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The properties of T^*-groups

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Abstract. This paper is a continuation of [1]. The aim of this work is the generalization of Zacher's theorem [2] for the solvable T^*-groups, a characterization of these groups by the normalizers of p-subgroups and the study of subnormal p-subgroups and of the Sylow subgroups of a solvable T^*-group.

Throughout this paper G will denote a finite group. A T-group is a group G whose subnormal subgroups are normal in G. In [1] M. Asaad introduced the concept of T^*-group. A group G is called T^*-group if every subnormal subgroup of G is quasinormal in G. A subgroup of G is quasinormal in G if it permutes with every Sylow subgroup of G. In [1] we proved theorems concerning T-groups for T^*-groups. Now we are continuing the study of solvable T^*-groups. The notation used in this paper is standard.

First we generalize a theorem of Zacher [2] for solvable T^*-groups.

Theorem 1. *Let G be a solvable group, the prime divisors of its order $p_1 > p_2 > \cdots > p_k$, and let $P_1, \dots, P_k$ be a Sylow system with $P_i \in Syl_{p_i}(G)$. G is a T^*-group if and only if it satisfies the following conditions:*

(i) If $1 \le i < j \le k$, then $P_j \le N_G(P_i)$.
(ii) For all $1 \le i < j \le k$, if $x \in P_i$, $y \in P_j$ then there exists a natural number n such that $x^y = x^n$.

Proof. 1. Suppose G is a solvable T^*-group. Then by Lemma 1 of [1] G is supersolvable whence it has a Sylow tower. So G satisfies (i). As every subgroup of a solvable T^*-group is again a T^*-group by Theorem 1

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of [1], it follows that $N_G(P_i)$ is a solvable T^*-group. We have that $\langle x\rangle$ is subnormal in $N_G(P_i)$, and using Lemma 2 of [1] $P_j < N_G(\langle x\rangle)$ is true. Thus G satisfies (ii).

2. Conversely, assume G satisfies (i) and (ii). We show that every subgroup of P_i is quasinormal in $N_G(P_i)$. Let B be an arbitrary subgroup of P_i. By the conditions $P_j < N_G(B)$ for all $j > i$. As $P_j^y < N_G(B^y)$ for every $y \in N_G(P_i)$, clearly any Sylow p_j-subgroup of $N_G(P_i)$ normalizes any subgroup of P_i. Let $\ell < i$ and let D be an arbitrary Sylow p_ℓ-subgroup of $N_G(P_i)$. By Hall's theorems $(P_iD)^z \leq P_\ell P_i$ for some $z \in G$. As $(P_iD)^z = P_i^z D^z$, $D^z \leq P_\ell$ and $P_i < N_G(P_\ell)$ clearly $P_i^z < N_G(D^z)$ follows, furthermore $D^z \leq N_G(P_i^z)$, whence D centralizes P_i.

Thus every subgroup of P_i is quasinormal in $N_G(P_i)$. Using Lemma 4 of [1] it follows that either $P_i \leq G'$ or each Sylow q-subgroup ($q \neq p_i$) of $N_G(P_i)$ centralizes P_i. So we can repeat the first part of the proof of Theorem 2 in [1]. Consequently $G = HK$, where H is a nilpotent normal Hall subgroup of G, K is a nilpotent Hall subgroup of G, $H \cap K = 1$, furthermore for arbitrary $x \in H$, $y \in K$ there exists a natural number i such that $x^y = x^i$. Then G is a solvable T^*-group by Theorem 2 of [1].

We need the following

Lemma. *Let U be a p-subgroup of G, $a \in N_G(U)$ such that $(|a|, |U|) = 1$ furthermore a normalizes every subgroup of U. If there is an element $b \neq 1$ of U such that $ab = ba$, then $a \in C_G(U)$ follows.*

PROOF. Clearly $C_U(a) \neq 1$. Assume $C_U(a) \neq U$.

(a) $Z(\Omega_1(U)) \not\geq C_U(a)$.

Denote $W = Z(\Omega_1(U))C_U(a)$. Clearly $\langle a\rangle$ normalizes every subgroup of W and each element of $\langle a\rangle$ induces the identity on $W/Z(\Omega_1(U))$ by conjugation. Applying Lemma 3 of [1] $\langle a\rangle \leq C_G(W)$ follows, a contradiction.

(b) $Z(\Omega_1(U)) > C_U(a)$.

Clearly there exists a subgroup $T \neq 1$ such that $Z(\Omega_1(U)) = C_U(a) \times T$. Let $b \in T$, $b \neq 1$ and $u \in C_U(a)$, $u \neq 1$. Clearly a normalizes $\langle b\rangle$ and $\langle bu\rangle$, consequently $(bu)^a = (bu)^m$ where $2 \leq m \leq p-1$. As $(bu)^a = b^a u = b^m u^m$, $u^{m-1} = (b^m)^{-1} b^a$ follows. We have $b^a = b^n$ where $2 \leq n \leq p-1$ so $u^{m-1} = b^{n-m}$ is true, but $\langle u\rangle \cap \langle b\rangle = 1$ thus $u^{m-1} = 1$, a contradiction.

So $Z(\Omega_1(U)) = C_U(a)$.

(c) $Z(\Omega_1(U)) < \Omega_1(U)$.

