# Notes on functional equations of polynomial form

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1. Let us consider the polynomial

$$(1) \qquad P(X, Y, Z) = \sum_{i+j+k \leq n} a_{ijk} X^i Y^j Z^k$$

of order $n$ over the real (or complex) field $K$. Let $f: G \to K$ be a mapping of an abelian group $G$ into $K$. By the polynomial $P$ one can form a functional equation

$$P[f(x), f(y), f(x+y)] = 0, \qquad x, y \in G.$$

If $P(X, Y, Z)$ is not symmetric in variables $X$ and $Y$, then one can find an other polynomial form functional equation of order $m < n$ satisfied by the same function $f$. In fact, applying the transformation $(x, y) \to (y, x)$ for the independent variables, we see that the functional equation

$$(2) \qquad P(X, Y, Z) = 0, \quad X = f(x), \quad Y = f(y), \quad Z = f(x+y)$$

allowes the transformation

$$T: (X, Y, Z) \to (Y, X, Z),$$

by which we have

$$P_1 = P - TP = (1-T)P = \sum_{i+j+k \leq n} a_{ijk}(X^i Y^j - X^j Y^i) Z^k =$$

$$= (X - Y) \sum_{i+j+k < n} b_{ijk} X^i Y^j Z^k = \sum_{i+j+k \leq n} (a_{ijk} - a_{jik}) X^i Y^j Z^k \not\equiv 0,$$

hence $f$ must satisfy the polynomial form functional equation

$$P_2[f(x), f(y), f(x+y)] = 0, \quad P_1 = (X - Y) P_2$$

of order $n-1$ for every $f(x) \neq f(y)$. This last restriction is not essential e. g. in the case where $G$ is a topological group and $f$ is continuous. Thus the reduction is useful in many cases for nonsymmetric polynomial form functional equations.

The reduction can not be applied only if $P$ is invariant under $T$, i. e. it is symmetric:

$$a_{ijk} = a_{jik}.$$

Remark further that this reduction by $T$ can be applied for nonsymmetric polynomial form functional equations only in a unique step as

$$P_1 = P - TP = (X-Y)P_2$$

is allready antisymmetric and so $P_2$ must be symmetric.

Clearly, for every nonsymmetric polynomial form functional equation (2) of order $n$ we can construct a symmetric one by

$$P_s = (1+T)P = P+TP.$$

This is invariant under $T$ and its order is not more than $n$. By this way, conversely, every symmetric polynomial $P_s$ can be represented with a suitable $P$, e. g. with $P = 1/2P_s$.

2. In many cases it is enough to look only for the odd solutions $f$ of (2) for which

$$f(-x) = -f(x), \qquad x \in G \tag{3}$$

is true. If e. g. we have

$$P(X, Y, Z) = a(X+Y)^n - \sum_{i+j+k=n} a_{ijk} X^i Y^j Z^k \qquad (n>0),$$

where

$$2^n a - \sum_{i+j+k=n} a_{ijk} \neq 0,$$

then, clearly, every solution $f$ of (2) must satisfy also the condition (3).

Assuming (3), the following transformations of independent variables are useful to reduce (2):

$$(x, y, x+y) \to (x+y, -y, x) \quad \text{resp.} \quad (-x, x+y, y) \quad \text{resp.} \quad (-x, -y, -x-y).$$

Then (2) allowes the transformations:

$$T_Y: (X, Y, Z) \to (Z, -Y, X),$$
$$T_X: (X, Y, Z) \to (-X, Z, Y),$$
$$I: (X, Y, Z) \to (-X, -Y, -Z).$$

The following properties of transformations can be observed:

1. commutability of $I$ by the others;
2. $T^2 = T_X^2 = T_Y^2 = I^2 = 1$;
3. $T_X T_Y T_X = T_Y T_X T_Y = IT$;
4. $(T_X T_Y)^2 = T_Y T_X, \qquad (T_X T_Y)^3 = 1.$

As we have seen, the antisymmetrization $(1-T)P$ can be applied for the reduction of the order of the functional equation (2) not more than once. Now, by combining e. g. with the transformation $T_X$, the antisymmetrization applied for $T_X P$ resp. for $T_X(1+T)P$ may be much more useful. Then we have

$$(1-T)T_X(1+T) = (T_X - T_Y)(1+T) = (T-1)T_Y(1+T)$$

therefore now the reduction can be used for the symmetric $(1+T)P$ also without making use of $T$. This shows that the last reduction my be useful also for general cases where $G$ is not necessarily abelian and the antisymmetrization by $(1-T)$ can not be applied.

