\bar{\mathbf{D}}(B), \tag{1.5}$$

$$\bar{\mathbf{D}}(\bar{s}) = 1. \tag{1.6}$$

2°. The transformation $\mathbf{T}^\sigma$ is defined in $\mathscr{M}$ for $\sigma \in \mathscr{K}$ as follows:

$$\mathbf{T}^\sigma(f) = \{e^{\sigma t} f(t)\} \quad \text{if} \quad f \in \mathscr{C}^+(-\infty, \infty)$$

$$\mathbf{T}^\sigma\left(\frac{f}{g}\right) = \frac{\mathbf{T}^\sigma(f)}{\mathbf{T}^\sigma(g)} \quad \text{if} \quad \frac{f}{g} \in \mathscr{M}.$$

Therefore, the equivalent of $\mathbf{T}^\sigma$ in $\mathscr{M}_\mu$ is

$$\overline{\mathbf{T}^\sigma}(F) = L_\mu \mathbf{T}^\sigma L_\mu^{-1}(F) = \{e^{\sigma\mu(x)} F(x)\}_\mu \quad \text{if} \quad F \in \mathscr{C}_\mu(\alpha, \beta)$$

$$\overline{\mathbf{T}^\sigma}\left(\frac{F}{G}\right) = \frac{\overline{\mathbf{T}^\sigma}(F)}{\overline{\mathbf{T}^\sigma}(G)} \quad \text{if} \quad \frac{F}{G} \in M\mu.$$

The basic properties of $\overline{\mathbf{T}^\sigma}$: If $A, B \in \mathscr{M}_\mu$, then

$$\overline{\mathbf{T}^\sigma}(A+B) = \overline{\mathbf{T}^\sigma}(A) + \overline{\mathbf{T}^\sigma}(B) \tag{1.7}$$

$$\overline{\mathbf{T}^\sigma}(AB) = \overline{\mathbf{T}^\sigma}(A)\overline{\mathbf{T}^\sigma}(B) \tag{1.8}$$

$$\overline{\mathbf{T}^\sigma}\,\overline{\mathbf{T}^\varrho} = \overline{\mathbf{T}^{\sigma+\varrho}} \qquad (\sigma, \varrho \in \mathscr{K}) \tag{1.9}$$

3°. The transformation $\mathbf{U}_k$ is defined in the following manner. Let $k > 0$, then

$$\mathbf{U}_k(f) = \{kf(kt)\} \quad \text{if} \quad f \in \mathscr{C}^+(-\infty, \infty),$$

$$\mathbf{U}_k\left(\frac{f}{g}\right) = \frac{\mathbf{U}_k(f)}{\mathbf{U}_k(g)} \quad \text{if} \quad \frac{f}{g} \in \mathscr{M}; \qquad f, g \in \mathscr{C}^+(-\infty, \infty). \tag{1.10}$$

It may be easily seen that

$$\overline{\mathbf{U}}_k(F) = \{kF[\mu^{-1}(k\mu(x))]\}_\mu \quad \text{if} \quad F \in \mathscr{C}_\mu(\alpha, \beta)$$

$$\overline{\mathbf{U}}_k\left(\frac{F}{G}\right) = \frac{\overline{\mathbf{U}}_k(F)}{\overline{\mathbf{U}}_k(G)} \in \mathscr{M}_\mu \quad \text{if} \quad \frac{F}{G} \in \mathscr{M}_\mu; \qquad F, G \in \mathscr{C}_\mu(\alpha, \beta). \tag{1.11}$$

The basic properties of $\overline{\mathbf{U}}_k$:

$$\overline{\mathbf{U}}_k(A+B) = \overline{\mathbf{U}}_k(A) + \overline{\mathbf{U}}_k(B), \qquad (A, B \in \mathscr{M}_\mu) \tag{1.12}$$

$$\overline{\mathbf{U}}_k(\lambda A) = \lambda \overline{\mathbf{U}}_k(A) \qquad (\lambda \in \mathscr{K}) \tag{1.13}$$

$$\overline{\mathbf{U}}_k(AB) = \overline{\mathbf{U}}_k(A)\overline{\mathbf{U}}_k(B) \tag{1.14}$$

$$\overline{\mathbf{U}}_{k_1}\overline{\mathbf{U}}_{k_2} = \overline{\mathbf{U}}_{k_1 k_2} \qquad (k_1, k_2 \in (0, \infty)) \tag{1.15}$$

Theorem 1. 2. *Let A be an element of $\mathcal{M}_\mu$ such that*

$$\lim_{n\to\infty} \overline{\mathbf{U}}_n(A) = \xi \qquad (n=1, 2, \ldots) \tag{1.16}$$

exists. Then the limit ξ is always a number.

Proof. In the case of $\mu(x)=x$, the theorem is proved in the paper [7]. It follows from the continuity of L_μ^{-1} that the limit

$$\lim_{n\to\infty} \mathbf{U}_n L_\mu^{-1}(A) = \lim_{n\to\infty} L_\mu^{-1} L_\mu \mathbf{U}_n L_\mu^{-1}(A) = \lim_{n\to\infty} L_\mu^{-1} \overline{\mathbf{U}}_n(A) = L_\mu^{-1}(\xi)$$

in $\mathcal{M}$ exists. Consequently, according to the quoted theorem of the paper [7], the operator $L_\mu^{-1}(\xi)$ is a number. Thus, by (0. 6), $\xi = L_\mu[L_\mu^{-1}(\xi)]$ is a number too.

4°. Since $\mathcal{M}$ is embedded in $\mathscr{C}\mathscr{T}$ ([7]), we can consider the equivalent transformation $\bar{c} = L_\mu c L_\mu^{-1}$ of an operator c of $\mathcal{M}$. If we denote the element $L_\mu(c)$ of $\mathcal{M}_\mu$ somewhat inexactly, by $\bar{c} = L_\mu(c)$, then we get

$$\bar{c}(F) = (L_\mu c L_\mu^{-1})(F) = L_\mu[c \cdot L_\mu^{-1}(F)] = L_\mu(c) \cdot L_\mu[L_\mu^{-1}(F)] = L_\mu(c) \cdot F = \bar{c} \cdot F$$

for all $F \in \mathcal{M}_\mu$. Thus, the equivalents of the operators of $\mathcal{M}$, (regarded as transformations) are the elements of $\mathcal{M}_\mu$ (regarded as transformations).

We have defined in [7] the derivative $\mathbf{F}'$ of a transformation $\mathbf{F}$ of $\mathscr{C}\mathscr{T}$ in the following manner:

$$\mathbf{F}' = s\mathbf{F} - \mathbf{F}s. \tag{1.17}$$

We introduce here, similarly, the following definition: By the derivative $\mathbf{F}_\mu'$ of the transformation $\mathbf{F}_\mu \in \overline{\mathscr{C}\mathscr{T}}$ we understand the transformation

$$\mathbf{F}_\mu' = \bar{s}\mathbf{F}_\mu - \mathbf{F}_\mu \bar{s}, \tag{1.18}$$

where $\bar{s} = L_\mu s L_\mu^{-1}$.

Theorem 1. 3. *Let $\mathbf{F} \in \mathscr{C}\mathscr{T}$ and let $\overline{\mathbf{F}} = L_\mu \mathbf{F} L_\mu^{-1}$ and $\overline{\mathbf{F}'} = L_\mu \mathbf{F}' L_\mu^{-1}$ be the equivalents of $\mathbf{F}$ and $\mathbf{F}'$, respectively. Then*

$$(\overline{\mathbf{F}})' = \overline{\mathbf{F}'}. \tag{1.19}$$

Proof. This theorem is an immediate consequence of (0. 17). Indeed

$$(\overline{\mathbf{F}})' = \bar{s}\overline{\mathbf{F}} - \overline{\mathbf{F}}\bar{s} = \overline{s\mathbf{F}} - \overline{\mathbf{F}s} = \overline{s\mathbf{F} - \mathbf{F}s} = \overline{\mathbf{F}'}.$$

Theorem 1. 4. *Let $\mathscr{T}^D$ be the set of all transformations $\mathbf{F}$ of $\mathscr{C}\mathscr{T}$ for which the equation holds: $\mathbf{FD} = \mathbf{DF}$. Let $\overline{\mathscr{T}^D}$ be the set of all transformations $\overline{\mathbf{F}} = L_\mu \mathbf{F} L_\mu^{-1}$ where $\mathbf{F} \in \mathscr{T}^D$. Then* (i) *$\overline{\mathscr{T}^D}$ is a commutative subring of $\overline{\mathscr{C}\mathscr{T}}$* (ii) *if $\overline{\mathbf{F}} \in \overline{\mathscr{T}^D}$, then $\overline{\mathbf{F}'} \in \overline{\mathscr{T}^D}$.*

Proof. It is clear that $\mathscr{T}^D$ is a subring of $\mathscr{C}\mathscr{T}$. Thus, in consequence of the equations (0. 17), $\overline{\mathscr{T}^D}$ is a subring of $\overline{\mathscr{C}\mathscr{T}}$. Moreover, since $\mathbf{D}s = 1 + s\mathbf{D}$, for

$\mathbf{F} \in \mathcal{T}^D$, we get

$$\mathbf{F}'\mathbf{D} = (s\mathbf{F} - \mathbf{F}s)\mathbf{D} = s\mathbf{F}\mathbf{D} - \mathbf{F}s\mathbf{D} = s\mathbf{D}\mathbf{F} - \mathbf{F}(\mathbf{D}s - 1) =$$
$$= (\mathbf{D}s - 1)\mathbf{F} - (\mathbf{F}\mathbf{D})s + \mathbf{F} = (\mathbf{D}s)\mathbf{F} - \mathbf{F} - (\mathbf{D}\mathbf{F})s + \mathbf{F} =$$
$$= \mathbf{D}(s\mathbf{F}) - \mathbf{D}(\mathbf{F}s) = \mathbf{D}(s\mathbf{F} - \mathbf{F}s) = \mathbf{D}\mathbf{F}'$$

Consequently, $\mathbf{F}' \in \mathcal{T}^D$. Hence, according to theorem 1. 3,

$$(\bar{\mathbf{F}})' = \overline{\mathbf{F}'} \in \overline{\mathcal{T}^D}.$$

The commutativity of $\overline{\mathcal{T}^D}$ is a consequence of theorem 2. 4 which will be proved in § 2.

