On a characterization of the Rényi—Shannon entropy for incomplete probability distributions

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RÉNYI [1] has defined axiomatically in the following way the information associated to an incomplete probability distribution: Let π denote the set of all finite sequences $P=(p_1, \ldots, p_n)$ of nonnegative numbers such that $0 < w(P) = \sum_{k=1}^{n} p_k \leqq 1$. If $P \in \pi$ and $Q \in \pi$ where $P=(p_1, \ldots, p_n)$, $Q=(q_1, \ldots, q_m)$ we put $P * Q = (p_1 q_1, \ldots, p_n q_1; \ldots, p_1 q_m, \ldots, p_n q_m)$. If $P \in \pi$ and $Q \in \pi$ where $P=(p_1, \ldots, p_n)$, $Q=(q_1, \ldots, q_m)$ and $w(P)+w(Q) \leqq 1$ we put $P \cup Q = (p_1, \ldots, p_n, q_1, \ldots, q_m)$. Let $I(P)=I(p_1, \ldots, p_n)$ be defined for $P \in \pi$ such that

A') $I(p)$ is continuous for $0<p \leqq 1$, and $I(1/2)=1$

B') $I(p_1, \ldots, p_n)$ is a symmetric function of its variables,

C') $I(P*Q) = I(P)+I(Q)$, further

D') if $w(P)+w(Q) \leqq 1$ then

$$I(P \cup Q) = \frac{w(P)I(P)+w(Q)I(Q)}{w(P)+w(Q)}.$$

Rényi has shown that under the conditions

$$I(P) = \frac{\sum_{k=1}^{n} p_k \log_2 \frac{1}{p_k}}{\sum_{k=1}^{n} p_k}.$$

A distribution $P \in \pi$ is called an *imconplete probability distribution:* the function $I(P)$ is called the information *associated to P.*

We want to show here the following.

Theorem. *If for every* $P \in \pi$:

A) $I(p)$ *is continuous for* $0<p \leqq 1$ *and nonnegative,*

B) $I(p_1, \ldots, p_n)$ *is a symmetric function of its variables,*

C) $I(p_1, \ldots, p_n, 0) = I(p_1, \ldots, p_n)$,

D) $I(p_1, \ldots, p_n) - I(p_1, \ldots, p_{n-2}, p_{n-1}+p_n) = \Phi(p_1+\ldots+p_{n-2}, p_{n-1}, p_n)$ *with* $\Phi \geqq 0$,

E) $I(P*Q) = I(P)+I(Q)$,

F) $\Phi(0, p, 1-p)$ *is a Lebesgue integrable function over the interval* $0 \leqq p \leqq 1$ *not identically equal to* 0, *then we have for some* $a>1$ *and* $n>1$

$$I(P) = \frac{\sum_{k=1}^{n} p_k \log_a \frac{1}{p_k}}{w(P)} + \log_a w(P) - \log_b w(P)$$

In particular if $a=b$, *then*

$$I(P) = \frac{\sum_{k=1}^{n} p_k \log_a \frac{1}{p}}{\sum_{k=1}^{n} p_k}.$$

PROOF. From the hypothesis C and D it follows immediately

(1) $$\Phi(p_1+\dots+p_{n-2}, p_{n-1}, 0) = I(p_1, \dots, p_{n-1}, 0) - I(p_1, \dots, p_{n-1}) = 0.$$

By the symmetry hypothesis, we have

$$I(p_1, \dots, p_{n-1}, p_n) = I(p_1, \dots, p_n, p_{n-1}).$$

From this and from D we obtain

(2) $$\Phi(p_1+\dots+p_{n-2}, p_{n-1}, p_n) = \Phi(p_1+\dots+p_{n-2}, p_n, p_{n-1}).$$

By the symmetry hypothesis, we get also:

$$I(p_1, \dots, p_{n-2}, p_{n-1}, p_n) = I(p_1, \dots, p_{n-1}, p_{n-2}, p_n).$$

From this and applying D twice we deduce that necessarily

(3) $$\Phi(p_1+\dots+p_{n-2}, p_{n-1}, p_n) + \Phi(p_1+\dots+p_{n-3}, p_{n-2}, p_{n-1}+p_n) =$$
$$= \Phi(p_1+\dots+p_{n-1}, p_{n-2}, p_n) + \Phi(p_1+\dots+p_{n-3}, p_{n-1}, p_{n-2}+p_n).$$

