Two isotopically equivalent varieties of groupoids

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We shall say that a grupoid $\langle A, \omega' \rangle$ is nomially derived from a groupoid $\langle A, \omega \rangle$ if there exists a ω-word $w(x, y)$ in x, y such that $xy\omega' = w(x, y)$ for all $x, y \in A$. If further $\langle A, \omega \rangle$ is also nomially derived from $\langle A, \omega' \rangle$ we shall say that $\langle A, \omega \rangle$, $\langle A, \omega' \rangle$ are nomially equivalent. Groupoids $\langle A, \omega \rangle$, $\langle A, \omega' \rangle$ will be called isotopically equivalent if each is an isotope ([1]) of the other.

Two classes K_1, K_2 of groupoids will be called nomially (isotopically) equivalent if every groupoid in one class is nomially (isotopically) equivalent to a groupoid in the other.

Nomial and isotopical equivalences can be defined for arbitrary classes of universal algebras of a given species without any difficulty.

Examples of nomially equivalent varieties of algebras are well-known. Thus varieties of Boolean algebras and Boolean rings are nomially equivalent.

In this note we prove that the variety of groupoids defined by the law $y = xt\omega xz\omega yz\omega\omega t\omega\omega$ is isotopically equivalent to the variety of abelian groups.

Theorem. *The varieties of groupoids defined by the laws*

(I) $$y = xt\omega\, xz\omega\, yz\omega\, \omega t\omega\omega$$

and

(II) $$y = x\, xz\omega\, yz\omega\, \omega\omega$$

are isotopically equivalent but not nomially equivalent.

The PROOF of the theorem is divided into the following four lemmas.

Lemma 1. *A groupoid satisfying* I *is a quasigroup.*

PROOF. In I replace t by $y_2 z\omega$, y by y_1 and x by $xz\omega$. We get the law

$$y_1 = xz\omega\, y_2 z\omega\, \omega xz\omega\, z\omega y_1\, z\omega\, \omega y_2\, z\omega\, \omega\omega.$$

If we write w for the word $xz\omega\, z\omega y_1\, z\omega\, \omega y_2 z\omega\, \omega$ and u for the word $xw\omega$ then this last law together with I gives

$$y_2 = xw\omega\, xz\omega\, y_2 z\omega\, \omega w\omega = xw\omega\, y_1\, \omega = uy_1\, \omega.$$

Hence I implies that $\forall y_1, y_2 \exists t\ (ty_1\omega = y_2)$. This can be expressed by saying that the equation $ty_1\omega = y_2$ has a solution in t in every groupoid satisfying I for all y_1, y_2.

The equation $y_1 t\omega = y_2$ is also solvable. This can be seen as follows. We have just proved that we can find x, t_1 such that $xt_1\omega = y_1$. Then, by I,

$$y_1 xz\omega\, y_2 z\omega\omega\, t_1\omega\omega = y_2$$

and if we put $t = xz\omega\, y_2 z\omega\, \omega t_1\omega$ we have $y_1 t\omega = y_2$.

We now prove the 'cancellation laws'. Let t_1, y, t_2 be such that $t_1 y\omega = t_2 y\omega$. Then, by (I),

$$t_1 = xt\omega\, xy\omega\, t_1 y\omega\, \omega t\omega\, \omega = xt\omega\, xy\omega\, t_2 y\omega\, \omega t\omega\, \omega = t_2$$

and the right cancellation law holds. Next let t_1, y, t_2 be such that $yt_1\omega = yt_2\omega$. Find x, z, y_1, y_2 such that $y = xz\omega$, $t_1 = y_1 z\omega$, $t_2 = y_2 z\omega$. Then

$$y_1 = xt\omega\, xz\omega\, y_1 z\omega\, \omega t\omega\omega = xt\omega\, yt_1\omega\, t\omega\omega = xt\omega\, yt_2\omega\, t\omega\omega =$$

$$= xt\omega\, xz\omega\, y_2 z\omega\, \omega t\omega\, \omega\, y_2$$

so that $t_1 = t_2$ and the left cancellation law also holds. This completes the proof of the lemma.

Lemma 2. *The law* (I) *is equivalent to the following statement. There exists* e *such that for all* x, y, z

(i) $xe\omega\, e\omega = x$,

(ii) $xz\omega\, yz\omega\, \omega = xy\omega\, e\omega$,

(iii) $xe\omega\, xy\omega\, \omega = y$.