Remark 1. 1. Let $\mathcal{T}_\mu^{\bar{D}}$ be the set of all transformations $\mathbf{F}_\mu$ of $\mathscr{C}\mathcal{T}_\mu$ for which $\mathbf{F}_\mu \overline{\mathbf{D}} = \overline{\mathbf{D}}\mathbf{F}_\mu$. Then, obviously, $\mathcal{T}_\mu^{\bar{D}} = \overline{\mathcal{T}^D}$.

Theorem 1. 5. *Let $\bar{\mathbf{F}}$ be a transformation of $\mathcal{T}_\mu^{\bar{D}}$. Then the function*

$$\varphi(\lambda) = \frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)} \tag{1.20}$$

is a numerical function defined on the interval $\alpha < \lambda < \beta$. Moreover, $\varphi(\lambda)$ is derivable with respect to $\mu(\lambda)$ in (α, β) and the derivative of $\varphi(\lambda)$ with respect to $\mu(\lambda)$ is given by

$$\frac{d\varphi(\lambda)}{d\mu(\lambda)} = \frac{\bar{\mathbf{F}}'[h(\lambda)]}{h(\lambda)}.$$

PROOF. According to a theorem of the paper [7] (p. 191) the function

$$\psi(\lambda) = \frac{\mathbf{F}(e^{-\lambda s})}{e^{-\lambda s}} \qquad (\mathbf{F} = L_\mu^{-1}\bar{\mathbf{F}}L_\mu) \tag{1.21}$$

is a derivable numerical function on the interval $-\infty < \lambda < \infty$ and the derivative of $\psi(\lambda)$ is given by

$$\psi'(\lambda) = \frac{\mathbf{F}'(e^{-\lambda s})}{e^{-\lambda s}}. \tag{1.22}$$

Consequently, the function $\psi[\mu(\lambda)]$ is a numerical function which is derivable with respect to $\mu(\lambda)$ in (α, β). Since $\psi[\mu(\lambda)]$ is a number for every $\lambda \in (\alpha, \beta)$, we get, by (0. 6), and (0. 11),

$$\psi[\mu(\lambda)] = L_\mu[\psi(\mu(\lambda))] = L_\mu\left[\frac{\mathbf{F}(e^{-\mu(\lambda)s})}{e^{-\mu(\lambda)s}}\right] = \frac{L_\mu[\mathbf{F}(e^{-\mu(\lambda)s})]}{L_\mu[e^{-\mu(\lambda)s}]} =$$
$$= \frac{(L_\mu \mathbf{F} L_\mu^{-1})(L_\mu[e^{-\mu(\lambda)s}])}{L_\mu[e^{-\mu(\lambda)s}]} = \frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)}$$

i.e.

$$\psi[\mu(\lambda)] = \varphi(\lambda), \tag{1.23}$$

and similarly, by (1. 22),

$$\frac{d\varphi(\lambda)}{d\mu(\lambda)} = \psi'[\mu(\lambda)] = L_\mu(\psi'[\mu(\lambda)]) = \frac{\overline{\mathbf{F}}'[h(\lambda)]}{h(\lambda)}$$

Thus the theorem is proved.

Remark 1. 2. It follows from theorems 1. 4 and 1. 5 by induction that the function $\varphi(\lambda)$, defined by (1. 20), is infinitely derivable with respect to $\mu(\lambda)$ and the derivative of order n of φ with respect to μ is given by

$$\frac{d^n\varphi}{d\mu^n} = \frac{\bar{\mathbf{F}}^{(n)}[h(\lambda)]}{h(\lambda)}$$

where $\bar{\mathbf{F}}^{(n)}$ is the derivative of order n of $\bar{\mathbf{F}}$ defined in the following manner:

$$\bar{\mathbf{F}}^{(n)} = (\bar{\mathbf{F}}^{(n-)})' \qquad (n=1, 2, \ldots, \ \bar{\mathbf{F}}^{(0)} = \bar{\mathbf{F}}).$$

Theorem 1. 6. *Let* $\overline{\mathbf{F}}$ *be a transformation of* $\mathscr{T}_\mu^D$ *and let*

$$\varphi(\lambda) = \frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)}.$$

Then

$$\bar{\mathbf{F}}(f) = \{\varphi(x)f(x)\}_\mu$$

for all $f \in \mathscr{L}_\mu(\alpha, \beta)$.

Proof. Let $\mathbf{F} = L_\mu^{-1}\overline{\mathbf{F}}L_\mu$. Since $\mathbf{F} \in \mathscr{T}^D$, it follows from a theorem of [7] (p. 191) that

$$\mathbf{F}(u) = \{\psi(t)u(t)\} \tag{1. 24}$$

for all $u \in \mathscr{C}^+(-\infty, \infty)$, where $\psi(\lambda) = \dfrac{\mathbf{F}(e^{-\lambda s})}{e^{-\lambda s}}$. We shall prove that (1. 24) remains valid for $u \in \mathscr{L}^+(-\infty, \infty)$. It follows from the embedding of $\mathscr{L}^+(-\infty, \infty)$ in $\mathscr{M}$ that

$$\{f(t)\} = s\Big\{\int_{-\infty}^{t} f(\tau)\,d\tau\Big\} \tag{1. 25}$$

for all $f \in \mathscr{L}^+(-\infty, \infty)$. We obtain, by (1. 17), that

$$\mathbf{F}\{f(t)\} = \mathbf{F}\Big[s\Big\{\int_{-\infty}^{t} f(\tau)\,d\tau\Big\}\Big] = (\mathbf{F}s)\Big\{\Big[\int_{-\infty}^{t} f(\tau)\,d\tau\Big\}\Big] =$$

$$= (s\mathbf{F} - \mathbf{F}')\Big[\Big\{\int_{-\infty}^{t} f(\tau)\,d\tau\Big\}\Big] = s\mathbf{F}\Big[\Big\{\int_{-\infty}^{t} f(\tau)\,d\tau\Big\}\Big] - \mathbf{F}'\Big[\Big\{\int_{-\infty}^{t} f(\tau)\,d\tau\Big\}\Big].$$

Since $\int\limits_{-\infty}^{t} f(\tau)\,d\tau \in \mathscr{C}^+(-\infty, \infty)$, from (1. 22) and (1. 24) we get

$$\mathbf{F}\{f(t)\} = s\{\psi(t) \int\limits_{-\infty}^{t} f(\tau)\,d\tau\} - \{\psi'(t) \int\limits_{-\infty}^{t} f(\tau)\,d\tau\} =$$
$$= s\{\psi(t) \int\limits_{-\infty}^{t} f(\tau)\,d\tau\} - s\{\int\limits_{-\infty}^{t} \psi'(\tau) \int\limits_{-\infty}^{\tau} f(\sigma)\,d\sigma\,d\tau\}.$$

Integration by parts gives

$$\int\limits_{-\infty}^{t} \psi'(\tau) \int\limits_{-\infty}^{\tau} f(\sigma)\,d\sigma\,d\tau = \psi(t) \int\limits_{-\infty}^{t} f(\sigma)\,d\sigma - \int\limits_{-\infty}^{t} \psi(\tau) f(\tau)\,d\tau,$$

and thus we have

$$(1.26) \quad \mathbf{F}(f) = s\{\psi(t) \int\limits_{-\infty}^{t} f(\tau)\,d\tau\} - s\{\psi(t) \int\limits_{-\infty}^{t} f(\sigma)\,d\sigma - \int\limits_{-\infty}^{t} \psi(\tau) f(\tau)\,d\tau\} =$$
$$= s\{\int\limits_{-\infty}^{t} \psi(\tau) f(\tau)\,d\tau\} = \{\psi(t) f(t)\}.$$

Now we shall prove that the equation

$$(1.27) \qquad L_\mu\{f(t)\} = \{f[\mu(x)]\}_\mu$$

is valid not only for functions of $\mathscr{C}^+(-\infty, \infty)$ but also for $f \in \mathscr{L}^+(-\infty, \infty)$. Since $\int\limits_{-\infty}^{t} f(\tau)\,d\tau \in \mathscr{C}^+(-\infty, \infty)$, it follows from the definition of L_μ that

$$L_\mu\{\int\limits_{-\infty}^{t} f(\tau)\,d\tau\} = \{\int\limits_{-\infty}^{\mu(x)} f(\tau)\,d\tau\}_\mu = \{\int\limits_{\alpha}^{x} f[\mu(t)]\,d\mu(t)\}_\mu.$$

Therefore, by the formula (0. 9), we get

$$L_\mu\{f(t)\} = L_\mu\left[s\{\int\limits_{-\infty}^{t} f(\tau)\,d\tau\}\right] = L_\mu(s)\, L_\mu\{\int\limits_{-\infty}^{t} f(\tau)\,d\tau\} =$$
$$= \bar{s}\{\int\limits_{-\infty}^{\mu(x)} f(\tau)\,d\tau\}_\mu = \bar{s}\{\int\limits_{\alpha}^{x} f[\mu(t)]\,d\mu(t)\}_\mu = \{f[\mu(x)]\}_\mu.$$

Let now $\{f(x)\}_\mu \in \mathscr{L}_\mu(\alpha, \beta)$. Then $\{f[\mu^{-1}(t)]\} \in \mathscr{L}^+(-\infty, \infty)$, and thus, by (1. 26), (1. 27) and (1. 23) we obtain

$$\bar{\mathbf{F}}(f) = L_\mu \mathbf{F} L_\mu^{-1}(f) = L_\mu \mathbf{F}\{f[\mu^{-1}(t)]\} = L_\mu\{\psi(t) f[\mu^{-1}(t)]\} =$$
$$= \{\psi[\mu(x)] f(x)\}_\mu = \{\varphi(x) f(x)\}_\mu,$$

this proves the theorem.

§ 2. The product of a generalized function by infinitely derivable functions

The elements of $\mathscr{M}_\mu$ will be called generalized functions. We shall denote generalized functions by symbols $\{f(x)\}_\mu$, $\{g(x)\}_\mu$ etc. This notation is purely symbolic and, in general, it is not allowed to substitute numbers for the variable x. Sometimes we shall write simply $f(x)$, $g(x)$ etc. instead of $\{f(x)\}_\mu$, $\{g(x)\}_\mu$ etc. if it leads no to misunderstanding.