From E we deduce, for $m=1$

(4) $$I(p_1 q, \dots, p_n q) = I(p_1, \dots, p_n) + I(q)$$

and thus in particular, for $n=1$

(4') $$I(pq) = I(p) + I(q)$$

The relation (4') must be true for all (p, q) with $0<p\leqq 1$, $0<q\leqq 1$; the function $I(p)$ must be continuous and nonnegative for all $p: 0<p\leqq 1$; this implies ([2]):

(5) $$I(p) = c \log p \quad \text{for} \quad 0<p\leqq 1,$$

where c is a negative constant. Applying D many times successively we obtain

(6) $$I(p_1, \dots, p_n) = \Phi(p_1+\dots+p_{n-2}, p_{n-1}, p_n) +$$
$$+ \Phi(p_1+\dots+p_{n-3}, p_{n-2}, p_{n-1}+p_n) + \dots + \Phi(0, p_1, p_2+\dots+p_n) +$$
$$+ I(p_1+p_2+\dots+p_n)$$

and also, bearing in mind the symmetry of $I(p_1, \dots, p_n)$,

$$(6') \qquad I(p_1 q_1, \dots, p_n q_1; \dots, p_1 q_m, \dots, p_n q_m) =$$

$$= \Phi[w(P)(q_1 + \dots + q_{m-1}) + (p_1 + \dots + p_{n-2}) q_m, p_{n-1} q_m, p_n q_m] + \dots +$$
$$+ \Phi[w(P)(q_1 + \dots + q_{m-1}), p_1 q_m, (p_2 + \dots + p_n) q_m] + \dots +$$
$$+ \Phi[w(P)(q_2 + \dots + q_m) + (p_1 + \dots + p_{n-2}) q_1, p_{n-1} q_1, p_n q_1] + \dots +$$
$$+ \Phi[w(P)(q_2 + \dots + q_m), p_1 q_1, (p_2 + \dots + p_n) q_1] + I(w(P) q_1, w(P) q_2, \dots, w(P) q_m).$$

From the relationships (4), (6), (6′) it follows from E, that for all $P \in \pi$ and $Q \in \pi$ one has

$$\Phi[w(P)(q_1 + \dots + q_{m-1}) + (p_1 + \dots + p_{n-2}) q_m, p_{n-1} q_m, p_n q_m] + \dots +$$
$$+ \Phi[w(P)(q_1 + \dots + q_{m-1}), p_1 q_m, (p_2 + \dots + p_n) q_m] + \dots +$$
$$(7) \qquad + \Phi[w(P)(q_2 + \dots + q_m) + (p_1 + \dots + p_{n-2}) q_1, p_{n-1} q_1, p_n q_1] + \dots +$$
$$+ \Phi[w(P)(q_2 + \dots + q_m), p_1 q_1, (p_2 + \dots + p_n) q_1] =$$
$$= \Phi[p_1 + \dots + p_{n-2}, p_{n-1}, p_n] + \dots + \Phi[0, p_1, p_2 + \dots + p_n].$$

From (7) we obtain, for $m=1$

$$(7') \qquad \Phi[(p_1 + \dots + p_{n-2}) q, p_{n-1} q, p_n q] + \dots + \Phi[0, p_1 q, (p_2 + \dots + p_n) q] =$$
$$= \Phi[p_1 + \dots + p_{n-2}, p_{n-1}, p_n] + \dots + \Phi[0, p_1, p_2 + \dots + p_n].$$

(7′) implies, for $n=2$

$$(7'') \qquad \Phi(0, p_1 q, p_2 q) = \Phi(0, p_1, p_2)$$

for all p_1, p_2, q with $p_1 \geqq 0$, $p_2 \geqq 0$, $0 < p_1 + p_2 \leqq 1$, $0 < q \leqq 1$; and for $n=3$

$$(7''') \qquad \Phi(p_1 q, p_2 q, p_3 q) + \Phi(0, p_1 q, (p_2 + p_3) q) =$$
$$= \Phi(p_1, p_2, p_3) + \Phi(0, p_1, p_2 + p_3)$$

for all

$$p_k, \quad q \quad \text{with} \quad p_k \geqq 0, \quad 0 < \sum_{k=1}^{3} p_k \leqq 1, \quad 0 < q \leqq 1 \qquad (k=1, 2, 3)$$

As for all (7″) and (7‴) are simultaneously true, it follows

$$(8) \qquad \Phi(p_1 q, p_2 q, p_3 q) = \Phi(p_1, p_2, p_3)$$

for all

$$p_k, \quad q \quad \text{with} \quad p_k \geqq 0, \quad 0 < \sum_{k=1}^{3} p_k \leqq 1, \quad 0 < q \leqq 1 \qquad (k=1, 2, 3)$$