Naturally, this reduction by $T_X$ and $T_Y$ can be used for a symmetric $P$ only if neither $T_X P$ nor $T_Y P$ are invariant under $T$, i. e.

$$(T_X - T_Y)P \not\equiv 0, \quad T_X T_Y P \not\equiv P.$$

Clearly, for every polynomial $P$,

$$(1+T_X+T_Y+T_X T_Y+T_Y T_X+T_X T_Y T_X)P$$

is invariant under $T_X$ and $T_Y$.

3. In order to solve a polynomial form functional equation (2) satisfying (3) it is enough to consider only symmetric invariant polynomials.

*Definition.* A polynomial $Q(X, Y, Z)$ is called *invariant*, if

$$T_X Q = T_Y Q, \quad \text{i. e.} \quad T_X T_Y Q = Q.$$

It is called a *symmetric* invariant, if, moreover, $TQ = Q$.

Clearly,

$$S = 1+T_X T_Y+T_Y T_X = 1+T_X T_Y+(T_X T_Y)^2$$

is an invariant operator as we have

$$T_X T_Y S = S, \quad \text{i. e.} \quad T_X S = T_Y S.$$

Conversely, every invariant $Q$ can be representes in the form

$$Q = SP$$

with a suitable polynomial $P$. In fact, we have

$$T_X T_Y Q = Q, \quad (T_X T_Y)^2 Q = Q,$$

hence

$$[1+T_X T_Y+(T_X T_Y)^2]Q = 3Q$$

and so e. g. $P = 1/3Q$ can represent $Q$ in the form $Q = SP$.

An invariant $Q = SP$ is symmetric if and only if $P$ is symmetric as

$$TS = ST = I(T_X+T_Y+T_X T_Y T_X).$$

Since every polynomial $P$ can be built up as the sum of homogeneous polynomials

$$H_r = \sum_{i+j+k=r} a_{ijk} X^i Y^j Z^k$$

and the order $r$ is unchanged by $T_X$, $T_Y$ and $T$, hence in order to determine all the (symmetric) invariants $Q$ of order $n$ it is enough to determine the (symmetric) homogeneous invariants for the orders $r \leqq n$.

Some examples:

The only linear homogeneous invariant is a $H_1$, where

$$H_1 = X+Y-Z = -SZ.$$

The quadratic homogeneous symmetric invariants are linear combinations of the following ones:

$$SZ^2 = X^2+Y^2+Z^2, \quad S(XY) = XY-Z(X+Y).$$

For $n=3$ we have:

$$SZ^3 = Z^3-X^3-Y^3,$$
$$S[(X+Y)Z^2] = (X+Y)Z^2+(Y-Z)X^2+(X-Z)Y^2,$$
$$S(XYZ)=3XYZ.$$

The other symmetric homogeneous invariants for $n>3$ have a similar form: $Q=SH$, where

$$H = (XY)^p(X^q+Y^q)Z^r$$

is the most general form of terms in a symmetric homogeneous polynomial.

4. An interesting invariant of order $n=4$ is e. g.

$$Q_4 = (X+Y-Z)(X+Z)(Y+Z)(X-Y).$$

This plays an important role e. g. in the reduction of the following polynomial type functional equation

$$P_n = (X+Y)^n - Z^n = 0, \quad X=f(x), \quad Y=f(y), \quad Z=f(x+y). \tag{4}$$

In fact, here the substitution $y=-x$ shows that (3) holds and $P_n$ can be factorized as

$$P_n = H_1 \sum_{i+j+k=n-1} a_{ijk}X^iY^jZ^k, \quad a_{ijk}=a_{jik},$$

where

$$H_1 = X+Y-Z$$

is an invariant:

$$T_XH_1 = T_YH_1 = -H_1.$$

Now we have

$$(T_X-T_Y)P_n = -H_1 \sum_{i+j+k=n-1} (-1)^i a_{ijk}(X^iY^k - X^kY^i)Z^j = H_1(X-Y)P_{n-1},$$

where

$$P_{n-2} = TP_{n-2}$$

is symmetric.

