# On generalized convolution quotients

To Professor B. Gyires on his 60[th] birthday

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## Intraduction

Let $\mu(x)$ be a continuous increasing function in $\alpha<x<\beta$ $(-\infty\leqq\alpha<\beta\leqq\infty)$ such that $\lim_{x\to\alpha+0}\mu(x)=-\infty$ and $\lim_{x\to\beta-0}\mu(x)=\infty$. $\mu(x)$ will be referred to as a base function on $(\alpha,\beta)$. Let $\mathscr{C}^+(\alpha,\beta)$ be the linear space of continuous functions on $(\alpha,\beta)$ which vanish in some right-sided neighbourhood of $\alpha$. By means of $\mu(x)$ we define the generalized convolition in $\mathscr{C}^+(\alpha,\beta)$ as follows

$$f*g=\int_\alpha^\beta f[\mu^{-1}(\mu(x)-\mu(t))]g(t)\,d\mu(t)\qquad (f,g\in\mathscr{C}^+(\alpha,\beta)). \tag{1}$$

In the case of $\mu(x)=x$, (1) becomes the usual convolution

$$\int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

in $\mathscr{C}^+(-\infty,\infty)$. If $\mu(x)=\log x$, then (1) is limited to the multiplicative convolution

$$\int_0^\infty f\left(\frac{x}{t}\right)g(t)\frac{1}{t}\,dt.$$

We prove that the space $\mathscr{C}^+(\alpha,\beta)$ endowed with the multiplication (1) is a ring without divisors of zero. This ring will be denoted by $\mathscr{C}_\mu(\alpha,\beta)$. Every $\mathscr{C}_\mu(\alpha,\beta)$ is isomorphic to $\mathscr{C}_{\{x\}}(-\infty,\infty)$. The quotient field of $\mathscr{C}_\mu(\alpha,\beta)$ will be denoted by $\mathscr{M}_\mu$, and it will be called the field of generalized convolution quotients. $\mathscr{M}_{\{x\}}$ is the field of Mikusiński operators. Every $\mathscr{M}_\mu$ is isomorphic to $\mathscr{M}_{\{x\}}$. § 1 is devoted to the definition and basic properties of the field $\mathscr{M}_\mu$. In § 2 the embedding of the locally Stieltjes integrable functions in $\mathscr{M}_\mu$ is given. In § 3 the properties of the shift operator in $\mathscr{M}_\mu$ is investigated. In § 4 the convergence is defined. Finally, in § 5, linear transformations of $\mathscr{M}_\mu$ are considered. Thus, it will be showed that every $\mathscr{M}_\mu$ gives an operational calculus for the operator $d/d\mu(t)$.

## § 1. The field $\mathcal{M}_\mu$

Definition 1. 1. A real valued function $\mu(x)$, defined in $\alpha<x<\beta$ $(-\infty\leqq\alpha<<\beta\leqq\infty)$, is called a base function if

1° $\mu(x)$ is continuous in $\alpha<x<\beta$,

2° $\mu(x)$ is increasing in $\alpha<x<\beta$,

3° $\lim\limits_{x\to\alpha+0}\mu(x)=-\infty$,

4° $\lim\limits_{x\to\beta-0}\mu(x)=\infty$.

We shall show that every base function $\mu(x)$ determines a field $\mathcal{M}_\mu$, which is isomorphic to the field $\mathcal{M}$ of the Mikusiński operators.

Definition 1. 2. Let $\alpha<\xi<\beta$ and let $\mathcal{C}^\xi(\alpha,\beta)$ be the class of the complex-valued functions, defined on $\alpha<x<\beta$ such that every function $f(x)$ of $\mathcal{C}^\xi(\alpha,\beta)$ is continuous in $\alpha<x<\beta$ and vanishes in $\alpha<x\leqq\xi$. Let

$$\mathcal{C}^+(\alpha,\beta)=\bigcup_{\alpha<\xi<\beta}\mathcal{C}^\xi(\alpha,\beta)$$

Obviously, $\mathcal{C}^+(\alpha,\beta)$ is a vector space (under addition and multiplication by scalars).