1. *The "ordinary" sum of two generalized functions and the "ordinary" product of a generalized function by a number*

Since the generalized functions are elements of the field $\mathscr{M}_\mu$, there is no need to define the algebraic operations on generalized functions. These operations are called, temporarily, field operations. Thus, the field sum $\{f_1(x)\}_\mu+\{f_2(x)\}_\mu$ of the generalized functions $\{f_1(x)\}_\mu=\dfrac{\{u_1(x)\}_\mu}{\{v_1(x)\}_\mu}$ and $\{f_2(x)\}_\mu=\dfrac{\{u_2(x)\}_\mu}{\{v_2(x)\}_\mu}$ (where u_1, v_1, u_2, v_2 are functions of $\mathscr{C}_\mu(\alpha,\beta)$) is the generalized function

$$\{f_1(x)\}+\{f_2(x)\}=$$

$$=\frac{\left\{\int\limits_\alpha^\beta u_1[\mu^{-1}(\mu(x)-\mu(t))]\,v_2(t)\,d\mu+\int\limits_\alpha^\beta v_1[\mu^{-1}(\mu(x)-\mu(t))]\,u_2(t)\,d\mu\right\}_\mu}{\left\{\int\limits_\alpha^\beta v_1[\mu^{-1}(\mu(x)-\mu(t))]\,v_2(t)\,d\mu\right\}_\mu}.$$

Similarly, the field product $\{f_1(x)\}_\mu\{f_2(x)\}_\mu$ is the generalized function

$$\{f_1(x)\}_\mu\{f_2(x)\}_\mu=\frac{\left\{\int\limits_\alpha^\beta u_1[\mu^{-1}(\mu(x)-\mu(t))]\,u_2(t)\,d\mu\right\}_\mu}{\left\{\int\limits_\alpha^\beta v_1[\mu^{-1}(\mu(x)-\mu(t))]\,v_2(t)\,d\mu\right\}_\mu}.$$

Now we shall introduce the notion of the "ordinary" sum of two generalized functions and of the "ordinary" product of a generalized function by a number.

Definition 2. 1. By the "ordinary" sum $\{f_1(x)+f_2(x)\}_\mu$ of the generalized functions $\{f_1(x)\}_\mu$ and $\{f_2(x)\}_\mu$ we understand the generalized function

$$\{f_1(x)+f_2(x)\}_\mu=\{f_1(x)\}_\mu+\{f_2(x)\}_\mu. \tag{2.1}$$

By the "ordinary" product $\{\lambda f(x)\}_\mu$ of a generalized function $\{f(x)\}_\mu$ by a number λ we understand the generalized function

$$\{\lambda f(x)\}_\mu=\lambda\{f(x)\}_\mu. \tag{2.2}$$

The adjective "ordinary" refers to the circumstance that in the case of functions the sum (2. 1) and the product (2. 2) becomes the sum and product, respectively, of functions in the ordinary sense.

In the general case the field product of two generalized functions defines the generalized convolution of the generalized functions:

Definition 2. 2. By the generalized convolution $\{f_1(x)*f_2(x)\}_\mu$ of the generalized functions $\{f_1(x)\}_\mu$ and $\{f_2(x)\}_\mu$ we understand the generalized function

$$\{f_1(x)*f_2(x)\}_\mu=\{f_1(x)\}_\mu\cdot\{f_2(x)\}_\mu. \tag{2. 3}$$

The definition of the "ordinary" product of a generalized function by a function is somewhat complicated and it will be given in the next section.

2. *The "ordinary" product of a generalized function by an infinitely derivable function*

Definition 2. 3. We denote by $\mathscr{C}_\mu^{(\infty)}(\alpha,\beta)$ the set of all functions

$$\varphi(\lambda)=\frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)}\qquad(\lambda\in(\alpha,\beta)), \tag{2. 4}$$

where $\bar{\mathbf{F}}\in\mathscr{T}_\mu^{\bar{D}}$. By the "ordinary" product $\{\varphi(x)f(x)\}_\mu$ of a generalized function $\{f(x)\}_\mu$ by the function $\varphi(x)\in\mathscr{C}_\mu^{(\infty)}(\alpha,\beta)$ we understand the generalized function

$$\{\varphi(x)f(x)\}_\mu=\bar{\mathbf{F}}(\{f(x)\}_\mu) \tag{2. 5}$$

where $\bar{\mathbf{F}}\in\mathscr{T}_\mu^{\bar{D}}$ is determined by (2. 4).

Theorem 2. 1. *If* $\varphi(x)=\gamma$ *is a constant function and* $\{f(x)\}_\mu$ *is an arbitrary generalized function, the product* $\{\varphi(x)f(x)\}_\mu$ *coincides with the ordinary product defied by* (2. 2).

Proof. Let $\bar{\mathbf{F}}_\gamma(A)=\gamma A$ for all $A\in\mathscr{M}_\mu$. Obviously, $\bar{\mathbf{F}}_\gamma\in\mathscr{T}_\mu$, moreover $\bar{\mathbf{F}}_\gamma\in\mathscr{T}_\mu^{\bar{D}}$, since $\gamma\bar{\mathbf{D}}=\bar{\mathbf{D}}\gamma$ by the linearity of $\bar{\mathbf{D}}$. Furthermore

$$\frac{\bar{\mathbf{F}}_\gamma[h(\lambda)]}{h(\lambda)}=\frac{\gamma h(\lambda)}{h(\lambda)}=\gamma=\varphi(\lambda)\qquad(\alpha<\lambda<\beta)$$

and hence, by (2. 5) and (2. 2), we get

$$\{\varphi(x)f(x)\}_\mu=\bar{\mathbf{F}}_\gamma(\{f(x)\}_\mu)=\gamma\{f(x)\}_\mu=\{\gamma f(x)\}_\mu,$$

which proves the theorem.

In the following theorems, if it leads no to misunderstanding, we shall write simply $f(x)$ for the generalized function $\{f(x)\}_\mu$.

Theorem 2. 2. If $\varphi(x)\in\mathscr{C}^{(\infty)}(\alpha,\beta)$ and $f(x),g(x)\in\mathscr{M}_\mu$, then

$$\varphi(x)[f(x)+g(x)]=\varphi(x)f(x)+\varphi(x)g(x). \tag{2. 6}$$

Proof. Let $\varphi(\lambda)=\frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)}$. Then, by the linearity of $\bar{\mathbf{F}}$, we obtain

$$\{\varphi(x)[f(x)+g(x)]\}_\mu=\bar{\mathbf{F}}[\{f(x)+g(x)\}_\mu]=\bar{\mathbf{F}}[\{f(x)\}_\mu+\{g(x)\}_\mu]=$$
$$=\bar{\mathbf{F}}[\{f(x)\}_\mu]+\bar{\mathbf{F}}[\{g(x)\}_\mu]=\{\varphi(x)f(x)\}_\mu+\{\varphi(x)g(x)\}_\mu=\{\varphi(x)f(x)+\varphi(x)g(x)\}_\mu.$$

Theorem 2. 3. *If $\varphi(x)$ and $\psi(x)$ are functions of $\mathscr{C}^{(\infty)}(\alpha,\beta)$ then $\varphi(x)+\psi(x)\in\mathscr{C}^{(\infty)}(\alpha,\beta)$ and holds*

(2. 7) $$[\varphi(x)+\psi(x)]f(x)=\varphi(x)f(x)+\psi(x)f(x)$$

for all $f(x)\in\mathscr{M}_\mu$.

Proof. Let $\varphi(\lambda)=\frac{\bar{\mathbf{F}}[h(\lambda)}{h(\lambda)}$ and $\psi(\lambda)=\frac{\bar{\mathbf{G}}[h(\lambda)]}{h(\lambda)}$ where $\bar{\mathbf{F}},\bar{\mathbf{G}}\in\mathscr{T}_\mu\bar{\text{D}}$.

Then

$$\varphi(\lambda)+\psi(\lambda)=\frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)}+\frac{\bar{\mathbf{G}}[h(\lambda)]}{h(\lambda)}=\frac{\bar{\mathbf{F}}[h(\lambda)]+\bar{\mathbf{G}}[h(\lambda)]}{h(\lambda)}=\frac{(\mathbf{F}+\bar{\mathbf{G}})[h(\lambda)]}{h(\lambda)}.$$

Since $\mathscr{T}_\mu^{\bar{\text{D}}}$ is a ring, it follows that $\bar{\mathbf{F}}+\bar{\mathbf{G}}\in\mathscr{T}_\mu^{\bar{\text{D}}}$. Consequently $\varphi(x)+\psi(x)\in\mathscr{C}^{(\infty)}(\alpha,\beta)$ and

$$\{[\varphi(x)+\psi(x)]f(x)\}_\mu=(\bar{\mathbf{F}}+\bar{\mathbf{G}})(\{f(x)\}_\mu)=\bar{\mathbf{F}}(\{f(x)\}_\mu)+\bar{\mathbf{G}}(\{f(x)\}_\mu)=$$
$$=\{\varphi(x)f(x)\}_\mu+\{\psi(x)f(x)\}_\mu=\{\varphi(x)f(x)+\psi(x)f(x)\}_\mu$$

for each $f(x)\in\mathscr{M}_\mu$. This proves the theorem.

Theorem 2. 4. *If $\varphi(x)$ and $\psi(x)$ are functions of $\mathscr{C}^{(\infty)}(\alpha,\beta)$ then $\varphi(x)\psi(x)\in\mathscr{C}^{\infty}(\alpha,\beta)$ and*

(2. 8) $$\varphi(x)[\psi(x)f(x)]=[\varphi(x)\psi(x)]f(x)$$

for all $f(x)\in\mathscr{M}_\mu$.

Proof. Let $\varphi(\lambda)=\frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)}$ and $\psi(\lambda)=\frac{\bar{\mathbf{G}}[h(\lambda)]}{h(\lambda)}$, where $\bar{\mathbf{F}},\bar{\mathbf{G}}\in\mathscr{T}_\mu\bar{\text{D}}$. Since $\psi(\lambda)$ is a number, it follows from the linearity of $\bar{\mathbf{F}}$ that

$$\frac{(\bar{\mathbf{F}}\bar{\mathbf{G}})[h(\lambda)]}{h(\lambda)}=\frac{\bar{\mathbf{F}}[\bar{\mathbf{G}}(h(\lambda))]}{h(\lambda)}=\frac{\bar{\mathbf{F}}\left[h(\lambda)\frac{\bar{\mathbf{G}}(h(\lambda))}{h(\lambda)}\right]}{h(\lambda)}=\frac{\bar{\mathbf{F}}[h(\lambda)\psi(\lambda)]}{h(\lambda)}=$$
$$=\frac{\bar{\mathbf{F}}[h(\lambda)]\psi(\lambda)}{h(\lambda)}=\varphi(\lambda)\psi(\lambda),$$

and thus $\varphi(x)\psi(x)\in\mathscr{C}^{(\infty)}(\alpha,\beta)$, since $\bar{\mathbf{F}}\bar{\mathbf{G}}\in\mathscr{T}_\mu^{\bar{\text{D}}}$, because $\mathscr{T}_\mu^{\bar{\text{D}}}$ is a ring. Using the definition 2. 3 we obtain

$$\{[\varphi(x)\psi(x)]f(x)\}_\mu=(\bar{\mathbf{F}}\bar{\mathbf{G}})(\{f(x)\}_\mu)=\bar{\mathbf{F}}[\bar{\mathbf{G}}(\{f(x)\}_\mu)]=$$
$$=\bar{\mathbf{F}}[\{\psi(x)f(x)\}_\mu]=\{\varphi(x)[\psi(x)f(x)]\}_\mu$$

for all $f(x)\in\mathscr{M}_\mu$, and the theorem is proved.

