The K-unitary convolution of certain arithmetical functions

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§ 1. A divisor d of n is called a k-unitary divisor of n if $(d, n/d)_k = 1$; here the symbol $(a, b)_k$ stands for the largest kth power divisor common to both a and b. For any arithmetical functions $g(n)$ and $h(n)$, let $f(n)$ be their k-unitary convolution, and $F(x)$ be the summatory function of $f(n)$: i. e.

$$f(n) = \sum_{\substack{d\delta=n \\ (d,\delta)_k=1}} g(d)h(\delta) \tag{1.1}$$

and

$$F(x) = \sum_{n \leqq x} f(n) \tag{1.2}$$

Functions involving 1-unitary divisors (unitary divisors) have been studied by ECKFORD COHEN and some others. In particular, in [2], Cohen considers (1.1) when $k=1$ and proves (Theorems 4.1 and 5.1 of [2]), that if

$$g(n) = O(1), \quad \text{and} \quad h(n) = n \tag{1.3}$$

$$F(x) = \frac{x^2}{2} \sum_{n=1}^{\infty} \frac{g(n)\varphi(n)}{n^3} + O(x \log^2 x) \tag{1.4}$$

and if

$$g(n) = O(1), \quad \text{and} \quad h(n) = n\mu^2(n) \tag{1.5}$$

$$F(x) = \frac{x^2}{2} \sum_{n=1}^{\infty} \frac{g(n)\beta(n)}{n^2} + O(x^{3/2}) \tag{1.6}$$

where

$$\beta(n) = \frac{n\varphi(n)}{\zeta(2)J(n)} \tag{1.7}$$

$\varphi(n)$ being the Euler's totient function and $J(n)$ the Jordan's totient function of order 2.

Here we consider the order of magnitude of $f(n)$ in the more general case when $k \geqq 1$, $g(n) = O(n^\varepsilon)$, $h(n) = n^r$ and $n^r\mu^2(n)$. (Theorems 3.1 and 3.2). In particular, these results would imply (See Corollaries 3.1.1 and 3.2.1 together with (3.6)) that the main terms in (1.4) and (1.6) remain the same even though we take unbounded $g(n)$ subject to the condition $g(n) = O(n^\varepsilon)$ with $\varepsilon < 1/2$ in (1.3) and $\varepsilon < 3/4$ in (1.5).

We will be naturally led to the generalised Euler's function $\varphi_k(n)$ which is defined as the number of integers in a resideu system mod n which are k-prime to n; i. e. which have 1 as their k th power G. C. D. with n. We remark that this function is different from the generalised Euler's function (although we are using the same symbol) defined by Cohen ([3]) as the number of numbers in a residue system mod n^k which are k-prime to n^k and that both these functions give Euler's function for $k=1$. In Section 2, we obtain some of the properties of this functions as generalisations of the corresponding properties of $\varphi(n)$.

§ 2. Let $\tau_k^*(n)$ denote the number of k-unitary divisors of n. It is easily seen that $\tau_k^*(n)$ is a multiplicative function of n and so it is completely determined by the values $\tau_k^*(p^\alpha)$ p ranging over primes and α over positive integers. Also since $(p^\beta, p^{\alpha-\beta})_k = 1$ if and only if $\min\{\beta, \alpha-\beta\} < k$, it follows that every divisor of p^α is a k-unitary divisor of p^α, if $\alpha < 2k$, and if $\alpha \geqq 2k$, p^β will be a k-unitary divisor of p^α if and only if $\beta \varepsilon \{0, 1, 2, \dots, k-1, \alpha-k+1, \dots, \alpha\}$. These observations give.

Theorem 2.1. *If the canonical decomposition of n is given by*

$$n = q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$$

where $O < \beta_i < 2k$, $i = 1, 2, \dots, s$ *and* $\alpha_i \geqq 2k$, $i = 1, 2, \dots, t$ *then*

$$\tau_k^*(n) = \left\{\prod_{i=1}^{s} (1+\beta_i)\right\}\left\{\prod_{i=1}^{t} (2k)\right\} \tag{2. 1}$$

When $k=1$ each $\beta_i = 1$ and the formula (2. 1) reduces to the well known formula for the number of unitary divisors of n.