It is well known ([1]) that the set $\mathcal{C}^+(-\infty,\infty)$ forms a commutative ring with respect to addition and multiplication in the following sense

$$f(x)*g(x)=\int_{-\infty}^{\infty}f(x-t)g(t)\,dt,\qquad (f,g\in\mathcal{C}^+(-\infty,\infty)). \tag{1. 1}$$

The absence of divisors of zero in the ring $\mathcal{C}^+(-\infty,\infty)$ makes it possible to extend this ring to a quotient field $\mathcal{M}$ and the elements of the field $\mathcal{M}$ are called Mikusiński operators.

Definition 1. 3. Let $\mu(x)$ be a base function in $(\alpha,\beta)$ and let $\mu^{-1}(t)$ be the inverse function of $\mu(x)$. The product of the functions $f(x)\in\mathcal{C}^+(\alpha,\beta)$ and $g(x)\in\in\mathcal{C}^+(\alpha,\beta)$ let be defined as follows

$$f(x)*g(x)=\int_\alpha^\beta f[\mu^{-1}(\mu(x)-\mu(t))]g(t)\,d\mu(t)\qquad (x\in(\alpha,\beta)), \tag{1. 2}$$

where the integral is understood in the sense of Stieltjes. The product (1. 2) is called the generalized convolution of $f$ and $g$.

Evidently, $f,g\in\mathcal{C}^+(\alpha,\beta)$ implies that $f*g\in\mathcal{C}^+(\alpha,\beta)$. The set $\mathcal{C}^+(\alpha,\beta)$ endowed with the operations of addition and multiplication defined by (1. 2), forms an algebraic system and it will be denoted by $\mathcal{C}_\mu(\alpha,\beta)$.

The symbol $\{f(x)\}_\mu$ denotes that the function $f(x)\in\mathcal{C}^+(\alpha,\beta)$ is regarded as an element of the algebraic system $\mathcal{C}_\mu(\alpha,\beta)$. Thus we may preserve the usual notation of the algebraic operations in $\mathcal{C}_\mu(\alpha,\beta)$ without misunderstandings:

$$\{f(x)\}_\mu+\{g(x)\}_\mu=\{f(x)+g(x)\}_\mu \tag{1. 3}$$

$$\{f(x)\}_\mu\{g(x)\}_\mu=\Big\{\int_\alpha^\beta f[\mu^{-1}(\mu(x)-\mu(t))]g(t)\,d\mu(t)\Big\}_\mu \tag{1. 4}$$

*Remark* 1. 1. The function $\mu(x)=x$ is a base function in $(-\infty, \infty)$. In this case the product (1. 2) reduces to the convolution (1. 1). The term "generalized convolution" is motivated by the above circumstance. In the case of $\mu(x)=x$ we shall preserve the original notations of Mikusiński and we shall write simply

$$\{f(t)\}\{g(t)\} = \Big\{ \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau \Big\} \tag{1. 5}$$

and $\mathscr{C}^+(-\infty, \infty)$ will denote the convolution ring in which the multiplication is defined by (1. 5).

**Theorem 1. 1.** *Let $\mu(x)$ be a base function in $(\alpha, \beta)$. Then the set $\mathscr{C}_\mu(\alpha, \beta)$ forms a commutative ring without zero divisors. Moreover, $\mathscr{C}_\mu(\alpha, \beta)$ is isomorphic to the convolution ring $\mathscr{C}^+(-\infty, \infty)$* (see remark 1. 1).

Proof. It will be sufficient to prove that the mapping

$$L_\mu\{f(t)\} = \{f[\mu(x)]\}_\mu \tag{1. 6}$$

of $\mathscr{C}^+(-\infty, \infty)$ onto $\mathscr{C}_\mu(\alpha, \beta)$ is an isomorphism. It is clear that (1. 6) definies a one-to-one correspondence between the elements of $\mathscr{C}^+(-\infty, \infty)$ and $\mathscr{C}_\mu(\alpha, \beta)$. We shall show that the mapping (1. 6) preserves the algebraic operations. In fact:

$$L_\mu(\{f(t)\}+\{g(t)\}) = L_\mu\{f(t)+g(t)\} = \{f[\mu(x)]+g[\mu(x)]\}_\mu =$$
$$= \{f[\mu(x)]\}_\mu + \{g[\mu(x)]\}_\mu = L_\mu\{f(t)\} + L_\mu\{g(t)\},$$

and

$$L_\mu(\{f(t)\}\{g(t)\}) = L_\mu\Big\{ \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau \Big\} = \Big\{ \int_{-\infty}^{\infty} f[\mu(x)-\tau]g(\tau)\,d\tau \Big\}_\mu =$$
$$= \Big\{ \int_{\alpha}^{\beta} f[\mu(x)-\mu(y)]g[\mu(y)]\,d\mu(y) \Big\}_\mu = \Big\{ \int_{\alpha}^{\beta} f[\mu(\mu^{-1}(\mu(x)-\mu(y)))]g[\mu(y)]\,d\mu(y) \Big\}_\mu =$$
$$= \{f[\mu(x)]\}_\mu \{g[\mu(x)]\}_\mu = L_\mu\{f(t)\}\cdot L_\mu\{g(t)\}$$