§ 3. Derivation of generalized functions

Let $\mu(x)$ be a base function on (α, β) which is derivable in (α, β). We shall suppose that $\mu'(x) \in \mathscr{C}_\mu^{(\infty)}(\alpha, \beta)$, as it happens for example in the case of $\mu(x) = x$, or else in the case of $\mu(x) = \log x$, etc. Then, by definition, there exists a linear transformation $\overline{\mathbf{M}}$ of $\mathscr{T}_\mu^{\overline{\mathbf{D}}}$ such that

$$\mu'(\lambda) = \frac{\overline{\mathbf{M}}[h(\lambda)]}{h(\lambda)} \qquad (\alpha < \lambda < \beta). \tag{3. 1}$$

Definition 3. 1. Let $\{f(x)\}_\mu$ be a generalized function of $\mathscr{M}_\mu$. By the derivative of a generalized function $\{f(x)\}_\mu$ we understand the generalized function $\overline{\mathbf{M}}[\bar{s}\{f(x)\}_\mu]$. We write in this case

$$\{f'(x)\}_\mu = \overline{\mathbf{M}}[\bar{s}\{f(x)\}_\mu]. \tag{3. 2}$$

Theorem 3. 1. *If a generalized function of $\mathscr{M}_\mu$ is a function of $\mathscr{C}_\mu(\alpha, \beta)$ with a continuous derivative in (α, β) then its derivative in the sense of definition 3.1 coincides with its derivative in the ordinary sense, provided $\mu'(x) \neq 0$ in (α, β).*

Proof. Let $\{f(x)\}_\mu \in \mathscr{C}_\mu(\alpha, \beta)$ such that $f'(x)$ is continuous in $\alpha < x < \beta$. Then, obviously, $\frac{df(x)}{d\mu(x)} = \frac{f'(x)}{\mu'(x)} \in \mathscr{C}_\mu(\alpha, \beta)$ and thus, by (0. 10), we have

$$\bar{s}\{f(x)\}_\mu = \left\{\frac{f'(x)}{\mu'(x)}\right\}_\mu.$$

It follows from theorem 1. 6 that

$$\overline{\mathbf{M}}[\bar{s}\{f(x)\}_\mu] = \overline{\mathbf{M}}\left\{\frac{f'(x)}{\mu'(x)}\right\}_\mu = \left\{\mu'(x)\frac{f'(x)}{\mu'(x)}\right\}_\mu = \{f'(x)\}_\mu,$$

and the theorem is proved.

Theorem 3. 2. *If $f(x)$ and $g(x)$ are generalized functions of $\mathscr{M}_\mu$, then*

$$[f(x)+g(x)]' = f'(x)+g'(x). \tag{3. 3}$$

Proof. It follows from the linearity of $\overline{\mathbf{M}}$ that

$$\{[f(x)+g(x)]'\}_\mu = \overline{\mathbf{M}}[\bar{s}\{f(x)+g(x)\}_\mu] = \overline{\mathbf{M}}[\bar{s}\{f(x)\}_\mu + \bar{s}\{g(x)\}_\mu] =$$

$$= \overline{\mathbf{M}}[\bar{s}\{f(x)\}_\mu] + \overline{\mathbf{M}}[\bar{s}\{g(x)\}_\mu] = \{f'(x)\}_\mu + \{g'(x)\}_\mu = \{f'(x)+g'(x)\}_\mu,$$

which proves the theorem.

Theorem 3. 3. *Let $\mu'(x) \in \mathscr{C}_\mu^{(\infty)}(\alpha, \beta)$ and $\mu'(x) \neq 0$ in (α, β). If $\varphi(x) \in \mathscr{C}_\mu^{(\infty)}(\alpha, \beta)$ and $f(x) \in \mathscr{M}_\mu$, then*

$$[\varphi(x) f(x)]' = \varphi'(x) f(x) + \varphi(x) f'(x). \tag{3. 4}$$

PROOF. Let $\bar{\mathbf{F}}$ be a transformation of $\mathscr{T}_\mu^{\bar{\mathbf{D}}}$ such that

$$\varphi(\lambda) = \frac{\bar{\mathbf{F}}[h(\lambda)]}{h(\lambda)} \qquad (\lambda \in (\alpha, \beta)).$$

We show that

$$\frac{(\overline{\mathbf{M}}\bar{\mathbf{F}}')[h(\lambda)]}{h(\lambda)} = \varphi'(\lambda) \qquad (\lambda \in (\alpha, \beta)). \tag{3. 5}$$

It follows from theorem 1. 5 that

$$\frac{\bar{\mathbf{F}}'[h(\lambda)]}{h(\lambda)} = \frac{d\varphi(\lambda)}{d\mu(\lambda)} = \frac{\varphi'(\lambda)}{\mu'(\lambda)}.$$

Hence, from the linearity of $\overline{\mathbf{M}}$ by (3. 1), we obtain

$$\frac{(\overline{\mathbf{M}}\bar{\mathbf{F}}')[h(\lambda)]}{h(\lambda)} = \frac{\overline{\mathbf{M}}[\bar{\mathbf{F}}'(h(\lambda))]}{h(\lambda)} = \frac{\overline{\mathbf{M}}\left[h(\lambda)\dfrac{\bar{\mathbf{F}}'(h(\lambda))}{h(\lambda)}\right]}{h(\lambda)} =$$

$$= \frac{\overline{\mathbf{M}}\left[h(\lambda)\dfrac{\varphi'(\lambda)}{\mu'(\lambda)}\right]}{h(\lambda)} = \frac{\overline{\mathbf{M}}[h(\lambda)]}{h(\lambda)} \cdot \frac{\varphi'(\lambda)}{\mu'(\lambda)} = \mu'(\lambda)\frac{\varphi'(\lambda)}{\mu'(\lambda)} = \varphi'(\lambda).$$

Thus is (3. 5) valid. Since $\overline{\mathbf{M}}\bar{\mathbf{F}}' \in \mathscr{T}_\mu^{\bar{\mathbf{D}}}$, it follows that

$$(\overline{\mathbf{M}}\bar{\mathbf{F}}')(f) = \{\varphi'(x)f(x)\}_\mu \tag{3. 6}$$

for every generalized function $f(x)$. Then

$$\begin{aligned}\{[\varphi(x)f(x)]'\}_\mu &= \overline{\mathbf{M}}[\bar{s}\{\varphi(x)f(x)\}_\mu] = \overline{\mathbf{M}}[\bar{s}\bar{\mathbf{F}}(f)] = \\ &= \overline{\mathbf{M}}[(\bar{s}\bar{\mathbf{F}} - \bar{\mathbf{F}}\bar{s})(f) + \bar{\mathbf{F}}(\bar{s}f)] = \\ &= \overline{\mathbf{M}}[\bar{\mathbf{F}}'(f)] + \overline{\mathbf{M}}\bar{\mathbf{F}}(\bar{s}f) = (\overline{\mathbf{M}}\bar{\mathbf{F}}')(f) + (\bar{\mathbf{F}}\overline{\mathbf{M}})(\bar{s}f) = \\ &= \{\varphi'(x)f(x)\}_\mu + \bar{\mathbf{F}}[\overline{\mathbf{M}}(\bar{s}f)] = \{\varphi'(x)f(x)\}_\mu + \bar{\mathbf{F}}\{f'(x)\}_\mu = \\ &= \{\varphi'(x)f(x)\}_\mu + \{\varphi(x)f'(x)\}_\mu = \{\varphi'(x)f(x) + \varphi(x)f'(x)\}_\mu.\end{aligned}$$

The theorem is proved.

§ 4. The limit of a generalized function as $x \to \beta$

Definition 4. 1. Let $\{f(x)\}$ be a generalized function of $\mathscr{M}_\mu$. If the limit

$$\lim_{n\to\infty} \overline{\mathbf{U}}_n(\bar{s}\{f(x)\}_\mu) = \xi$$

exists in $\mathscr{M}_\mu$, then the number ξ is called the limit of $\{f(x)\}_\mu$ as $x \to \beta$. We write in this case

$$\lim_{n\to\infty} \overline{\mathbf{U}}_n(\bar{s}\{f(x)\}_\mu) = \operatorname*{Lim}_{x\to\beta} f(x). \tag{4. 1}$$

Theorem 4. 1. *Let $f(x)$ be a function of $\mathscr{C}_\mu(\alpha, \beta)$. If*

$$\lim_{x\to\beta} f(x) \tag{4. 2}$$

exists in the usual sense, then $\operatorname{Lim}_{x\to\beta} f(x)$ *exists in the sense of definition* 4. 1 *and*

$$\operatorname{Lim}_{x\to\beta} f(x) = \lim_{x\to\beta} f(x). \tag{4. 3}$$

PROOF. It follows from the definition of $\mu(x)$ and (4. 2) that

$$\lim_{t\to\infty} f[\mu^{-1}(t)] = \lim_{x\to\beta} f(x). \tag{4. 4}$$

Then, by a theorem of the paper [4] (Theorem 1.),[1] the sequence $\mathbf{U}_n(sf)$ is convergent in $\mathscr{M}$ and

$$\lim_{n\to\infty} \mathbf{U}_n\big(s\{f[\mu^{-1}(t)]\}\big) = \lim_{t\to\infty} f[\mu^{-1}(t)].$$

Thus, by the continuity of L_μ, we obtain

$$\operatorname{Lim}_{x\to\beta} f(x) = \lim_{n\to\infty} \overline{\mathbf{U}}_n\big(\bar{s}\{f(x)\}_\mu\big) = \lim_{n\to\infty} L_\mu \mathbf{U}_n L_\mu^{-1}\big(L_\mu(s)\cdot L_\mu L_\mu^{-1}\{f(x)\}_\mu\big) =$$

$$= \lim_{n\to\infty} L_\mu \mathbf{U}_n L_\mu^{-1}\big[L_\mu\big(s\{f[\mu^{-1}(t)]\}\big)\big] = \lim_{n\to\infty} L_\mu\big[\mathbf{U}_n\big(s\{f[\mu^{-1}(t)]\}\big)\big] =$$

$$= L_\mu\Big[\lim_{t\to\infty} f[\mu^{-1}(t)]\Big] = \lim_{t\to\infty} f[\mu^{-1}(t)] = \lim_{x\to\beta} f(x).$$

The theorem is proved.