PROOF. I implies (i)—(iii). We have, by (I)

$$x = xt\omega\, xz\omega\, xz\omega\omega\, t\omega\omega.$$

By Lemma 1 $xz\omega$ can be any element t_1 for suitable z. Hence

(iv) $x = xt_1\omega\, tt_1\omega\, t\omega\omega$, for all x, t_1, t.

This gives

$$xx\omega = xx\omega\, t\omega x\, x\omega t\, \omega\omega$$

and if t is such that $xx\omega\, t\omega = y$,

$xx\omega = yy\omega = e$, some fixed element.

We can now write (iv) in the form

$$x = xt\omega\, et\omega\, \omega.$$

In the last equation if t is taken equal to e we arrive at (i).

To prove (ii) we note that

$$y = xt\omega\, xz_1\omega\, yz_1\omega\, \omega t\omega\, \omega = xt\omega\, xz_2\omega\, yz_2\omega\, \omega t\omega\, \omega.$$

Hence by Lemma 1

$$xz_1\omega\, yz_1\omega\, \omega = xz_2\omega\, yz_2\omega\, \omega$$

so that $xz\omega\, yz\omega\, \omega = xy\omega\, \omega yy\, \omega = xy\omega\, e\omega$.
By (I) and (i)

$$y = xe\omega\, xy\omega\, yy\omega\, \omega e\omega\, \omega = xe\omega\, xy\omega\, \omega.$$

This proves (iii).

Now we show that (i)—(iii) imply (I). We have

$$\begin{aligned} xt\omega\, yz\omega\, yz\omega\, \omega t\omega\omega &= xt\omega\, xy\omega\, e\omega t\, \omega\omega &&, \text{ by (ii) of Lemma 2}\\ &= xe\omega\, xy\omega\, e\omega e\, \omega\omega &&, \text{ by (ii) of Lemma 2}\\ &= xe\omega\, xy\omega\, \omega &&, \text{ by (i) of Lemma 2}\\ &= y &&, \text{ by (iii) of Lemma 2}\end{aligned}$$

Hence I holds and the lemma is proved.

Lemma 3. *A groupoid* $\langle A, \omega\rangle$ *satisfies* (I) *if and only if* $\langle A, \omega'\rangle$ *satisfies* (II) *where*

$$(1) \qquad xy\omega = xy\omega'\gamma, \qquad \text{for all } x, y \in A,$$

where $xy\omega' = xy\omega\, xx\omega\, \omega$ *for all* $x, y \in A$ *and* γ *is an involution of* $\langle A, \omega'\rangle$. *(We recall that* γ *is called an involution if* γ *is an automorphism and* γ^2 *is the identity map.) Further a groupoid* $\langle A, \omega'\rangle$ *satisfies* (II) *if and only if* $\langle A, \omega\rangle$ *satisfies* (I) *where*

$$(2) \qquad xy\omega' = xy\omega\, xx\omega\, \omega \qquad \text{for all } x, y \in A.$$

PROOF. *Part 1.* Let $\langle A, \omega'\rangle$ satisfy (II) and $\langle A, \omega\rangle$ be defined by (1). Then $\langle A, \omega\rangle$ satisfies (I). For,

$$\begin{aligned} & xt\omega\, xz\omega\, yz\omega\, \omega t\omega\, \omega \\ &= xt\omega'\, \gamma xz\omega'\gamma\, yz\omega'\gamma\, \omega'\gamma\, t\omega'\gamma\, \omega'\gamma, && \text{by (1)}\\ &= xt\omega'\, xz\omega'\, yz\omega'\, \omega' t\omega'\, \omega', && \text{since } \gamma \text{ is an involution}\\ &= y. \end{aligned}$$

In this last step we have used the result [2] that II characterizes abelian groups in terms of the operation of subtraction.