Since $\mathscr{C}^+(-\infty, \infty)$ is a commutative ring which has no zero divisor, it follows from the isomorphism, that $\mathscr{C}_\mu(\alpha, \beta)$ is a commutative ring without zero divisors too. Thus the theorem is proved.

Corollary 1. The ring $\mathscr{C}_\mu(\alpha, \beta)$ can be extended in the usual way to a quotient field $\mathscr{M}_\mu$. The elements of $\mathscr{M}_\mu$ will be called generalized convolution quotients and denoted by

$$\frac{\{f(x)\}_\mu}{\{g(x)\}_\mu}, \ \dots \ \text{etc.}$$

Corollary 2. Every element $\bar{a}$ of $\mathscr{M}_\mu$ has a representative

$$\bar{a} = \frac{\{f_0(x)\}_\mu}{\{g_0(x)\}_\mu}$$

where $\{f_0(x)\}_\mu$ and $\{g_0(x)\}_\mu$ are functions of the class $\mathscr{C}^{x_0}(\alpha, \beta)$ and $x_0 \in (\alpha, \beta)$ is the zero of $\mu(x) : \mu(x_0)=0$.

Corollary 3. The mapping

$$\{f(x)\}_\mu \leftrightarrow \frac{\{f(x)\}_\mu \{g(x)\}_\mu}{\{g(x)\}_\mu} \qquad (g \in \mathscr{C}_\mu(\alpha, \beta))$$

defines an embedding of $\mathscr{C}_\mu(\alpha, \beta)$ in $\mathscr{M}_\mu$.

Corollary 4. Let $\mathscr{K}$ be the field of complex numbers. The mapping

$$\lambda \leftrightarrow \frac{\{\lambda f(x)\}_\mu}{\{f(x)\}_\mu} \qquad (\lambda \in \mathscr{K},\ f \in \mathscr{C}_\mu(\alpha, \beta)) \tag{1. 7}$$

defines an embedding of $\mathscr{K}$ in $\mathscr{M}_\mu$.

Corollary 5. The field $\mathscr{M}_\mu$ is isomorphic to the field $\mathscr{M}$ of Mikusiński operators. This isomorphism is the extension of the mapping (1. 6) in the following manner:

$$L_\mu\left(\frac{\{f(t)\}}{\{g(t)\}}\right) = \frac{\{f[\mu(x)]\}_\mu}{\{g[\mu(x)]\}_\mu} = \frac{L_\mu(f)}{L_\mu(g)} \qquad \left(\frac{f}{g} \in \mathscr{M}\right) \tag{1. 8}$$

*Remark* 1. 2. We may regard the numbers on the one hand as elements of $\mathscr{M}$, on the other as elements of $\mathscr{M}_\mu$. We show that $L_\mu$ preserves the numbers:

$$L_\mu(\lambda) = \lambda \qquad (\lambda \in \mathscr{K}) \tag{1. 9}$$

Indeed, let $f(t)$ be a function of $\mathscr{C}^+(-\infty, \infty)$, then $\bar{f}(x) = f[\mu(x)]$ is a function of $\mathscr{C}_\mu(\alpha, \beta)$ and we have

$$L_\mu(\lambda) = L_\mu\left(\frac{\{\lambda f(t)\}}{\{f(t)\}}\right) = \frac{L_\mu\{\lambda f(t)\}}{L_\mu\{f(t)\}} = \frac{\{\lambda f[\mu(x)]\}_\mu}{\{f[\mu(x)]\}_\mu} = \frac{\{\lambda \bar{f}(x)\}_\mu}{\{\bar{f}(x)\}_\mu} = \lambda$$

## § 2. The embedding of locally integrable functions

Let $\mathscr{L}_\mu(\alpha, \beta)$ be the class of functions $f(x)$ defined in $\alpha < x < \beta$ such that

(i) $f(x)$ vanishes identically in a right-sided neighbourhood of $\alpha$.