Since $a \leqq n$, $(a, n)_k = d^k$ if and only if $a/d^k \leqq n/d^k$ and $(a/d^k, n/d^k)_k = 1$, it follows from the familiar argument that the

Theorem 2.2.

$$\sum_{d^k|n} \varphi_k(n/d^k) = n.$$

is true.

Theorem 2.3. For any arithmetical functions $f_1(n)$ and $f_2(n)$

$$f_2(n) = \sum_{d^k|n} f_1(n/d^k) \quad \text{if and only if} \quad f_1(n) = \sum_{d^k|n} f_2(n/d^k)\mu(d).$$

Proof. If $f_2(n) = \sum_{d^k|n} f_1(n/d^k)$, then

$$\sum_{d^k|n} f_2(n/d^k)\mu(d) = \sum_{d^k\delta=n} f_2(\delta)\mu(d) = \sum_{d^k\delta=n} \mu(d) \sum_{D^k\Delta=\delta} f_1(\Delta) =$$

$$= \sum_{r^k\Delta=n} f_1(\Delta) \sum_{d^k|r^k} \mu(d) = f_1(n),$$

since $\sum_{d^k|r^k} \mu(d) = 1$ or O according as $r=1$ or $r>1$. The proof of the other half is similar

We note that Theorem 2.3 when $k=1$ is the well known Möbius inversion formula.

From Theorems 2. 2 and 2. 3, we get

Theorem 2.4.

$$\varphi_k(n) = n \sum_{d^k|n} \frac{\mu(d)}{d^k} \tag{2. 2}$$

Formula (2. 2) implies that $\varphi_k(n)$ is multiplicative. Taking $n=p^\alpha$ in (2. 2), we get

$$\varphi_k(p^\alpha) = \begin{cases} p^\alpha \text{ if } & \alpha < k, \\ p^{\alpha-k}(p^k-1), \text{ if } & \alpha \geqq k. \end{cases} \tag{2. 3}$$

Lemma 2.1.

(i) $$\sum_{m \leqq x} m^r = \frac{x^{r+1}}{r+1} + O(x^r), \quad r \geqq 0, \quad x \geqq 1;$$

(ii) $$\sum_{m \leqq x} 1/m^r = \begin{cases} O(1), & r>1,\ x \geqq 1; \\ O(\log x), & r=1,\ x \geqq 2; \\ O(x^{1-r}), & r<1,\ x \geqq 1; \end{cases}$$

(iii) $$\sum_{m > x} 1/m^r = O(1/x^{r-1}), \quad r>1,\ x \geqq 1.$$

This lemma is well known and we omit the proof.

Theorem 2.5.

$$\Phi_k(x) = \sum_{n \leqq x} \varphi_k(n) = \frac{x^2}{2} \frac{1}{\zeta(2k)} + O(x \log x).$$

PROOF.

$$\Phi_k(x) = \sum_{n \leqq x} \varphi_k(n),$$

which by Theorem 2. 4 is equal to

$$\sum_{d^k \delta \leqq x} \delta \mu(d) = \sum_{d^k \leqq x} \mu(d) \sum_{\delta \leqq x/d^k} \delta,$$

which by (i) of Lemma 2. 1

$$= \sum_{d^k \leqq x} \mu(d) \{1/2)(x/d^k)^2 + O(x/d^k)\}$$

$$= \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{2k}} - \frac{x^2}{2} \sum_{d > x^{1/k}} \frac{\mu(d)}{d^{2k}} + \sum_{d \leqq x^{1/k}} \mu(d) O(x/d^k) \tag{2. 4}$$

Now, by using Lemma 2. 1, the second term in (2. 4) is $O(x^{1/k})$ and the third term is $O(x \log x)$ and the theorem is clear in virtue of Theorem 287 of [4].