Part 2. Let $\langle A, \omega\rangle$ satisfy (I) and ω' be defined by (2). Then $\langle A, \omega'\rangle$ satisfies (II). For by the proof of Lemma 2 $xx\omega = e$ for some e and for all x in A and hence

$$\begin{aligned} & xxz\, \omega' yz\, \omega'\omega'\omega' \\ &= xx\, z\omega\, e\omega yz\, \omega e\omega\, \omega e\omega\omega e\, \omega &&, \text{ by (2)}\\ &= x\, xz\omega\, yz\omega\, \omega e\omega\, e\omega\omega\, e\omega &&, \text{ by (ii) of Lemma 2}\\ &= x\, xz\omega\, yz\omega\, \omega e\omega\, \omega &&, \text{ by (i) of Lemma 2}\\ &= xe\omega\, xz\omega\, yz\omega\, \omega e\omega\, \omega &&, \text{ by (ii) of Lemma 2}\\ &= y &&, \text{ by I.}\end{aligned}$$

Part 3. Let $\langle A, \omega\rangle$ satisfy (I) and let $\langle A, \omega'\rangle$ be defined by (2). We show that (1) holds for some involution γ of $\langle A, \omega'\rangle$. Define γ by $x\gamma = xe\omega$ where e is the constant value of $xx\omega$ in $\langle A, \omega\rangle$. Then γ is an involution of $\langle A, \omega'\rangle$. For

$$\begin{aligned} xy\omega'\gamma &= xy\omega\; e\omega e\; \omega \\ &= xe\omega\; ye\omega\; \omega e\omega, \qquad \text{by (ii) of Lemma 2} \\ &= x\gamma y\; \gamma\omega e\; \omega = x\gamma y\; \gamma\omega' \end{aligned}$$

and

$$x\gamma\gamma = xe\omega\; e\omega = x, \qquad \text{by (i) of Lemma 2.}$$

The involution γ satisfies (1); $xy\omega'\gamma = xy\omega\; e\omega e\; \omega = xy\omega$, by (i) of Lemma 2.

Part 4. Let $\langle A, \omega'\rangle$ satisfy II and $\langle A, \omega\rangle$ be defined (1). We show that ω, ω' are related by (2). This is easy:

$$xy\omega\; xx\omega\omega = xy\omega'\gamma\; xx\omega'\gamma\; \omega'\gamma = yx\omega'\; xx\omega'\; \omega' = xy\omega'.$$

Here we have used the fact that ω' is an operation of subtraction in an abelian group and the assumption that γ is an involution of $\langle A, \omega'\rangle$.

The proof of the lemma is complete.

We observe that Part 3 of the proof of Lemma 3 shows that if $\langle A, \omega\rangle$ satisfies I and ω' is defined by (2) then $\langle A, \omega'\rangle$ is an isotope of $\langle A, \omega\rangle$. In view of this remark Lemma 3 shows that the varieties defined by I and II are isotopically equivalent.

Lemma 4. *The varieties defined by* (I) *and* (II) *are not nomially equivalent.*

Proof. Suppose that the lemma is not true. Let $\langle A, \omega\rangle$ be a countably free groupoid satisfying (I) and let $\langle A, \omega'\rangle$ be nomially equivalent to $\langle A, \omega\rangle$ and suppose that $\langle A, \omega'\rangle$ satisfies (II). Since endomorphisms of $\langle A, \omega\rangle$ and $\langle A, \omega'\rangle$ are the same it is easy to see that $\langle A, \omega'\rangle$ is a free abelian group. Let k_1, k_2 be integers such that $xy\omega = k_1x + k_2y$ for all $x, y \in A$. Here we have used the usual operation of addition in an abelian group and usual notation. If we substitute $xy\omega = k_1x + k_2y$ in (I) and use the fact that $\langle A, \omega'\rangle$ is free we find that $k_1k_2 = \pm 1$. This implies that $\langle A, \omega\rangle$ satisfies (II) or the law

(II*) $$y = zy\omega\; zx\omega\; \omega x\omega.$$

But $\langle A, \omega\rangle$ is free. Hence (I) implies (II) or (II*).

However there exist groupoids that satisfy (I) but not (II) or (II*). An example is the groupoid consisting of four elements 0, 1, 2, 3 in which the multiplication is defined by the table:

	0	1	2	3
0	0	3	2	1
1	3	0	1	2
2	2	1	0	3
3	1	2	3	0

This contradiction proves the lemma.

Incidentally the law (I) has the interesting property that if $\langle A, \omega\rangle$ satisfies (I) then so does the dual groupoid $\langle A, \omega^*\rangle$ where $xy\omega = yx\omega^*$ for all $x, y \in A$. The law (II) does not have this property.

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References

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