(ii) $f(x)$ is integrable with respect to $\mu(x)$ in every subinterval $(\alpha_1, \beta_1)$ $(\alpha \leqq \alpha_1 < \beta_1 < \beta)$ in the sense of Lebesgue—Stieltjes.

If $\mu(x) = x$, then $\mathscr{L}_\mu(-\infty, \infty)$ is the class of the locally Lebesgue-integrable functions $f$ vanishing in any interval $(-\infty, \lambda)$, where the number $\lambda$ depends on $f$. We write in this case $\mathscr{L}^+(-\infty, \infty)$ instead of $\mathscr{L}_\mu(-\infty, \infty)$.

**Theorem 2. 1.** *$f(x)$ is a function of $\mathscr{L}_\mu(\alpha, \beta)$ if and only if $f[\mu^{-1}(t)]$ is a function of $\mathscr{L}^+(-\infty, \infty)$.*

This theorem is an immediate consequence of the well known connection between the Lebesgue and Lebesgue—Stieltjes integrals (see [2]).

By the embedding of $\mathscr{L}_\mu(\alpha, \beta)$ in $\mathscr{M}_\mu$ we make use of the fact that $\mathscr{L}^+(-\infty, \infty)$ is embedded in $\mathscr{M}$. Let $f(x) \in \mathscr{L}_\mu(\alpha, \beta)$. Then, by theorem 2. 1, $\{f[\mu^{-1}(t)]\} \in \mathscr{M}$. We identify the element $L_\mu\{f[\mu^{-1}(t)]\}$ of $\mathscr{M}_\mu$ with the function $f(x)$ and we write in this case

$$\{f(x)\}_\mu = L_\mu\{f[\mu^{-1}(t)]\}. \tag{2. 1}$$

For $-\infty<\lambda<\infty$ let

$$H_\lambda(x)=\begin{cases}0 & \text{if} \quad x<\lambda\\ 1 & \text{if} \quad \lambda\leqq x.\end{cases} \tag{2.2}$$

Obviously, $H_\lambda(x)$ is a function of $\mathscr{L}_\mu(\alpha,\beta)$ and thus $\{H_\lambda(x)\}_\mu$ is an element of $\mathscr{M}_\mu$, provided $\alpha<\lambda<\beta$.

For the zero $x_0$ of $\mu(x)$, the function $\bar{l}=\{H_{x_0}(x)\}_\mu$ is called the operator of integration with respect to $\mu(x)$. This definition is justified by

$$\bar{l}\{f(x)\}_\mu=\Big\{\int_\alpha^x f(t)\,d\mu(t)\Big\}_\mu \tag{2.3}$$

for $f\in\mathscr{L}_\mu(\alpha,\beta)$. Indeed, we obtain from (2.1) that

$$\bar{l}\{f(x)\}_\mu=L_\mu\{H_{x_0}[\mu^{-1}(t)]\}\cdot L_\mu\{f[\mu^{-1}(t)]\}=L_\mu(\{H_0(t)\}\{f[\mu^{-1}(t)]\})=$$

$$=L_\mu\Big\{\int_{-\infty}^{\infty}H_0(t-\tau)f[\mu^{-1}(\tau)]\,d\tau\Big\}=L_\mu\Big\{\int_{-\infty}^{t}f[\mu^{-1}(\tau)]\,d\tau\Big\}=$$

$$=\Big\{\int_{\mu(\alpha)}^{\mu(x)}f[\mu^{-1}(\tau)]\,d\tau\Big\}_\mu=\Big\{\int_\alpha^x f(t)\,d\mu(t)\Big\}_\mu.$$

Let $f(x)$ and $g(x)$ be defined in the neighborhood of the point $x$. The derivative of $f$ with respect to $g$ in the point $x$ is the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{g(x+h)-g(x)}$$

if it exists and it will be denoted by

$$\frac{df(x)}{dg(x)}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{g(x+h)-g(x)}. \tag{2.4}$$

If $f$ and $g$ have derivatives in $x$, then, obviously,

$$\frac{df(x)}{dg(x)}=\frac{\dfrac{df(x)}{dx}}{\dfrac{dg(x)}{dx}}=\frac{f'(x)}{g'(x)}.$$