Clearly, Theorem 2. 5 reduces to Merten's Theorem (Theorem 330 of [4]) when $k=1$.

§ 3. Let now, for any real numbers x and r, $r \geqq 0$, $x \geqq 1$

$$\varphi_{k,r}(x, n) = \sum_{\substack{m \leqq x \\ (m,n)_k = 1}} m^r \tag{3. 1}$$

$$\varphi'_{k,r}(x, n) = \sum_{\substack{m \leqq x \\ (m,n)_k = 1}} m^r \mu^2(m) \tag{3. 2}$$

Lemma 3.1. *For* $r \geqq 0,\ x \geqq 1,\ n \geqq 1$

(i)
$$\varphi_{k,r}(x,n) = \frac{x^{r+1}}{r+1}\frac{\varphi_k(n)}{n} + O\big(\tau_k'(n)x^r\big)$$

(ii)
$$\varphi_{1,r}'(x,n) = \frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\frac{\varphi(n)\eta_1(n)}{n} + O\big(\tau_1'(n)x^{r+1/2}\big)$$

(iii)
$$\varphi_{k,r}'(x,n) = \frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\frac{\varphi_2(n)\eta_2(n)}{n} + O\big(\tau_2'(n)x^{r+1/2}\big), \quad k \geqq 2,$$

where

(3. 3)
$$\eta_k(n) = \sum_{d^k|n} \frac{\mu^2(d)}{J(d)}$$

and $\tau_k'(n)$ *denotes the number of square free divisors of* n *whose* k *th powers also divide* n.

PROOF:

(i)
$$\varphi_{k,r}(x,n) = \sum_{\substack{m \leqq x \\ (m,n)_k = 1}} m^r = \sum_{m \leqq x} m^r \sum_{d^k|(m,n)_k} \mu(d) = \sum_{d^k|n} \mu(d) \sum_{j=1}^{[x|d^k]} (jd^k)^r,$$

which by Lemma 2. 1 is equal to

$$\sum_{d^k|n} \mu(d) d^{kr} \left\{\left(\frac{x}{d^k}\right)^{r+1} \frac{1}{r+1} + O\left(\left(\frac{x}{d^k}\right)^r\right)\right\}$$

form which the first part of the lemma follows in virtue of Theorem 2. 4.

ii) This is in fact i) of Lemma 3. 2 in [1] (see also, Lemma 5. 2 of [2]) where the coefficient of x^{r+1} is

$$\frac{\beta(n)}{r+1}; \quad \text{here} \quad \beta(n) = \frac{\varphi(n)}{n}\frac{n^2}{\zeta(2)J(n)},$$

and by using the evaluation formula for $J(n)$ (formula (3. 6) of [2]), it can easily be seen that $\eta_1(n) = \dfrac{n^2}{J(n)}$.

(iii)
$$\varphi_{k,r}'(x,n) = \sum_{\substack{m \leqq x \\ (m,n)_2 = 1}} m^r \mu^2(m) = \sum_{\substack{m \leqq x \\ (m,n)_2 = 1}} m^r \sum_{d^2\delta = m} \mu(d) =$$

$$= \sum_{\substack{d^2\delta \leqq x \\ (d^2,n)_2 = (\delta,n)_2 = 1}} (d^2\delta)^r \mu(d) = \sum_{\substack{d \leqq \sqrt{x} \\ (d^2,n)_2 = 1}} d^{2r}\mu(d)\varphi_{2,r}(x/d^2, n)$$

which by i) of the present lemma is

(3. 4)
$$\frac{x^{r+1}}{r+1}\frac{\varphi_2(n)}{n} \sum_{\substack{d \leqq \sqrt{x} \\ (d^2,n)_2 = 1}} \frac{\mu(d)}{d^2} + \sum_{\substack{d \leqq \sqrt{x} \\ (d^2,n)_2 = 1}} d^{2r}\mu(d) O\left(\tau_2'(n)\left(\frac{x}{d^2}\right)^r\right).$$