It follows from the linearity of $\overline{\mathbf{U}}_n$ that

$$\operatorname{Lim}_{x\to\beta}\big(f(x)+g(x)\big) = \operatorname{Lim}_{x\to\beta} f(x) + \operatorname{Lim}_{x\to\beta} g(x)$$

$$\operatorname{Lim}_{x\to\beta} \lambda f(x) = \lambda \operatorname{Lim}_{x\to\beta} f(x) \qquad (\lambda\in\mathscr{K})$$

§ 5. The Stieltjes integral of a generalized function

We have seen in § 0 that

$$\frac{1}{\bar{s}}\{f(x)\}_\mu = \Big\{\int_\alpha^x f(t)\,d\mu(t)\Big\}_\mu \tag{5. 1}$$

for $\{f(x)\}_\mu \in \mathscr{L}_\mu(\alpha, \beta)$. If $\{f(x)\}_\mu$ is a generalized function, which is not a function, we make use of the left side of (5. 1) for the definition of the right side. We denote

[1]) See also [8].

the indefinite Stieltjes-integral of a generalized function $\{f(x)\}$ with respect to $\mu(x)$ "from α to x" by $\left\{\int_{\alpha}^{x} f(t)\,d\mu(t)\right\}_{\mu}$ and

$$\left\{\int_{\alpha}^{x} f(t)\,d\mu(t)\right\}_{\mu} = \frac{1}{\bar{s}}\,\{f(x)\}_{\mu}.$$

Definition 5. 1. By the Stieltjes-integral of a generalized function $f(x)$ with respect to $\mu(x)$ from α to β we understand the number

$$\operatorname*{Lim}_{x\to\beta} \int_{\alpha}^{x} f(t)\,d\mu(t),$$

if the limit exists in the sense of definition 4. 1. We write in this case

$$\operatorname*{Lim}_{x\to\beta} \int_{\alpha}^{x} f(t)\,d\mu(t) = \int_{\alpha}^{\beta} f(t)\,d\mu(t). \tag{5. 2}$$

According to the definition of the indefinite Stieltjes-integral, we may write also

$$\int_{\alpha}^{\beta} f(t)\,d\mu(t) = \lim_{n\to\infty} \bar{\mathbf{U}}_n \left(\bar{s}\frac{1}{\bar{s}}\,\{f(x)\}_{\mu}\right) = \lim_{n\to\infty} \bar{\mathbf{U}}_n \{f(x)\}_{\mu} \tag{5. 3}$$

Theorem 5. 1. *Let $f(x)$ be a function defined in (α, β) such that $f(x)$ vanishes in some interval $\alpha < x < \xi$. If $\int_{\alpha}^{\beta} f(t)d\mu(t)$ exists as a Lebesgue—Stieltjes integral, then the limit $\lim_{n\to\infty} \bar{\mathbf{U}}_n\{f(x)\}_{\mu}$ exists in $\mathscr{M}_{\mu}$ and holds*[2])

$$\lim_{n\to\infty} \bar{\mathbf{U}}_n \{f(x)\}_{\mu} = \int_{\alpha}^{\beta} f(t)\,d\mu(t). \tag{5. 4}$$

Proof. It follows from the continuity of $\mu(x)$ that

$$\int_{\alpha}^{x} f(t)\,d\mu(t) \in \mathscr{C}_{\mu}(\alpha, \beta) \tag{i}$$

$$\lim_{x\to\beta} \int_{\alpha}^{x} f(t)\,d\mu(t) = \int_{\alpha}^{\beta} f(t)\,d\mu(t). \tag{ii}$$

Hence, by theorem 4. 1, we obtain

$$\lim_{n\to\infty} \bar{\mathbf{U}}_n \{f(x)\}_{\mu} = \lim_{n\to\infty} \bar{\mathbf{U}}_n \left(\bar{s}\left\{\int_{\alpha}^{x} f(t)\,d\mu(t)\right\}_{\mu}\right) = \operatorname*{Lim}_{x\to\beta} \int_{\alpha}^{x} f(t)\,d\mu(t) = \int_{\alpha}^{\beta} f(t)\,d\mu(t).$$

[2]) It is clear that $f(x) \in \mathscr{L}_{\mu}(\alpha, \beta)$.

Remark. The above theorem goes to show that the notion of the integral, given by definition 5. 1, is a generalization of the Lebesgue—Stieltjes integral.

Theorem 5. 2 *Let $f(x)$ and $g(x)$ be generalized functions of $\mathcal{M}_\mu$. If $\int_\alpha^\beta f(t)\,d\mu(t)$ and $\int_\alpha^\beta g(t)\,d\mu(t)$ exist then*

$$\int_\alpha^\beta [c_1 f(t)+c_2 g(t)]\,d\mu(t) = c_1\int_\alpha^\beta f(t)\,d\mu(t)+c_2\int_\alpha^\beta g(t)\,d\mu(t) \tag{5. 5}$$

for all numbers $c_1, c_2 \in \mathcal{K}$.

PROOF. Since $\bar{\mathbf{U}}_n(f)\to\int_\alpha^\beta f(t)\,d\mu$ and $\bar{\mathbf{U}}_n(g)\to\int_\alpha^\beta g(t)\,d\mu$ as $n\to\infty$ it follows that

$$\bar{\mathbf{U}}_n(c_1 f+c_2 g) = c_1\bar{\mathbf{U}}_n(f)+c_2\bar{\mathbf{U}}_n(g)\to c_1\int_\alpha^\beta f(t)\,d\mu+c_2\int_\alpha^\beta g(t)\,d\mu.$$

§ 6. Change of variable

Definition 6. 1. Let $\{f(t)\}$ be a generalized function of $\mathcal{M}$. Let $\mu(x)$ be a base function in (α, β). By the generalized function $\{f[\mu(x)]\}_\mu$ of $\mathcal{M}_\mu$ we understand the generalized function $L_\mu\{f(t)\}$, i.e.

$$\{f[\mu(x)]\}_\mu = L_\mu\{f(t)\}. \tag{6. 1}$$

Theorem 6. 1. *Let $\{f(t)\}$ be a generalized function of $\mathcal{M}$. Let $\mu(x)$ be a base function in (α, β). If $\int_{-\infty}^{\infty} f(\tau)d\tau$ exists in $\mathcal{M}$, then $\int_\alpha^\beta f[\mu(t)]\,d\mu(t)$ exists in $\mathcal{M}_\mu$ and*

$$\int_{-\infty}^{\infty} f(\tau)\,d\tau = \int_\alpha^\beta f[\mu(t)]\,d\mu(t). \tag{6. 2}$$

PROOF. Since $\int_{-\infty}^{\infty} f(\tau)d\tau = \lim_{n\to\infty}\mathbf{U}_n(f)$, it follows from the continuity of L_μ that

$$\bar{\mathbf{U}}_n\{f[\mu(x)]\}_\mu = \bar{\mathbf{U}}_n L_\mu(f) = L_\mu \mathbf{U}_n L_\mu^{-1} L_\mu(f) =$$

$$= L_\mu \mathbf{U}_n(f)\to L_\mu\Big(\int_{-\infty}^{\infty} f(\tau)\,d\tau\Big) = \int_{-\infty}^{\infty} f(\tau)\,d\tau \qquad (n\to\infty).$$

This shows the validity of (6. 2).

§ 7. The definite integral of a generalized function

It is clear from the classical definition of the integral that if $\varphi(x)$ is a numerical function with a continuous first derivative and $f(x)$ is continuous in $\alpha \leqq x \leqq \beta$, then

$$\int_\alpha^\beta f(x)\, d\varphi(x) = \int_\alpha^\beta f(x)\, \varphi'(x)\, dx$$

In this section we shall derive an analogous result for the generalized integral.

Definition 7. 1. The base function $\mu(x)$ is called normal if both $\mu'(x)$ and $\frac{1}{\mu'(x)}$ are functions of $\mathscr{C}_\mu^{(\infty)}(\alpha, \beta)$.

Theorem 7. 1. *Let $\mu(x)$ be a normal base function and let $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$ be transformations of $\mathscr{T}_\mu^{\overline{\mathbf{D}}}$ such that*

$$\mu'(\lambda) = \frac{\overline{\mathbf{M}}[h(\lambda)]}{h(\lambda)}$$

and

$$\frac{1}{\mu'(\lambda)} = \frac{\overline{\mathbf{N}}[h(\lambda)]}{h(\lambda)}$$

for all $\lambda \in (\alpha, \beta)$. Then $\overline{\mathbf{M}}$ has an inverse $\overline{\mathbf{M}}^{-1}$ and

$$\overline{\mathbf{M}}^{-1} = \overline{\mathbf{N}}. \tag{7. 1}$$

Proof. It follows from theorem 1. 6 that

$$\overline{\mathbf{M}}(u) = \{\mu'(x)u(x)\}_\mu$$

and

$$\overline{\mathbf{N}}(u) = \left\{\frac{1}{\mu'(x)}\, u(x)\right\}_\mu$$

for all $u \in \mathscr{C}_\mu(\alpha, \beta)$. Since both $\{\mu'(x)u(x)\}_\mu$ and $\left\{\frac{1}{\mu'(x)}\, u(x)\right\}_\mu$ are functions of $\mathscr{C}_\mu(\alpha, \beta)$, we have

$$(\overline{\mathbf{N}}\,\overline{\mathbf{M}})(u) = \overline{\mathbf{N}}\{\mu'(x)u(x)\}_\mu = \left\{\frac{1}{\mu'(x)}\, \mu'(x)u(x)\right\}_\mu = \{u(x)\}_\mu \tag{7. 2}$$

and

$$(\overline{\mathbf{M}}\,\overline{\mathbf{N}})(u) = \overline{\mathbf{M}}\left\{\frac{1}{\mu'(x)}\, u(x)\right\}_\mu = \left\{\mu'(x)\, \frac{1}{\mu'(x)}\, u(x)\right\}_\mu = \{u(x)\}_\mu . \tag{7. 2}$$

It follows from a theorem of the paper [7] (p. 182.), by the continuity of L_μ, that $\mathscr{C}_\mu(\alpha, \beta)$ is dense in $\mathscr{M}_\mu$. Since the transformations $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$ are continuous, we obtain from (7. 2) that

$$(\overline{\mathbf{N}}\,\overline{\mathbf{M}})(f) = f$$

and

$$(\overline{\mathbf{M}}\,\overline{\mathbf{N}})(f) = f$$

for every element f of $\mathcal{M}_\mu$. That is

$$\bar{\mathbf{N}}\overline{\mathbf{M}} = \overline{\mathbf{M}}\bar{\mathbf{N}} = 1$$

which shows that the inverse of $\overline{\mathbf{M}}$ is $\mathbf{N}$.