**Theorem 2.2.** *Let $\bar{f}(x)$ be a function of $\mathscr{C}_\mu(\alpha,\beta)$. If $\bar{f}(x)$ is derivable with respect to $\mu(x)$ in all points of the interval $\alpha<x<\beta$, then the function*

$$f(t)=\bar{f}[\mu^{-1}(t)]$$

*is also derivable in $-\infty<t<\infty$ and*

$$\frac{d\bar{f}(x)}{d\mu(x)}=f'[\mu(x)]. \tag{2.5}$$

*holds.*

PROOF. Let $t=\mu(x)$. Since, by the continuity of $\mu(x)$, $\Delta t = \mu(x+h)-\mu(x)\to 0$ as $h\to 0$, we get

$$\frac{d\bar{f}(x)}{d\mu(x)} = \lim_{h\to 0}\frac{\bar{f}(x+h)-\bar{f}(x)}{\mu(x+h)-\mu(x)} = \lim_{h\to 0}\frac{f[\mu(x+h)]-f[\mu(x)]}{\mu(x+h)-\mu(x)} =$$

$$= \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t} = f'(t) = f'[\mu(x)],$$

and the theorem is proved.

The element

$$\bar{s} = \frac{1}{\bar{l}} \tag{2. 6}$$

is called the operator of differentiation with respect to $\mu(x)$. This definition is motivated by the following theorem:

**Theorem 2. 3.** *Let $\bar{f}(x)$ be a function of $\mathscr{C}_\mu(\alpha, \beta)$.* If $\dfrac{d\bar{f}}{d\mu}$ *exists and* $\dfrac{d\bar{f}}{d\mu}\in\mathscr{C}_\mu(\alpha, \beta)$, *then*

$$\bar{s}\{\bar{f}(x)\}_\mu = \left\{\frac{d\bar{f}(x)}{d\mu(x)}\right\}_\mu . \tag{2. 7}$$

PROOF. It follows from theorem 2. 2 that $\{f'(t)\}\in\mathscr{C}^+(-\infty, \infty)$ for $f(t)= =\bar{f}[\mu^{-1}(t)]$ and thus, by a known result of the Mikusiński's operational calculus (see [3] p. 192.),

$$s\{f(t)\} = \{f'(t)\}+f[\Lambda(f)]e^{-\Lambda(f)s}.$$

It follows from the continuity of $f$ in $(-\infty, \infty)$ that $f[\Lambda(f)]=0$. Therefore

$$s\{f(t)\}=\{f'(t)\}. \tag{2. 8}$$

Since

$$\bar{s} = \frac{1}{\bar{l}} = \frac{1}{\{H_{x_0}(x)\}_\mu} = \frac{1}{\{H_0[\mu(x)]\}_\mu} = \frac{L_\mu(1)}{L_\mu\{H_0(t)\}} = L_\mu\left(\frac{1}{\{H_0(t)\}}\right) = L_\mu(s),$$

we get

$$\bar{s}\{\bar{f}(x)\}_\mu = L_\mu(s)L_\mu(f) = L_\mu(sf) = L_\mu\{f'(t)\} = \{f'[\mu(x)]\}_\mu = \left\{\frac{d\bar{f}(x)}{d\mu(x)}\right\}_\mu$$

and the theorem is proved.

## § 3. The shift operator

Definition 3. 1. Let

$$\{H_\lambda(x)\}_\mu \overset{\text{def}}{=} \begin{cases} 0 & \text{if} \quad \alpha<x<\lambda \\ 1 & \text{if} \quad \lambda\leqq x<\beta, \end{cases} \qquad (\alpha<\lambda<\beta).$$

The function

$$h(\lambda) = \bar{s}\{H_\lambda(x)\}_\mu \tag{3. 1}$$

is called the *shift operator* in $\mathscr{M}_\mu$.