Now,

$$\sum_{\substack{d \leqq \sqrt{x} \\ (d^2, n)_2 = 1}} \frac{\mu(d)}{d^2} = \sum_{d \leqq \sqrt{x}} \frac{\mu(d)}{d^2} \sum_{D^2 | (d^2, n)_2} \mu(D) =$$

$$= \sum_{D^2 | n} \mu(D) \sum_{j=1}^{[\sqrt{x}/D]} \frac{\mu(jD)}{j^2 D^2} = \sum_{D^2 | n} \frac{\mu^2(D)}{D^2} \sum_{\substack{j=1 \\ (j, D)=1}}^{[\sqrt{x}/D]} \frac{\mu(j)}{j^2} =$$

$$= \sum_{D^2 | n} \frac{\mu^2(D)}{D^2} \left\{ \sum_{\substack{j=1 \\ (j,D)=1}}^{\infty} \frac{\mu(j)}{j^2} - \sum_{\substack{j > \sqrt{x}/D \\ (j,D)=1}} \frac{\mu(j)}{j^2} \right\}$$

The first term above, by Lemma 5. 1 of [2] is $\dfrac{\eta_2(n)}{\zeta(2)}$ and since for any D such that $D^2 | n$,

$$\frac{\mu^2(D)}{D^2} \sum_{\substack{j > \sqrt{x}/D \\ (j,D)=1}} \frac{\mu(j)}{j^2} = O\left(\frac{1}{\sqrt{x}}\right),$$

iii) is clear.

Actually, making use of Theorem 2. 4, (3. 3), and the evaluation formula for $J(n)$, we get

Lemma 3.2.

$$\frac{\varphi_k(n)\eta_k(n)}{n} = \prod_{\substack{p^\alpha \| n \\ \alpha \geqq k}} \frac{p^{2-k}(p^k-1)}{p^2-1} \tag{3. 5}$$

where $p^\alpha \| n$, means that $p^\alpha | n$, and $p^{\alpha+1} \nmid n$.

In particular, we note

(3. 6)

(i) $$\frac{\varphi(n)\eta_1(n)}{n} = \prod_{p|n} \frac{p}{p+1} = \zeta(2)\beta(n)$$

(ii) $$\frac{\varphi_2(n)\eta_2(n)}{n} = 1$$

Theorem 3.1. *If in* (1. 1), $g(n) = O(n^\varepsilon)$, $O < \varepsilon < 1$, *and* $h(n) = n^r$, $r > \varepsilon$, *then*

$$F(x) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{g(n)\varphi_k(n)}{n^{r+2}} + E_r(x) \tag{3. 7}$$

where

$$E_r(x) = \begin{cases} O(x^r), & r > 1+\varepsilon,\ x \geqq 1 \\ O(x^r \log^2 x), & r = 1+\varepsilon,\ x \geqq 2 \\ O(x^{2-r+2\varepsilon}), & \frac{1}{2}+\varepsilon < r < 1+\varepsilon,\ x \geqq 1. \end{cases}$$

PROOF.

$$F(x) = \sum_{\substack{d\delta \leqq x \\ (d,\delta)_k = 1}} g(d)\delta^r = \sum_{d \leqq x} g(d) \sum_{\substack{\delta \leqq x/d \\ (d,\delta)_k = 1}} \delta^r = \sum_{d \leqq x} g(d)\varphi_{k,r}(x/d, d),$$

which by (i) of Lemma (3. 1), is equal to

$$\sum_{d\leqq x} g(d)\left\{\left(\frac{x}{d}\right)^{r+1}\frac{1}{r+1}\frac{\varphi_k(d)}{d}+O\left[\tau_k'(d)\left(\frac{x}{d}\right)^r\right]\right\}=$$

$$=\frac{x^{r+1}}{r+1}\sum_{d=1}^{\infty}\frac{g(d)\varphi_k(d)}{d^{r+2}}+O\left(x^{r+1}\sum_{d>x}\frac{g(d)\varphi_k(d)}{d^{r+2}}\right)+O\left(x^r\sum_{d\leqq x}\frac{\tau_k'(d)}{d^{r-\varepsilon}}\right).$$

