Note on a conjecture of P. Erdős

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The function f defined on the natural numbers is called *additive* if $(m, n)=1$ implies

$$f(mn) = f(m)+f(n) \tag{1}$$

(as usual, (m, n) denotes the greatest common divisor of m and n). If (1) holds for all pairs m, n then f is called *completely additive*. In [1], P. ERDŐS conjectured that if f is additive and

$$f(n+1)-f(n) = O(1), \tag{2}$$

then there is a constant α such that $f(n) = \alpha \log n + O(1)$.

The following lemma is a partial result which shows that it suffices to prove that if f is completely additive and (2) holds, then $f(n) = \alpha \log n$.

Lemma. *If f is an additive function and $f(n+1)-f(n) = O(1)$, then $g(n) = \lim_{l\to\infty} \frac{f(n^l)}{l}$ exists. Moreover*

$$f(n) = g(n) + O(1),$$

g is completely additive, and $g(n+1)-g(n) = O(1)$.

PROOF. In the following, the constants in O symbols are absolute; $\zeta(n, k, l)$ and $\theta(n, k, l)$ denote expressions which are bounded, with bounds depending on n and k but not on l.

For arbitrary m and q we have

$$f(m^q+m^{q-1}+\dots+1) = f(m^q+m^{q-1}+\dots+m)+O(1) =$$
$$= f(m^{q-1}+m^{q-2}+\dots+1)+f(m)+O(1),$$

whence, by repeated application

$$f(m^q+m^{q-1}+\dots+1) = q\big(f(m)+O(1)\big).$$

Thus, if $(m-1, q) = 1$, then

$$f(m^q-1) = f(m-1)+f(m^{q-1}+m^{q-2}+\dots+1) = q\big(f(m)+O(1)\big) \tag{3}$$

Let n and k be fixed, and let $m = n^k$. If $l > mk$, then we can write l as $l = qk + r$ with $q \equiv 1 \mod (m-1)$ and $0 \leqq r < k(m-1)$. Then by (2) and (3) we have

$$f(n^l) = f(n^l - n^r) + O(n^r) = f(n^{qk} - 1) + \zeta(n, k, l) = q\big(f(n^k) + O(1)\big) + \zeta(n, k, l).$$

It follows that

$$l^{-1} f(n^l) - k^{-1} f(n^k) = l^{-1} q\big(f(n^k) + O(1)\big) + l^{-1}\zeta(n, k, l) - k^{-1} f(n^k) =$$
$$= l^{-1} q O(1) + l^{-1}\zeta(n, k, l) - l^{-1} k^{-1} r f(n^k).$$

Hence

$$l^{-1} f(n^l) - k^{-1} f(n^k) = (qk + r)^{-1} q O(1) + l^{-1}\theta(n, k, l). \tag{4}$$

Now (4) implies that $\{l^{-1} f(n^l)\}_{l=1}^{\infty}$ *is a Cauchy sequence*. In fact, first k can be chosen so as to make the first term of the right hand side small, and then it follows that for sufficiently large values of l the corresponding terms of the sequence will have small differences. We therefore may conclude that $g(n) = \lim_{l \to \infty} l^{-1} f(n^l)$ exists, and by taking limits in (4) we find

$$g(n) = k^{-1} f(n^k) + O(k^{-1}).$$

In particular, for $k = 1$ we find

$$f(n) = g(n) + O(1)$$

which, with (2), implies $g(n+1) - g(n) = O(1)$.

Since f is additive, so is g, and it follows from the definition of g that $g(n^k) = kg(n)$ for all n and k, so the function g is completely additive. This completes the proof.

References

[1] P. Erdős, On the distribution function of additive functions, *Ann. of Math.* **47** (1946), 1—20.

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