In the following theorems we shall suppose that $\mu(x)$ is a normal base function.[3]

Definition 7. 2. Let $\mu(x)$ be a normal base function, and let $\{f(x)\}_\mu$ be a generalized function of $\mathcal{M}_\mu$. By the definite integral

$$\int_\alpha^\beta f(t)\,dt$$

of $\{f(x)\}_\mu$ we understand the integral of the generalized function $\left\{\frac{1}{\mu'(x)} f(x)\right\}_\mu$ with respect to $\mu(x)$, if it exists. We write in this case

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^\beta \frac{f(t)}{\mu'(t)}\,d\mu(t). \tag{7. 3}$$

The principal properties of the integral are summarized in the following theorem.

Theorem 7. 2. *If* $\int_\alpha^\beta f(t)dt$ *and* $\int_\alpha^\beta g(t)dt$ *exist for the generalized functions* $f(x)$ *and* $g(x)$ *of* $\mathcal{M}_\mu$, *then* $\int_\alpha^\beta [f(t)+g(t)]dt$ *exists too and*

$$\int_\alpha^\beta [f(t)+g(t)]\,dt = \int_\alpha^\beta f(t)\,dt + \int_\alpha^\beta g(t)\,dt. \tag{7. 4}$$

If $\int_\alpha^\beta f(t)\,dt$ *exists, then* $\int_\alpha^\beta cf(t)\,dt$ exists too and

$$\int_\alpha^\beta cf(t)\,dt = c\int_\alpha^\beta f(t)\,dt, \tag{7. 5}$$

where c is an arbitrary number.

This theorem follows immediately from theorems 5. 2, 2. 3 and 2. 4.

Definition 7. 3. By the indefinite integral $\left\{\int_\alpha^x f(t)dt\right\}_\mu$ of the generalized function $\{f(x)\}_\mu$ we understand the generalized function

$$\left\{\int_\alpha^x \frac{f(t)}{\mu'(t)}\,d\mu(t)\right\}_\mu$$

[3]) In § 8 we shall show that, for example, $\mu(x) = \log x$ is a normal base function. It is trivial that $\mu(x) = x$ is normal.

Remark. It is easily to show that

$$\int_{\alpha}^{x} f'(t)\,dt = f(x)$$

for every $f(x) \in \mathcal{M}_\mu$. Indeed, by definition 3. 1,

$$\{f'(x)\}_\mu = \overline{\mathbf{M}}(\bar{s}\{f(x)\}_\mu)$$

and thus, by definition 2. 3 according to theorem 7. 1, we get

$$\left\{\frac{f'(x)}{\mu'(x)}\right\}_\mu = \overline{\mathbf{M}}^{-1}\{f'(x)\}_\mu = \overline{\mathbf{M}}^{-1}[\overline{\mathbf{M}}(\bar{s}\{f(x)\}_\mu)] = \bar{s}\{f(x)\}_\mu.$$

Consequently, by definition 7. 3,

$$\left\{\int_{\alpha}^{x} f'(t)\,dt\right\}_\mu = \left\{\int_{\alpha}^{x} \frac{f'(t)}{\mu'(t)}\,d\mu(t)\right\}_\mu = \frac{1}{\bar{s}}\left\{\frac{f'(x)}{\mu'(x)}\right\}_\mu = \frac{1}{\bar{s}}\,\bar{s}\{f(x)\}_\mu = \{f(x)\}_\mu.$$

Theorem 7. 3. *Let $f(x)$ be a generalized function of $\mathcal{M}_\mu$. The existence of the integral $\int_{\alpha}^{\beta} f(t)\,dt$ in the sense of definition 7.2 is equivalent to the existence of the generalized limit $\operatorname{Lim}_{x\to\beta} \int_{\alpha}^{x} f(t)\,dt$ and the following equation holds:*

$$\operatorname*{Lim}_{x\to\beta} \int_{\alpha}^{x} f(t)\,dt = \int_{\alpha}^{\beta} f(t)\,dt$$

This theorem is an immediate consequence of definitions 7. 2, 7. 3 and 5. 1.

Theorem 7. 4. *Let $f(x)$ be a numerical function defined on $\alpha < x < \beta$ and vanishing in some right-sided neighbourhood of α. Let $\mu(x)$ be a normal base function on (α, β). If $\int_{\alpha}^{\beta} f(t)\,dt$ exists as an improper Lebesgue integral, i.e.*

$$\int_{\alpha}^{\beta} f(t)\,dt = \lim_{x\to\beta} \int_{\alpha}^{x} f(t)\,dt \tag{7. 6}$$

then $f(x)$ is integrable in the sense of definition 7. 2, and the generalized integral of $f(x)$ is equal to the improper Lebesgue integral (7. 6).

Proof. We prove first that $f(x)$ is a function of $\mathcal{L}_\mu(\alpha, \beta)$. Since $f(x)$ vanishes in some right-sided neighbourhood of α, we have only to show that $\int_{\alpha_1}^{\beta_1} f(t)\,d\mu(t)$ exists as a Lebesgue—Stieltjes integral for all $\alpha < \alpha_1 < \beta_1 < \beta$. Since $\mu(x)$ is a normal base function on (α, β), both $\mu'(x)$ and $\dfrac{1}{\mu'(x)}$ are continuous in (α, β). Thus, the functions $f(x)$ and $f(x)\mu'(x)$ are both Lebesgue integrable. Moreover, by a familiar connection between the Stieltjes and Lebesgue integrals ([9]), we have

$$\int_{\alpha_1}^{\beta_1} f(t)\mu'(t)\,dt = \int_{\alpha_1}^{\beta_1} f(t)\,d\mu(t),$$

that is $f(x)$ is inegrable with respect to $\mu(x)$ in every interval $(\alpha_1, \beta_1)\subset(\alpha, \beta)$ and thus $f(x)$ is a function of $\mathscr{L}_\mu(\alpha, \beta)$.

We prove now that the indefinite integral of $f(x)$ in the Lebesgue sense coincides with the indefinite integral in the sense of definition 7. 3. We have to show that

$$l\overline{\mathbf{M}}^{-1}(f) = \Big\{\int_{\alpha}^{x} f(t)\,dt\Big\}_\mu \tag{7. 7}$$

where the integral of the right side of (7. 7) is undestood as a Lebesgue integral. It follows from theorem 1. 6 that

$$\overline{\mathbf{M}}^{-1}(f) = \left\{\frac{f(x)}{\mu'(x)}\right\}_\mu$$

where $\dfrac{f(x)}{\mu'(x)}\in\mathscr{L}_\mu(\alpha, \beta)$. Since $\mu(x)$ is a normal base function, $\mu'(x)$ is continuous in (α, β). Therefore the Stieltjes integral with respect to $\mu(x)$ reduces to a Lebesgue integral. Consequently,

$$l\overline{\mathbf{M}}^{-1}(f) = l\left\{\frac{f(x)}{\mu'(x)}\right\}_\mu = \left\{\int_{\alpha}^{x} \frac{f(t)}{\mu'(t)}\,d\mu(t)\right\}_\mu = \left\{\int_{\alpha}^{x} \frac{f(t)}{\mu'(t)}\,\mu'(t)\,dt\right\}_\mu = \left\{\int_{\alpha}^{x} f(t)\,dt\right\}_\mu,$$

and thus (7. 7) is proved.

Since $\int_{\alpha}^{x} f(t)\,dt\in\mathscr{C}_\mu(\alpha, \beta)$, it follows from (7. 6), by theorem 4. 1, that

$$\operatorname*{Lim}_{x\to\beta}\int_{\alpha}^{x} f(t)\,dt = \lim_{x\to\beta}\int_{\alpha}^{x} f(t)\,dt = \int_{\alpha}^{\beta} f(t)\,dt.$$

Thus, by theorem 7. 3, we have established that the generalized integral of $f(x)$ is (7. 6) and the proof of the theorem is completed.

§ 8. Extension of the concept of the generalized integral for functions which are not elements of the field $\mathscr{M}_\mu$

Till now we have ourselves restricted to functions which vanish identically in some right-sided neighbourhood of α. Namely, if a function $f(x)$ does not vanish identically in the right-sided neighbourhood of α, then $f(x)$ cannot be an element of $\mathscr{M}_\mu$. For example, the constant function $\{c\}\neq 0$ cannot be a generalized function of $\mathcal{M}_\mu$. Let us suppose, on the contrary, that $c \neq 0$ and $\{c\}_\mu \in \mathcal{M}_\mu$. Then there exist functions $u(x)$ and $v(x)$ of $\mathscr{C}_\mu(\alpha, \beta)$ such that

$$\Big\{\int_\alpha^\beta cv(t)\, d\mu(t)\Big\}_\mu = \{c\}_\mu \{v(x)\}_\mu = \{u(x)\}_\mu .$$

Thus the function $u(x)$ should be a constant function of $\mathscr{C}_\mu(\alpha, \beta)$. Hence $u(x) \equiv 0$ in (α, β), and thus we have the contradiction

$$\{c\}_\mu = \frac{\{0\}_\mu}{\{v(x)\}_\mu} = 0.$$

Nevertheless we shall show in this section, that the results, established in the previous sections, may be used to extend the notion of the generalized integral for functions, which are defined on (α, β), and these are not assumed to be necessarily vanishing in the neighbourhood of α.

Definition 8. 1. A numerical function $f(x)$ is said to be locally integrable on the interval $\alpha < x < \beta$ if (i) $f(x)$ is defined on (α, β) almost everywhere, (ii) the integral of $f(x)$ over any closed subinterval of (α, β) exists as a Lebesgue integral. The class of locally integrable functions on (α, β) will be denoted by $\mathscr{L}(\alpha, \beta)$.