Here applying once more the transformations $T_X$, $T_Y$, we can reduce (4) to

$$H_1(T_X-T_Y)P_{n-2} = 0,$$

supposed that

$$(X-Y), \quad T_X(X-Y) = -X-Z, \quad T_Y(X-Y) = Z+Y$$

are different from O. Thus we see that (4) is equivalent to

$$H_1(X-Y)(X+Z)(Y+Z)(T_X-T_Y)P_{n-2} = 0,$$

i. e. to a functional equation of the form

$$Q_4P_{n-3} = 0.$$

E. g., for $n=4$, by this way (4) can be reduced to

$$(X+Y-Z)(X-Y)(X-Z)(X+Z)(X+Y-Z)^2 = 0,$$

i. e. to

$$Q_4 = (X+Y-Z)(X-Y)(X+Z)(Y+Z) = 0,$$

where

$$X=f(x), \quad Y=f(y), \quad Z=f(x+y).$$

5. As another example let us consider the functional equation

$$\sum_{i+j+k<2} a_{ijk} X^i Y^j Z^k, \quad X=f(x), \quad Y=f(y), \quad Z=f(x+y). \tag{5}$$

Supposed that (3) holds, we have the following possibilities:

1. (5) is of the form

$$a(X^2+Y^2+Z^2)+b(XY-XZ-YZ) = 0;$$

2. (5) can be reduced to

$$(X-Y)(aX+bY+cZ) = 0.$$

In the case 1. by putting

$$y=0 \quad \text{i. e.} \quad Y=0, \quad Z=X,$$

we see that

$$f(x)=0 \quad \text{for} \quad 2a-b \neq 0;$$

$$a(X+Y-Z)^2 = 0 \quad \text{for} \quad 2a-b=0.$$

In the case 2. similarly

$$f(x)=0 \quad \text{for} \quad a+c \neq 0 \quad \text{or} \quad b+c \neq 0;$$

$$a(X-Y)(X+Y-Z) = 0 \quad \text{for } a=b=-c.$$

In this last case we must jet solve the functional equation

$$f(x+y) = f(x)+f(y) \quad \text{if} \quad f(x) \neq f(y), \qquad x, y \in G.$$

Let us consider a fixed pair $(x, y)$. If the set of values of the function $f$ is enouhg large, we can find a $z \in G$ such that

$$f(z) \neq f(y), \qquad f(z) \neq f(x+y), \qquad f(y+z) \neq f(x).$$

But then we have

$$f(x+y+z) = f(x+y)+f(z) = f(x)+f(y+z) = f(x)+f(y)+f(z),$$

consequently,

$$f(x+y) = f(x)+f(y)$$

holds for every fixed $x, y \in G$.

Remark that

$$XY-XZ-YZ = 0, \qquad X=f(x), \qquad Y=f(y), \qquad Z=f(x+y)$$

has a solution

$$f(x)=a(x)^{-1}$$

where $a(x)$ is an arbitrary additive function. This solution satisfies also (3), however, it is not defined for $x=0$.

## NOTES ON FUNCTIONAL EQUATIONS OF POLYNOMIAL FORM

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*Summary*

Supposed that $f(-x)=-f(x)$, the functional equation

$$P(X, Y, Z)=0, \quad X=f(x), \quad Y=f(y), \quad Z=f(x+y)$$

allowes the transformations

$$T_X: \ (X, Y, Z) \to (-X, Z, Y), \quad T_Y: \ (X, Y, Z) \to (Z, -Y, X).$$

This can be seen by the substitutions $(x, y) \to (-x, x+y)$ and $(x, y) \to (x+y, -y)$ of independent variables. By the transformations $T_X$ and $T_Y$, the functional equation $P=0$ can be reduced in many cases to a simpler one. However, this reduction cannot be applied if $P$ is invariant under $T_X$ and $T_Y$. In the classification theory of functional equations these invariants play an important role. In the present paper all the polynomial invariantes for which $T_X P=T_Y P$ holds are determined. Some remarks about the solutions of invariant equations are given.