Prime and maximal ideals in lattices

By DON ADAMS (Melbourne)

1. Introduction. It is a crucial property of a Boolean lattice that an ideal in the lattice is prime if and only if it is maximal. The proof of Stone's representation theorem (that any Boolean lattice is isomorphic to the lattice of open and closed subsets of a topological space) rests on this property. It would be interesting to find out how far this property is characteristic of Boolean lattices. NACHBIN ([2]) has shown that any distributive lattice with 0 and 1 that satisfies this condition is a Boolean lattice. In this paper we find that a much weaker condition than Nachbin's is all that is needed for a lattice, in which prime and maximal ideals and dual ideals coincide, to be a Boolean lattice (Theorem 2). Before we do this we show that any lattice with 0 and 1 in which the ideals coincide as above is complemented (Theorem 1).

It is easy to see that the coincidence of maximal and prime ideals alone is not enough for the lattice to be Boolean — the five element nonmodular lattice is a counter-example.

2. Definitions and preliminary results. For the definitions of fundamental concepts such as lattice, ideal, prime, distributive etc., we refer the reader to BIRKHOFF [1], especially chapters IX and X.

We shall summarise in the following lemma all the standard results that we need. Their proof is quite straightforward, and may be found in [1] if necessary.

Lemma. *Let L be a lattice with* 0 *and* 1. *We have:*

(i) *If L is complemented, every prime* (*dual*) *ideal is maximal.*
(ii) *If L is distributive, every maximal* (*dual*) *ideal is prime.*
(iii) *Any proper ideal in L is contained in a maximal ideal.*
(iv) *The set complement $L\setminus P$ of a prime ideal P of L is a prime dual ideal.*

3. The first theorem. Before proceeding to the proof of Theorem 1, we need the following

Lemma 1. *Let M be a maximal dual ideal in a lattice L with least element* 0, *and let $a\in L$ such that $a\notin M$. Then there exists $z\in M$ such that $a\wedge z=0$.*

PROOF. Assume that no such z exists. That is, assume that $a\wedge x\neq 0$ for all $x\in M$. Then the set

$$M'=\{y:y\geqq a\wedge x,\ x\in M\}$$

is a proper dual ideal, and M is a proper subset of M'. So M is not maximal, contradicting the assumption of the lemma. This completes the proof.

We are now in a position to prove

Theorem 1. *Let L be a lattice with greatest element* 1 *and least element* 0 *in which an ideal, or dual ideal, is maximal if and only if it is prime. Then L is complemented.*

Proof. Let a be any element of L. If a is 0 or 1, then it has complement 1 or 0. So we may assume that a is not 0 or 1.

Put $I = \{x : a \wedge x = 0,\ x \in L\}$. Then I is a proper subset of L. We shall show that I is an ideal in L. Suppose that, contrarily, there exist two elements x, y in I such that $x \vee y$ does not belong to I, i. e. $a \wedge x = 0$ and $a \wedge y = 0$, but $a \wedge (x \vee y) \neq 0$. As $a \wedge (x \vee y) \neq 0$, there is a maximal dual ideal F which contains $a \wedge (x \vee y)$. F is prime, by assumption. Hence, as $x \vee y \in F$, either $x \in F$ or $y \in F$. As $a \in F$, either $a \wedge x \in F$ or $a \wedge y \in F$. But, as $a \wedge x = a \wedge y = 0$, this means that F is not proper. So our assumption that $x \vee y \in I$ is false, and I is an ideal in L.

Assume that a has no complement. Then $a \vee x \neq 1$ for all x in I and the set

$$J = \{t : t \leqq a \vee x,\ x \in I\}.$$

is a proper ideal in L. Thus there is a maximal ideal M containing J, and $a \in M$, I is a subset of M.

As M is a maximal ideal in L, it is also prime. Hence the set complement, $L \setminus M$, of M in L is a prime dual ideal. By assumption, $L \setminus M$ is thus a maximal dual ideal. As a belongs to M, a does not belong to $L \setminus M$. Hence, using lemma 1, there exists $z \in L \setminus M$ such that $a \wedge z = 0$. As $a \wedge z = 0$, $z \in I$, and as $I \subset M$, $z \in M$. This is a contradiction, so the assumption that a has no complement is false.

This completes the proof of the theorem.

4. The second theorem. A lattice L with least element 0 is said to be weakly complemented if it satisfies the condition:

(w) If a, $b \in L$ and $a < b$, then there exists $c \in L$ such that $a \wedge c = 0$, $b \wedge c \neq 0$.

Any relatively complemented lattice is weakly complemented. It should be noted that the condition (w) is not a consequence of the lattice being complemented.

We shall need the following lemma.

Lemma 2. *Let L be a complemented and weakly complemented lattice in which all maximal ideals and maximal dual ideals are prime. Then, if x, $y \in L$ $(x, y \neq 1)$ and if*

(*) *$x \in M$ if and only if $y \in M$ for all maximal prime ideals M then $x = y$.*

Proof. Suppose that x and y satisfy the conditions of the lemma. If $y < x \vee y$, then there exists, by (w), an element c such that

$$c \wedge y = 0 \quad \text{and} \quad c \wedge (x \vee y) \neq 0.$$

There is a maximal prime dual ideal F containing $c \wedge (x \vee y)$. The set $M = L \setminus F$ is then a maximal prime ideal and $y \wedge c \in M$, $c \wedge (x \vee y) \notin M$.

The element c does not belong to M, since $c \in M$ would imply $(x \vee y) \wedge c \in M$.

As M is prime, $y \wedge c \in M$ implies $y \in M$. Further $x \in M$ would imply $x \vee y \in M$ and $(x \vee y) \wedge c \in M$, so $x \in M$.

That is, M is a maximal prime ideal that contains y but not x. This contradicts the assumption (*) of the lemma.

Thus we have shown that $y = x \vee y$. We may similarly show that $x = x \vee y$, and then we have $x = y$.

We are now in a position to state and prove our main theorem.

Theorem 2. *Let L be a weakly complemented lattice with greatest element* 1 *and least element* 0 *and let an ideal, or dual ideal, in L be maximal if and only if it is prime. Then L is a Boolean lattice.*

PROOF. By Theorem 1, L is complemented. We shall show that L is distributive. Let a, x and y be arbitrary elements in L. Put

$$s = a \wedge (x \vee y) \quad \text{and} \quad t = (a \wedge x) \vee (a \wedge y).$$

It is easy to see that $s \geqq t$. We shall show that $s = t$.

If $t = 1$, $s \geqq t$, then $s = t$.

If $t \neq 1$, suppose $t \in M$, a maximal prime ideal. Then $a \wedge x \in M$ and $a \wedge y \in M$. Then either $a \in M$, or, if it doesn't, x and y both belong to M. That is, either $a \in M$ or $x \vee y \in M$. So, $a \wedge (x \vee y) \in M$, i. e. $s \in M$.

We have shown that s belongs to every maximal prime ideal that t does. As $s \geqq t$, t belongs to every maximal prime ideal containing s. If we now apply lemma 2, we see that $s = t$. This completes the proof of the theorem.

References

[1] G. BIRKHOFF, Lattice Theory, 2nd. edition, Amer. Math. Soc. Coll. Publ., vol. 25, 1948.
[2] L. NACHBIN, Une propriété caractéristique des algèbres Booléiennes, *Portugal. Math.* **6** (1947), 115—118.

(Received April 7, 1968.)