If $\mu(x)$ is a normal base function, then clearly

$$\mathscr{L}_\mu(\alpha, \beta) \subset \mathscr{L}(\alpha, \beta).$$

Moreover. if $f \in \mathscr{L}(\alpha, \beta)$ and $\alpha < \xi < \beta$ then

$$f(x) H_\xi(x) \in \mathscr{L}_\mu(\alpha, \beta),$$

where

$$H_\xi(x) = \begin{cases} 0 & \text{if} \quad x < \xi \\ 1 & \text{if} \quad \xi \leqq x. \end{cases}$$

Definition 8. 2. The function $f(x)$, defined on (α, β), is said to be integrable in the neighbourhood of α if $\int_\alpha^\xi f(t)\, dt$ exists as a Lebesgue integral for all $\alpha < \xi < \beta$.

If $f \in \mathscr{L}(\alpha, \beta)$ and if the Lebesgue integral $\int_\alpha^{\xi_0} f(t)\, dt$ exists for some $\alpha < \xi_0 < \beta$, then, obviously, $f(x)$ is integrable in the neighborhood of α.

Every function of $\mathscr{L}_\mu(\alpha, \beta)$ is integrable in the neighbourhood of α, provided $\mu(x)$ is a normal base function.

Definition 8. 3. Let $f(x)$ be a function of $\mathscr{L}(\alpha, \beta)$ and let $\alpha < \xi < \beta$. By the integral of $f(x)$ from ξ to β we understand the integral $\int_\alpha^\beta f(t) H_\xi(t)\, dt$, if it exists in the sense of definition 7. 2. In such a case we write

$$\int_\alpha^\beta f(t) H_\xi(t)\, dt = \int_\xi^\beta f(t)\, dt.$$

Definition 8. 4. Let $f(x)\in\mathscr{L}(\alpha,\beta)$ be integrable in the neighbourhood of α. $f(x)$ is said to be integrable on (α,β) if $\int_{\xi}^{\beta} f(t)dt$ exists in the sense of definition 8. 3 for some $\xi\in(\alpha,\beta)$. The integral of $f(x)$ from α to β is given by

$$\int_{\alpha}^{\beta} f(t)\,dt=\int_{\alpha}^{\xi} f(t)\,dt+\int_{\xi}^{\beta} f(t)\,dt \tag{8. 1}$$

where $\int_{\alpha}^{\xi} f(t)dt$ exists as a Lebesgue integral.

Theorem 8. 1. (i) $\int_{\alpha}^{\beta} f(t)dt$, *if it exists in the sense of definition* 8. 4, *is unique.* (ii) *If* $\int_{\alpha}^{\beta} f(t)dt$ *exists in the sense of definition* 8. 4, *then* $\int_{\gamma}^{\beta} f(t)dt$ *exists in the sense of definition* 8. 3 *for every* $\alpha<\gamma<\beta$ *and the following equation holds:*

$$\int_{\alpha}^{\beta} f(t)\,dt=\int_{\alpha}^{\gamma} f(t)\,dt+\int_{\gamma}^{\beta} f(t)\,dt$$

Proof. (i) Suppose that there exist numbers ξ_1 and ξ_2 $(\alpha<\xi_1<\xi_2<\beta)$, such that $\int_{\xi_1}^{\beta} f(t)\,dt$ and $\int_{\xi_2}^{\beta} f(t)\,dt$ exist in the sense of definition 8. 3. Then $\int_{\alpha}^{\beta} f(t)H_{\xi_1}(t)\,dt$ and $\int_{\alpha}^{\beta} f(t)H_{\xi_2}(t)\,dt$ exist in the sense of definition 7. 2. Thus, by theorem 7. 2, also $f(x)H_{\xi_1}(x)-f(x)H_{\xi_2}(x)$ is integrable in the sense of definition 7. 2 and

$$\int_{\xi_1}^{\beta} f(t)\,dt-\int_{\xi_2}^{\beta} f(t)\,dt=\int_{\alpha}^{\beta} f(t)H_{\xi_1}(t)\,dt-\int_{\alpha}^{\beta} f(t)H_{\xi_2}(t)\,dt=\int_{\alpha}^{\beta} f(t)[H_{\xi_1}(t)-H_{\xi_2}(t)]\,dt.$$

Since the function $f(x)[H_{\xi_1}(x)-H_{\xi_2}(x)]$ is integrable on (α,β) in the Lebesgue sense too, it follows from theorem 7. 4 that

$$\int_{\alpha}^{\beta} f(t)[H_{\xi_1}(t)-H_{\xi_2}(t)]\,dt=\int_{\xi_1}^{\xi_2} f(t)\,dt, \tag{8. 2}$$

where $\int_{\xi_1}^{\xi_2} f(t)\,dt$ is understood in the Lebesgue sense. Thus we have

$$\Big[\int_{\alpha}^{\xi_2} f(t)\,dt+\int_{\xi_2}^{\beta} f(t)\,dt\Big]-\Big[\int_{\alpha}^{\xi_1} f(t)\,dt+\int_{\xi_1}^{\beta} f(t)\,dt\Big]=$$

$$=\Big[\int_{\alpha}^{\xi_2} f(t)\,dt-\int_{\alpha}^{\xi_1} f(t)\,dt\Big]-\Big[\int_{\xi_1}^{\beta} f(t)\,dt-\int_{\xi_2}^{\beta} f(t)\,dt\Big]=\int_{\xi_1}^{\xi_2} f(t)\,dt-\int_{\xi_1}^{\xi_2} f(t)\,dt=0.$$

(ii) It will be sufficient to prove that, if $\int_{\xi_1}^{\beta} f(t)\,dt$ exists in the sense of definition 8. 3 for a number $\alpha < \xi_1 < \beta$, then the integral $\int_{\gamma}^{\beta} f(t)\,dt$ exists for every number $\alpha < \gamma < \beta$, too. Since $f(x)[H_\gamma(x) - H_{\xi_1}(x)]$ is integrable in the sense of Lebesgue, it follows from theorem 7. 4, that $\int_{\alpha}^{\beta} f(t)[H_\gamma(t) - H_{\xi_1}(t)]\,dt$ exists in the sense of definition 7. 2. Since $\int_{\xi_1}^{\beta} f(t)\,dt = \int_{\alpha}^{\beta} f(t) H_{\xi_1}(t)\,dt$ exists by hypothesis, it follows from theorem 7. 2 that $\int_{\gamma}^{\beta} f(t)\,dt$ exists, for,

$$\int_{\alpha}^{\beta} f(t)[H_\gamma(t) - H_{\xi_1}(t)]\,dt + \int_{\alpha}^{\beta} f(t) H_{\xi_1}(t)\,dt = \int_{\alpha}^{\beta} f(t) H_\gamma(t)\,dt = \int_{\gamma}^{\beta} f(t)\,dt.$$

Thus the result is established.

Theorem 8. 2. *If $f(x)$ is integrable on (α, β) in the Lebesgue sense, then $f(x)$ is integrable in the sense of definition* 8. 4, *and the generalized integral of $f(x)$ is equal to the Lebesgue integral of $f(x)$.*

PROOF. Since $f(x)H_\xi(x)$ is Lebesgue integrable, it follows from theorem 7. 4, that

$$\int_{\xi}^{\beta} f(t)\,dt = \int_{\alpha}^{\beta} f(t) H_\xi(t)\,dt \qquad (\alpha < \xi < \beta) \tag{8. 3}$$

exists in the sense of definition 8. 3. Moreover, since $f(x)$ is obviously integrable in the neighbourhood of α, $f(x)$ is integrable in the sense of definition 8. 4 and the equation (8. 1) holds. Since, in this case, (8. 3) is equal to the Lebesgue integral of $f(x)$ from ξ to β, it follows that the integral in (8. 1) is equal to the Lebesgue integral of $f(x)$ from α to β. The theorem is proved.

It should be remarked, that the validity of theorem 8. 2 does not depend on the choice of the base function $\mu(x)$ on (α, β). However, as the following examples show, the integral, in general, depends on the choice of the base function.

Example 1. To prove that $\int_{b}^{\infty} t^m\,dt\,(b > 0)$ does not exist for the integer $m \geqq 1$, if the base function is $\mu(x) = x$.

In this case $\mathcal{M}_\mu = \mathcal{M}$ is the field of Mikusiński operators. Suppose, on the contrary, that

$$\int_{b}^{\infty} \tau^m\,d\tau = \int_{-\infty}^{\infty} \tau^m H_b(\tau)\,d\tau = \lim_{n\to\infty} \mathbf{U}_n\{t^m H_b(t)\}. \tag{8. 4}$$

Since

$$e^{bs}\{t^m H_b(t)\} = \{(t+b)^m H_b(t+b)\} = \{(t+b)^m H_0(t)\} = \sum_{k=0}^{m} k!\binom{m}{k} b^{m-k}\frac{1}{s^{k+1}},$$

it follows from (8. 4) that

$$\mathbf{U}_n\{t^m H_b(t)\} = \mathbf{U}_n\left[e^{-bs}\sum_{k=0}^{m} k!\binom{m}{k} b^{m-k}\frac{1}{s^{k+1}}\right] =$$

$$= e^{-b\frac{s}{n}}\sum_{k=0}^{m} k!\binom{m}{k} b^{m-k}\frac{n^{k+1}}{s^{k+1}} \to \int_b^\infty \tau^m\, d\tau \qquad (n\to\infty).$$

Thus, necessarily

$$\frac{1}{n^{m+1}}\mathbf{U}_n\{t^m H_b(t)\}\to 0 \quad \text{as} \quad n\to\infty, \tag{8. 5}$$

for, $\frac{1}{n^{m+1}}\to 0$ as $n\to\infty$. In order to obtain a contradiction, we observe that

$$\frac{1}{n^{m+1}}\mathbf{U}_n\{t^m H_b(t)\} = e^{-b\frac{s}{n}}\sum_{k=0}^{m-1}\frac{1}{n^{m-k}} k!\binom{m}{k} b^{m-k}\frac{1}{s^{k+1}} + e^{-b\frac{s}{n}} m!\binom{m}{m}\frac{1}{s^{k+1}} \to$$

$$\to \frac{m!}{s^{k+1}} \neq 0$$

as $n\to\infty$. This shows that $\int_b^\infty t^m\,dt$ does not exist in $\mathcal{M}$.

Example 2. To prove that

$$\int_b^\infty t^\lambda\, dt = -\frac{b}{\lambda+1} \qquad (b>0) \tag{8. 6}$$

for every complex number $\lambda\neq -1$, provided $\mu(x)=\log x$.