On the other hand, putting

$$p_1 q = p_1', \quad p_2 q = p_2', \quad p_3 q = p_3', \quad \frac{1}{q} = q'$$

from (8) it follows that

$$\Phi(p_1', p_2', p_3') = \Phi(p_1'q', p_2'q', p_3'q') \tag{8'}$$

for all

$$p_k', \quad q' \quad \text{with} \quad p_k' \geqq 0, \quad 0 < \sum_{k=1}^{3} p_k' \leqq 1, \quad q' \geqq 1 \qquad (k=1, 2, 3).$$

Now we can notice that the relationships (8) and (8′) equal the unique condition

$$\Phi(p_1 q, p_2 q, p_3 q) = \Phi(p_1, p_2, p_3) \tag{8''}$$

for all

$$p_k, \quad q \quad \text{with} \quad p_k \geqq 0, \quad 0 < \sum_{k=1}^{3} p_k \leqq 1, \quad q > 0 \qquad (k=1, 2, 3).$$

Let us suppose now that in (7) $n=2$. (7) itself becomes in such a hypothesis 0

$$\sum_{r=1}^{m} \Phi[(p_1+p_2)(w(Q)-q_r), p_1 q_r, p_2 q_r] = \Phi(0, p_1, p_2) \tag{9}$$

We put in (9) $q_r = w(Q)x_r$. Being $q_r \geqq 0$ for $1 \leqq r \leqq m$, $\sum_{r=1}^{m} q_r = w(Q) > 0$, we will have $x_r \geqq 0$, for $1 \leqq r \leqq m$ and $\sum_{r=1}^{m} x_r = 1$. Then (9) becomes

$$\sum_{r=1}^{m} \Phi[(p_1+p_2)w(Q)(1-x_r), p_1 w(Q)x_r, p_2 w(Q)x_r] = \Phi(0, p_1, p_2) \tag{9'}$$

from which we deduce, bearing in mind (8″)

$$\sum_{r=1}^{m} \Phi[(p_1+p_2)(1-x_r), p_1 x_r, p_2 x_r] = \Phi(0, p_1, p_2). \tag{9''}$$

The relation (9″) having to be true however, if we choose the nonnegative numbers $x_1, \ldots, x_m$ such that $\sum_{r=1}^{m} x_r = \mathrm{q}$, and whatever the naturae number m may be, we deduce that necessarily

$$\begin{aligned} \Phi[(p_1+p_2)(1-x_1-x_2), p_1(x_1+x_2), p_2(x_1+x_2)] = \\ = \Phi[(p_1+p_2)(1-x_1), p_1 x_1, p_2 x_1] + \Phi[(p_1+p_2)(1-x_2), p_1 x_2, p_2 x_2] \end{aligned} \tag{10}$$

Fixing p_1 and p_2, $\Phi[(p_1+p_2)(1-x), p_1 x, p_2 x]$ is a function of the variable x only, defined over the intervall $0 \leqq x \leqq 1$, there being nonnegative. Putting

$$\varphi(x) = \Phi[(p_1+p_2)(1-x), p_1 x, p_2 x]$$

(10) assumes the form

$$\varphi(x_1+x_2) = \varphi(x_1) + \varphi(x_2) \tag{10'}$$

It is known (see [2]), that the only function $\varphi(x)$ satisfying these conditions is of the form

$$\varphi(x) = cx, \; x \in [0, 1] \quad \text{with} \quad c \geqq 0, \quad \text{constant}.$$

Correspondingly we obtain

(11) $$\Phi[(p_1+p_2)(1-x), p_1 x, p_2 x] = cx$$

From the comparison of (11) with (9″) we deduce, remembering that $\sum_{r=1}^{m} x_r = 1$

$$c = \Phi(0, p_1, p_2)$$

It is then

(11′) $$\Phi[(p_1+p_2)(1-x), p_1 x, p_2 x] = \Phi(0, p_1, p_2)x$$

$$p_1 \geqq 0, \quad p_2 \geqq 0, \quad 0 < p_1+p_2 \leqq 1, \quad 0 \leqq x \leqq 1.$$

Now we put

$$(p_1+p_2)(1-x) = \pi_1, \quad p_1 x = \pi_2, \quad p_2 x = \pi_3$$

from which it results

$$p_1 = (\pi_1+\pi_2+\pi_3)\frac{\pi_2}{\pi_2+\pi_3}, \quad p_2 = (\pi_1+\pi_2+\pi_3)\frac{\pi_3}{\pi_2+\pi_3}, \quad x = \frac{\pi_2+\pi_3}{\pi_1+\pi_2+\pi_3}.$$