**Theorem 3. 1.**

(3. 2) $$h(\lambda) = L_\mu(e^{-\mu(\lambda)s})$$

PROOF.
$$L_\mu(e^{-\mu(\lambda)s}) = L_\mu(s\{H_{\mu(\lambda)}(t)\}) = L_\mu(s)L_\mu\{H_{\mu(\lambda)}(t)\} =$$
$$= \bar{s}\{H_{\mu(\lambda)}[\mu(x)]\}_\mu = \bar{s}\{H_\lambda(x)\} = h(\lambda).$$

**Theorem 3. 2.** *The function $h(\lambda)$ satisfies the functional equation*

(3. 3) $$h(\mu^{-1}[\mu(\xi)+\mu(\eta)]) = h(\xi)h(\eta) \qquad (\xi, \eta \in (\alpha, \beta)).$$

PROOF. It follows from theorem 3. 1 that
$$h(\mu^{-1}[\mu(\xi)+\mu(\eta)]) = L_\mu(e^{-[\mu(\xi)+\mu(\eta)]s}) = L_\mu(e^{-\mu(\xi)s}e^{-\mu(\eta)s}) =$$
$$= L_\mu(e^{-\mu(\xi)s})L_\mu(e^{-\mu(\eta)s}) = h(\xi)h(\eta).$$

**Theorem 3. 3.** *For $F \in \mathscr{C}_\mu(\alpha, \beta)$ and $\lambda \in (\alpha, \beta)$,*

(3. 4) $$h(\lambda)\{F(x)\}_\mu = \{F(\mu^{-1}[\mu(x) - \mu(\lambda)])\}_\mu.$$

PROOF. If $f \in \mathscr{C}^+(-\infty, \infty)$, then $e^{-\lambda s}\{f(t)\} = \{f(t-\lambda)\}$. Thus
$$h(\lambda)\{F(x)\}_\mu = L_\mu(e^{-\mu(\lambda)s})L_\mu(L_\mu^{-1}\{F(x)\}_\mu) = L_\mu(e^{-\mu(\lambda)s})L_\mu\{F[\mu^{-1}(t)]\} =$$
$$= L_\mu(e^{-\mu(\lambda)s}\{F[\mu^{-1}(t)]\}) = L_\mu\{F[\mu^{-1}(t-\mu(\lambda))]\} = \{F(\mu^{-1}[\mu(x)-\mu(\lambda)])\}_\mu.$$

## § 4. The convergence in $\mathscr{M}_\mu$

In this section we shall define the notion of the convergence in $\mathscr{M}_\mu$ and we shall show that the mapping $L_\mu$ is continuous.

Definition 4. 1. A sequence of functions $f_n \in \mathscr{C}^+(\alpha, \beta)$ is said to be convergent in $\mathscr{C}^+(\alpha, \beta)$ to the function $f \in \mathscr{C}^\xi(\alpha, \beta)$ $(\xi \in (\alpha, \beta))$, if $f_n \in \mathscr{C}^\xi(\alpha, \beta)$ for all $n = 1, 2, \ldots$ and if the sequence $f_n$ is convergent to the limit $f$ uniformly in any closed subinterval $[\xi, \eta]$ of $[\xi, \beta)$. We write in this case

(4. 1) $$f_n \rightrightarrows f \quad \text{in} \quad \mathscr{C}^+(\alpha, \beta) \quad \text{as} \quad n \to \infty.$$

**Lemma 4. 1.** *Let $\mu(x)$ be a base function in $(\alpha, \beta)$. If $f_n \rightrightarrows f$ in $\mathscr{C}^+(-\infty, \infty)$ as $n \to \infty$, then $L_\mu(f_n) \rightrightarrows L_\mu(f)$ in $\mathscr{C}^+(\alpha, \beta)$ as $n \to \infty$.*

PROOF. Let $f \in \mathscr{C}^\lambda(-\infty, \infty)$. Then $f_n \in \mathscr{C}^\lambda(-\infty, \infty)$ for all $n = 1, 2, \ldots$. Consequently, $L_\mu(f_n) = \{f_n[\mu(x)]\}_\mu \in \mathscr{C}^\xi(\alpha, \beta)$ and $L_\mu(f) = \{f[\mu(x)]\}_\mu \in \mathscr{C}^\xi(\alpha, \beta)$ for $\xi = \mu^{-1}(\lambda)$. Fix $\varepsilon > 0$ and $\eta$ such that $\xi < \eta < \beta$. Since the sequence of functions $f_n$ is convergent to $f$ uniformly in the segment $\lambda \leqq t \leqq \mu(\eta) < \infty$, therefore there is an integer $N$ so that $|f_n(t) - f(t)| < \varepsilon$ whenever $n > N$ and $\lambda \leqq t \leqq \mu(\eta)$. Consequently,
$$|f_n[\mu(x)] - f[\mu(x)]| < \varepsilon$$
ehenever $n > N$ and $\xi \leqq x \leqq \eta$. This proves the lemma.