Now, by Lemma 2. 1, and the fact $\varphi_k(n)\leqq n$,

$$\sum_{d>x}\frac{g(d)\varphi_k(d)}{d^{r+2}}=O\left(\frac{1}{x^{r-\varepsilon}}\right),$$

and

$$\sum_{d\leqq x}\frac{\tau_k'(d)}{d^{r-\varepsilon}}\leqq\sum_{d\leqq x}\frac{\tau(d)}{d^{r-\varepsilon}}=\sum_{\delta_1\delta_2\leqq x}\frac{1}{(\delta_1\delta_2)^{r-\varepsilon}}\leqq\left(\sum_{m\leqq x}\frac{1}{m^{r-\varepsilon}}\right)^2,$$

and Theorem 3. 1 is clear.

We have the following corollaries:

Corollary 3. 1. 1. If in (1. 1) $g(n)=O(n^{\varepsilon})$ for some $\varepsilon<1/2$, and $h(n)=n^r$, then for $r\geqq 1$,

$$F(x)\sim\frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{g(n)\varphi_k(n)}{n^{r+2}}$$

Corollary 3. 1. 1. may also be stated as

Corollary 3. 1. 2. If in (1. 1) $g(n)=O(n^{\varepsilon})$ for some $\varepsilon<1/2$, and $h(n)=n^r$, then for $r\geqq 1$, the average order of $f(n)$ is

$$\left(\sum_{n=1}^{\infty}\frac{g(n)\varphi_k(n)}{n^{r+2}}\right)n^r.$$

If, in particular, we take $g(n)=1$, $h(n)=n^r$ in Theorem 3. 1, we get

Corollary 3. 1. 3.

$$\sum_{n\leqq x}\sigma_{k,r}^*(n)=\frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{\varphi_k(n)}{n^{r+2}}+E_r(x)$$

where $E_r(x)$ is given by

$$E_r(x)=\begin{cases}O(x^{2-r}), & \text{if } \frac{1}{2}<r<1,\ x\geqq 1\\ O(x\log^2 x), & \text{if } r=1,\ x\geqq 2\\ O(x^r), & \text{if } r>1,\ x\geqq 1;\end{cases}$$

and $\sigma_{k,r}^*(n)$ is the sum of the r th powers of the k-unitary divisors of n.

Theorem 3.2. *If in* (1. 1), $g(n)=O(n^{\varepsilon})$, $O<\varepsilon<1$, *and* $h(n)=n^r\mu^2(n)$, $r>\varepsilon$, *then*

$$\text{(i)}\qquad F(x)=\frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{g(n)\varphi(n)\eta_1(n)}{n^{r+2}}+E_r(x),\quad \text{if}\quad k=1;$$

$$\text{(ii)}\qquad F(x)=\frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{g(n)}{n^{r+1}}+E_r(x),\quad \text{if}\quad k>1,$$

where $E_r(x)$ *is given by*

$$E_r(x)=\begin{cases}O(x^{r+1/2}), & r>\frac{1}{2}+\varepsilon,\\ O(x^{r+1/2}\log^2 x), & r=\frac{1}{2}+\varepsilon,\ x\geqq 2,\\ O(x^{3/2-r+2\varepsilon}), & \frac{1}{4}+\varepsilon<r<\frac{1}{2}+\varepsilon,\ x\geqq 1.\end{cases}$$

PROOF. We prove the second part, the proof of the first part being similar.