We recognize immediately that $0 \leqq \pi_r \leqq 1$, for $r = 1, 2, 3$ and that $0 < \sum_{r=1}^{3} \pi_r \leqq 1$. It is also evident that (11′) is equivalent to

$$\Phi(\pi_1, \pi_2, \pi_3) = \frac{\pi_2+\pi_3}{\pi_1+\pi_2+\pi_3}\Phi\left[0, (\pi_1+\pi_2+\pi_3)\frac{\pi_2}{\pi_2+\pi_3}, (\pi_1+\pi_2+\pi_3)\frac{\pi_3}{\pi_2+\pi_3}\right].$$

On the other hand, (8″) and the preceding relationship imply

(12) $$\Phi(\pi_1, \pi_2, \pi_3) = \frac{\pi_2+\pi_3}{\pi_1+\pi_2+\pi_3}\Phi\left(0, \frac{\pi_2}{\pi_2+\pi_3}, \frac{\pi_3}{\pi_2+\pi_3}\right)$$

for all

$$\pi_1 : \pi_2 \geqq 0, \quad 0 < \sum_{r=1}^{3} \pi_r \leqq 1 \qquad (r = 1, 2, 3).$$

From (3) for $n = 3$ we deduce

(3′) $$\Phi(p_1, p_2, p_3) + \Phi(0, p_1, p_2+p_3) = \Phi(p_2, p_1, p_3) + \Phi(0, p_2, p_1+p_3),$$

$$p_k \geqq 0 \quad (k = 1, 2, 3), \quad 0 < \sum_{k=1}^{3} p_k \leqq 1,$$

and thus, bearing in mind (12)

(3″) $$\frac{p_2+p_3}{p_1+p_2+p_3}\Phi\left(0, \frac{p_2}{p_2+p_3}, \frac{p_3}{p_2+p_3}\right) + \Phi\left(0, \frac{p_1}{p_1+p_2+p_3}, \frac{p_2+p_3}{p_1+p_2+p_3}\right) =$$
$$= \frac{p_1+p_3}{p_1+p_2+p_3}\Phi\left(0, \frac{p_1}{p_1+p_3}, \frac{p_3}{p_1+p_3}\right) + \Phi\left(0, \frac{p_2}{p_1+p_2+p_3}, \frac{p_1+p_3}{p_1+p_2+p_3}\right)$$

Let us put

$$p = \frac{p_1}{p_1+p_2+p_3}, \quad q = \frac{p_2}{p_1+p_2+p_3}, \quad f(p) = \Phi(0, p, 1-p)$$

The function $f(p)$ is defined over the intervall $0 \leq p \leq 1$; it is Lebesgue integrable over this intervall, according to F; by (2) it has the property

$$f(p) = f(1-p) \qquad \text{for} \qquad p \in [0, 1]$$

It follows further from (3″)

$$(1-p) f\left(\frac{q}{1-p}\right) + f(p) = (1-q) f\left(\frac{p}{1-q}\right) + f(q)$$

for all p, q with $0 \leq p \leq 1$, $0 \leq q \leq 1$, $0 \leq p+q \leq 1$
This is Tveberg's equation ([3]). It follows that

$$f(p) = -p \log_a p - (1-p) \log_a (1-p) \quad \text{for} \quad p \in [0, 1].$$

From the definition of $f(p)$ and from (12) we then deduce

$$\Phi(p_1, p_2, p_3) = \frac{1}{p_1+p_2+p_3} [(p_2+p_3) \log_a (p_2+p_3) - p_2 \log_a p_2 - p_3 \log_a p_3]$$

Let us substitute this expression together with (5) in (6), observing that in (5) the constant c is linked to the logarithmic base and that therefore we can assume $I(p) = -\log_b p$. Thus we obtain

$$I(p_1, \dots, p_n) = \frac{-\sum_{k=1}^{n} p_k \log_a p_k}{\sum_{k=1}^{n} p_k} + \log_a \left(\sum_{k=1}^{n} p_k\right) - \log_b \left(\sum_{k=1}^{n} p_k\right)$$

If[1]) $a = b$, then in particular

$$I(p_1, \dots, p_n) = \frac{-\sum_{k=1}^{n} p_k \log_a p_k}{\sum_{k=1}^{n} p_k}$$

Thus our statement is proved.

References

[1] A. Rényi, On the foundations of information theory. *Rev. of the Intern. Statistical Institute*, **33** I (1965).
[2] J. Aczél, Lectures on functional equations. (1966).
[3] H. Tverberg, A new derivation of information function. *Math. Scand.* **6** (1958), 225—29.

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[1]) It is easy to see, that the condition

G) $\qquad I(p_1, p_2) = I(p)$

is with the property $a = b$ equivalent.