Definition 4. 2. A sequence of elements $A_n \in \mathcal{M}_\mu$ is said to be convergent in $\mathcal{M}_\mu$ to the limit $A \in \mathcal{M}_\mu$, if there exist representatives

$$\frac{F_n}{G_n} = A_n \qquad (F_n, G_n \in \mathcal{C}_\mu(\alpha, \beta), \quad n = 1, 2, \ldots)$$

and

$$\frac{F}{G} = A \qquad (F, G \in \mathcal{C}_\mu(\alpha, \beta))$$

such that

$$F_n \rightrightarrows F \quad \text{in} \quad \mathcal{C}^+(\alpha, \beta) \quad \text{as} \quad n \to \infty$$

and

$$G_n \rightrightarrows G \quad \text{in} \quad \mathcal{C}^+(\alpha, \beta) \quad \text{as} \quad n \to \infty.$$

**Theorem 4. 1.** *If a sequence of operators $a_n$ of the field $\mathcal{M}$ is convergent in $\mathcal{M}$ to the limit $a \in \mathcal{M}$, then the sequence of elements $L_\mu(a_n) \in \mathcal{M}_\mu$ is convergent in $\mathcal{M}_\mu$ to the limit $L_\mu(a)$:*

(4. 2) $$L_\mu(a_n) \to L\ (a) \qquad (n \to \infty).$$

PROOF. Let $\frac{f_n}{g_n} = a_n$ be a sequence of representatives such that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ in $\mathcal{C}^+(-\infty, \infty)$ as $n \to \infty$. Then $\frac{f}{g} = a$ and, by lemma 4. 1, we have $L_\mu(f_n) \rightrightarrows$ $\rightrightarrows L_\mu(f)$ and $L_\mu(g_n) \rightrightarrows L_\mu(g)$ in $\mathcal{C}^+(\alpha, \beta)$ as $n \to \infty$. Thus, by definition 4. 2,

$$L_\mu\left(\frac{f_n}{g_n}\right) = \frac{L_\mu(f_n)}{L_\mu(g_n)} \to \frac{L_\mu(f)}{L_\mu(g)} = L_\mu\left(\frac{f}{g}\right) \qquad (n \to \infty)$$

and the theorem is proved.

Using theorem 4. 1, the basic properties of the limit of a sequence in $\mathcal{M}_\mu$ can be easily deduced from the corresponding properties of the limit of a sequence in $\mathcal{M}$.

We remark that a similar theorem holds for $L_\mu^{-1}$.

## § 5. Linear transformations of $\mathcal{M}_\mu$

We consider maps $\bar{\mathbf{F}}$ of $\mathcal{M}_\mu$ into $\mathcal{M}_\mu$. These are called transformations of $\mathcal{M}_\mu$.

Definition 5. 1. Let $\mathbf{F}$ be an operator transformation of $\mathcal{M}$ (see [3]). The transformation

(5. 1) $$\bar{\mathbf{F}} = L_\mu \mathbf{F} L_\mu^{-1}$$

of $\mathcal{M}_\mu$ is called the equivalent of $\mathbf{F}$ in $\mathcal{M}_\mu$.

**Theorem 5. 1.** *If $\mathbf{F}$ and $\mathbf{G}$ are transformations of $\mathcal{M}$, then*

$$\overline{\mathbf{F}+\mathbf{G}} = \bar{\mathbf{F}} + \bar{\mathbf{G}}$$

$$\overline{\mathbf{FG}} = \bar{\mathbf{F}}\bar{\mathbf{G}}$$

PROOF. $\overline{\mathbf{F}+\mathbf{G}} = L_\mu(\mathbf{F}+\mathbf{G})L_\mu^{-1} = L_\mu(\mathbf{F}L_\mu^{-1}+\mathbf{G}L_\mu^{-1}) =$

$$= L_\mu\mathbf{F}L_\mu^{-1}+L_\mu\mathbf{G}L_\mu^{-1} = \bar{\mathbf{F}}+\bar{\mathbf{G}}$$

and similarly

$$\overline{\mathbf{FG}} = L_\mu(\mathbf{FG})L_\mu^{-1} = (L_\mu\mathbf{F}L_\mu^{-1})(L_\mu\mathbf{G}L_\mu^{-1}) = \bar{\mathbf{F}}\bar{\mathbf{G}}.$$

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