$$F(x)=\sum_{\substack{d\delta\leqq x\\ (d,\delta)_k=1}}g(d)\delta^r\mu^2(\delta)=\sum_{d\leqq x}g(d)\sum_{\substack{\delta\leqq x/d\\ (\delta,d)_k=1}}\delta^r\mu^2(\delta)$$

which by (3. 2), iii) of Lemma 3. 1, and ii) of (3. 6) is

$$\sum_{d\leqq x}g(d)\left\{\left(\frac{x}{d}\right)^{r+1}\frac{1}{r+1}\frac{1}{\zeta(2)}+O\left(\tau_2'(d)\left(\frac{x}{d}\right)^{r+1/2}\right)\right\}=$$

$$=\frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\sum_{d=1}^{\infty}\frac{g(d)}{d^{r+1}}+O\left(x^{r+1}\sum_{d>x}\frac{g(d)}{d^{r+1}}\right)+O\left(x^{r+1/2}\sum_{d\leqq x}\frac{\tau_2'(d)}{d^{r+1/2-\varepsilon}}\right).$$

The second term above, by iii) of Lemma 2. 1 is $O(x^{1+\varepsilon})$ and a proof similar to that employed in the proof of Theorem 3. 1 shows that the third term is $O(x^{r+1/2})$, $O(x^{r+1/2}\log^2 x)$, $O(x^{3/2-r+2\varepsilon})$ according as $r\gtreqless 1/2+\varepsilon$. The result follows now since

$$1+\varepsilon\leqq r+\tfrac{1}{2}\quad\text{if}\quad r\geqq\tfrac{1}{2}+\varepsilon,\quad 1+\varepsilon<\tfrac{3}{2}-r+2\varepsilon\quad\text{if}\quad r<\tfrac{1}{2}+\varepsilon,$$
$$\text{and}\quad r+1>\tfrac{3}{2}-r+2\varepsilon\quad\text{if}\quad r>\tfrac{1}{4}+\varepsilon.$$

Corollary 3. 2. 1. If in (1. 1), $g(n)=O(n^{\varepsilon})$ for some $\varepsilon<3/4$, and $h(n)=n^r\mu^2(n)$, then for $r\geqq 1$,

$$\text{(i)}\qquad F(x)\sim\frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{g(n)\varphi(n)\eta_1(n)}{n^{r+2}},\qquad k=1,$$

$$\text{(ii)}\qquad F(x)\sim\frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{g(n)}{n^{r+1}},\qquad k>1.$$

Corollary 3. 2. 1 can also be stated as

Corollary 3. 2. 2. If in (1. 1), $g(n)=O(n^{\varepsilon})$ for some $\varepsilon<3/4$, and $h(n)=n^r\mu^2(n)$, then for $r\geqq 1$, the average order of $f(n)$ is

$$\left(\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{g(n)\varphi(n)\eta_1(n)}{n^{r+2}}\right)n^r,\quad\text{if}\quad k=1,$$

and is

$$\left(\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{g(n)}{n^{r+1}}\right)n^r, \quad \text{if} \quad k>1.$$

Taking $g(n)=1$, we get, in particular, from Theorem 3. 2.

Corollary 3. 2. 3.

$$\sum_{n\leqq x}\sigma_{1,r}^{*\prime}(n)=\left(\frac{x^{r+1}}{r+1}\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{\varphi(n)\eta_1(n)}{n^{r+2}}+E_r(x)\right.$$

and

$$\sum_{n\leqq x}\sigma_{k,r}^{*\prime}(n)=\frac{x^{r+1}}{r+1}\frac{\zeta(r+1)}{\zeta(2)}+E_r(x), \qquad k\geqq 2,$$

where

$$E_r(x)=\begin{cases} O(x^{r+1/2}), & \text{if} \quad r>\frac{1}{2}, \\ O(x^{r+1/2}\log^2 x), & \text{if} \quad r=\frac{1}{2}, \\ O(x^{3/2-r}) & \text{if} \quad \frac{1}{4}<r<\frac{1}{2}, \end{cases}$$

$\sigma_{k,r}^{*\prime}(n)$ being the sum of the r th powers of the square free k-unitary divisors of n.

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(Received April 23, 1968.)