Let $\mathcal{M}_l$ be the field determined by the base function $\mu(x)=\log x$. We show first that the base function $\log x$ is normal. We have to prove that

$$\mu'(x) = \frac{1}{x}\in\mathscr{C}_l^{(\infty)}(0,\infty) \tag{8. 7}$$

and

$$\frac{1}{\mu'(x)} = x\in\mathscr{C}_l^{(\infty)}(0,\infty). \tag{8. 8}$$

Let $\overline{\mathbf{T}}^\sigma$ be the transformation defined in § 1, 2°. This transformation is the equivalent of $\mathbf{T}^\sigma$. Since ([7]) $\mathbf{T}^\sigma\mathbf{D}=\mathbf{D}\mathbf{T}^\sigma$, the transformation $\mathbf{T}^\sigma$ is an element of $\mathscr{T}^{\mathbf{D}}$. Consequently, $\overline{\mathbf{T}^\sigma}$ is a transformation of $\mathscr{T}_\mu^{\mathbf{D}} = \mathscr{T}_\mu^{\overline{\mathbf{D}}}$. Since

$$\frac{\mathbf{T}^\sigma(e^{-\lambda s})}{e^{-\lambda s}} = e^{\lambda\sigma} \qquad (-\infty<\lambda<\infty)$$

it follows that

$$\frac{\overline{\mathbf{T}^\sigma}[h(\lambda)]}{h(\lambda)} = \frac{(L_\mu \mathbf{T}^\sigma L_\mu^{-1})[L_\mu(e^{-\log \lambda s})]}{L_\mu(e^{-\log \lambda s})} = \frac{L_\mu[\mathbf{T}^\sigma(e^{-\log \lambda s})]}{L_\mu(e^{-\log \lambda s})} =$$

$$= L_\mu\left[\frac{\mathbf{T}^\sigma(e^{-\log \lambda s})}{e^{-\log \lambda s}}\right] = L_\mu(e^{\log \lambda \sigma}) = L_\mu(\lambda^\sigma) = \lambda^\sigma$$

for $\lambda > 0$. Thus we have proved that

(8. 9) $$x^\sigma \in \mathscr{C}_l^{(\infty)}(0, \infty)$$

for every number σ, and (8. 7) and (8. 8) are both special cases of (8. 9). Consequently, $\log x$ is a normal base function.

Now let λ be an arbitrary fixed complex number. Since, by (0. 13) and (0. 14),

$$\bar{s}\{x^\lambda H_1(x)\}_l = \left\{\frac{\lambda x^{\lambda-1} H_1(x)}{\dfrac{1}{x}}\right\}_l + 1 = \lambda\{x^\lambda H_1(x)\}_l + 1,$$

we obtain

$$\{x^\lambda H_1(x)\}_l = \frac{1}{\bar{s}-\lambda}.$$

It follows from (0. 12) that

$$\frac{1}{h(b)}\{x^\lambda H_b(x)\}_l = h\left(\frac{1}{b}\right)\{x^\lambda H_b(x)\}_l = \{(bx)^\lambda H_b(bx)\}_l = b^\lambda\{x^\lambda H_1(x)\}_l$$

for $b > 0$. Consequently $\{x^\lambda H_b(x)\}_l = h(b)\dfrac{b^\lambda}{\bar{s}-\lambda}$.

Let now $\lambda \neq -1$. Then we get

$$\int_b^\infty t^\lambda\,dt = \int_0^\infty t^\lambda H_b(t)\,dt = \int_0^\infty t^{\lambda+1} H_b(t)\,d\log t = \lim_{n\to\infty} \bar{\mathbf{U}}_n\{x^{\lambda+1} H_b(x)\}_l =$$

$$= \lim_{n\to\infty} \bar{\mathbf{U}}_n\left[h(b)\frac{b^{\lambda+1}}{\bar{s}-(\lambda+1)}\right] = \lim_{n\to\infty}\left(\bar{\mathbf{U}}_n[h(b)]\bar{\mathbf{U}}_n\left[\frac{b^{\lambda+1}}{\bar{s}-(\lambda+1)}\right]\right) =$$

$$= \lim_{n\to\infty} \bar{\mathbf{U}}_n[h(b)] \lim_{n\to\infty} \frac{b^{\lambda+1}}{\dfrac{\bar{s}}{n}-(\lambda+1)} = -\frac{b^{\lambda+1}}{\lambda+1},$$

for, by the continuity of L_μ,

$$\lim_{n\to\infty} \bar{\mathbf{U}}_n[h(b)] = \lim_{n\to\infty}(L_\mu \mathbf{U}_n L_\mu^{-1})[L_\mu(e^{-\log bs})] = \lim_{n\to\infty} L_\mu \mathbf{U}_n(e^{-\log bs}) =$$

$$= \lim_{n\to\infty} L_\mu\left(e^{-\frac{s}{n}\log b}\right) = L_\mu\left(\lim_{n\to\infty} e^{-\frac{s}{n}\log b}\right) = L_\mu(1) = 1.$$

Thus (8. 6) is proved.

Since, x^λ is obviously integrable in the neighbourhood of 0, we obtain by definition 8. 4,

(8. 10)
$$\int_0^\infty t^\lambda\,dt = \int_0^b t^\lambda\,dt + \int_b^\infty t^\lambda\,dt = \frac{b^{\lambda+1}}{\lambda+1} - \frac{b^{\lambda+1}}{\lambda+1} = 0$$

for $\lambda \neq -1$. Thus we have obtained the same result which is deduced in the theory of distributions (see [6] p. 97.).

Example 3. To prove that

(8. 11)
$$\int_0^\infty t^m\,dt = \frac{(-1)^{m+1}}{m+1} \qquad (m=1, 2, \dots)$$

if $\mu(x)=\log(x+1)$. Let $\mathcal{M}_{l_1}$ be the field determined by the base function $\mu(x)= =\log(x+1)$. It can be shown, similarly to the case of $\log x$, that $\log(x+1)$ is a normal base function.

Let m be a positive integer and let $s=\frac{1}{\{H_0(x)\}_{l_1}}$. We prove by induction that

(8. 12)
$$\{x^m H_0(x)\} = \frac{m!}{(s-m)(s-m+1)\dots(s-1)s}.$$

Since
$$s\{xH_0(x)\}_{l_1} = \left\{\frac{1}{\frac{1}{x+1}} H_0(x)\right\}_{l_1} = \{xH_0(x)\}_{l_1} + \{H_0(x)\}_{l_1} = \{xH_0(x)\} + \frac{1}{s}$$

we obtain
$$\{xH_0(x)\}_{l_1}(s-1) = \frac{1}{s}$$

and hence
$$\{xH_0(x)\}_{l_1} = \frac{1}{(s-1)s}.$$

Thus (8. 12) is true for $m=1$. Let us suppose that (8. 12) is valid for $m \geqq 1$. Since

$$s\{x^{m+1}H_0(x)\}_{l_1} = (m+1)\{x^{m+1}H_0(x)\}_{l_1} + (m+1)\{x^m H_0(x)\}_{l_1},$$

we obtain from (8. 12) that

$$\{x^{m+1}H_0(x)\}_{l_1} = \frac{m+1}{[s-(m+1)]}\{x^m H_0(x)\}_{l_1} = \frac{(m+1)!}{[s-(m+1)](s-m)\dots(s-1)s}.$$

and the proof of (8. 12) is completed.

On the basis of definition 8. 3 we get

$$\int_0^\infty t^m\,dt = \int_{-1}^\infty t^m H_0(t)\,dt = \int_{-1}^\infty t^m(t+1)H_0(t)\,d\log(t+1) =$$

$$= \lim_{n\to\infty} \bar{\mathbf{U}}_n\{(x^{m+1}+x^m)H_0(x)\}_{l_1} =$$

$$= \lim_{n\to\infty} \bar{\mathbf{U}}_n\left(\frac{(m+1)!}{[s-(m+1)](s-m)\dots(s-1)s}+\frac{m!}{(s-m)\dots(s-1)s}\right) =$$

$$= \lim_{n\to\infty} \frac{m!}{\left(\frac{s}{n}-(m+1)\right)\left(\frac{s}{n}-m\right)\dots\left(\frac{s}{n}-1\right)} = \frac{(-1)^{m+1}}{m+1}$$

and (8. 11) is established.

Example 4. In the quantum theory of radiation is raised the following divergent integral: $\int_1^\infty \frac{1}{t}\,dt$. We shall show that

$$\int_1^\infty \frac{1}{t}\,dt = 0 \tag{8. 13}$$

in $\mathcal{M}_\mu$ if $\mu(x)=\log\log x$ is the base function on $(1, \infty)$. In this case we shall denote $\mathcal{M}_\mu$ by $\mathcal{M}_{l_2}$. The zero of $\mu(x)=\log\log x$ is $x_0=e$. For the sake of simplicity we shall write $\{f(x)\}$ instead of $\{f(x)H_e(x)\}_\mu$ for $f(x)\in\mathcal{L}(1, \infty)$. Thus, for instance, $\{1\}=\{H_e(x)\}_{l_2}$. Moreover, we shall write simply $s=\frac{1}{\{1\}}$ instead of $\bar{s}=\frac{1}{\{H_e(x)\}_{l_2}}$

Since

$$s\{\log x\} = \left\{\frac{\frac{1}{x}}{\frac{1}{x\log x}}\right\} + \log e = \{\log x\}+1$$

we have

$$\{\log x\} = \frac{1}{s-1}$$

and

$$\bar{\mathbf{U}}_n\{\log x\} = \frac{1}{\frac{s}{n}-1}.$$

Thus, by definition 8. 3, we get

$$\int_e^\infty \frac{1}{t}\,dt = \int_1^\infty \frac{1}{t} H_e(t)\,dt = \int_1^\infty \frac{\frac{1}{t}}{\frac{1}{t\log t}} H_e(t)\,d\log\log t =$$

$$\int_1^\infty \log t\, H_e(t)\,d\log\log t = \lim_{n\to\infty} \bar{\mathbf{U}}_n\{\log x\} = \lim_{n\to\infty} \frac{1}{\frac{s}{n}-1} = -1.$$

Since $\frac{1}{x}$ is integrable in the neighbourhood of 1, it follows from definition 8. 4 that

$$\int_1^\infty \frac{1}{t}\,dt = \int_1^e \frac{1}{t}\,dt + \int_e^\infty \frac{1}{t}\,dt = 1-1 = 0.$$

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