Similarly to case (b) we can show that this case is impossible too.

Thus $C_U(a) = Z(\Omega_1(U)) = \Omega_1(U)$. Let $\ell \in U \backslash \Omega_1(U)$. By the conditions a normalizes $\langle \ell\rangle$. As a centralizes $\Omega_1(\langle \ell\rangle)$ consequently a centralizes $\langle \ell\rangle$. A contradiction.

Theorem 2. *G is a solvable T^*-group if and only if every p-subgroup A (for all prime divisors p of the order of G) is quasinormal in $N_G(P_0)$ where P_0 is a p-subgroup containing the subgroup A.*

Proof. Assume G is a solvable T^*-group. Then by Theorem 1 of [1] $N_G(P_0)$ is a solvable T^*-group too. Clearly A is subnormal in $N_G(P_0)$, whence A is quasinormal in $N_G(P_0)$.

Conversely, let p_1 be the smallest prime divisor of the order of G. We show that G has a normal p_1-complement. Let P_1 be a Sylow p_1-subgroup of G and let H be an arbitrary subgroup of P_1. We prove that $N_G(H)/C_G(H)$ is a p_1-group. Assume there is an element b of $N_G(H)\backslash C_G(H)$ of order q with $q \neq p_1$. Let a be an element of H of order p_1. By the conditions $\langle a\rangle$ is quasinormal in $N_G(H)$. It is easy to see $b \in N_H(\langle a\rangle)$. As $q > p_1$, $b \in C_G(a)$ follows. Clearly every subgroup of H is quasinormal in $N_G(H)$ by the conditions, whence b normalizes every subgroup of H. Using our Lemma $b \in C_G(H)$ is true, a contradiction. Thus $N_G(H)/C_G(H)$ is a p_1-group, consequently G has a normal p_1-complement. So $G = P_1K$, $K \lhd G$ and $P_1 \cap K = 1$. Consider the smallest prime divisor p_2 of the order of K. Similarly we can prove that K has a normal p_2-complement.

Thus G has a tower such that the prime divisors of the order of G are $p_1 < p_2 < \cdots < p_k$ and for arbitrary $1 \leq i \leq k$ there is a Sylow p_i-subgroup such that $P_i < N_G(P_j)$ for all $1 \leq i < j \leq k$. If i and j are such as above and $x \in P_j$, $y \in P_i$, then $\langle x\rangle$ is quasinormal in $N_G(P_j)$ by the conditions, whence it is easy to see that $y \in N_G(\langle x\rangle)$, consequently $x^y = x^n$ for some natural number n. Applying Theorem 1 G is a solvable T^*-group.

Theorem 3. *Let G be a solvable T^*-group. Then an arbitrary subnormal p-subgroup of G (for all prime divisors p of the order of G) is either normal or it is centralized by all Sylow q-subgroups of G with $q \neq p$.*

Proof. Let A be a subnormal p-subgroup of G. By Theorem 7 of [1] $G = MN$ where M is a nilpotent normal Hall subgroup of G, N is a nilpotent Hall subgroup of G, $M \cap N = 1$ furthermore every subgroup of prime power order of M is normal in G

(a) $A \leq M$. By the above A is normal in G

(b) $A \leq N^y$ for some $y \in G$.

Let Q be a Sylow q-subgroup of M with $q \neq p$. By the subnormality of A there is a chain $A \lhd A_1 \lhd \ldots \lhd A_\ell \lhd A_{\ell+1} \lhd \ldots \lhd A_m = G$.

Let A_ℓ be such that $Q \leq A_\ell$ but $Q \nleq A_{\ell-1}$. A normalizes every subgroup of Q. Since $A_{\ell-1} \lhd A_\ell$, $Q \lhd A_\ell$ and $A \leq A_{\ell-1}$ it follows that each element of A induces the identity on $Q/Q \cap A_{\ell-1}$ by conjugation. Using Lemma 3 of [1] $A \leq C_G(Q)$ follows. As Q is an arbitrary Sylow subgroup of M, $A \leq C_G(M)$ is true. We have $G = M \cdot N^y$, N^y is a nilpotent Hall subgroup of G and $N^y = P \times T$ where P is a Sylow p-subgroup of G, whence $C_G(M) \geq M \cdot T$. As $MT \lhd G$ it is easy to see that A is centralized by an arbitrary Sylow q-subgroup of G with $q \neq p$.

Theorem 4. *G is a solvable T^*-group if and only if every Sylow subgroup P satisfies one of the following conditions:*

(a) every subgroup of P is normal in G

(b) every Sylow subgroup of $N_G(P)$ different from P centralizes P.

Proof. Assume G is a solvable T^*-group . By the Theorem 7 of [1] $G = MN$ where M is a nilpotent normal Hall subgroup of G, N is a nilpotent Hall subgroup of G, $M \cap N = 1$ and every subgroup of prime power order of M is normal in G. Let R be an arbitrary Sylow subgroup of G.

Assume $R \leq M$. By the above every subgroup of R is normal in G.

Assume $R \leq N^y$ for some $y \in G$. Clearly $N_G(R) = N^y \cdot (N_G(R) \cap M)$. The structure of G yields $B = N_G(R) \cap M \leq C_G(R)$ and $N^y = R \times L$ where L is a nilpotent Hall subgroup of G, consequently $N_G(R) = R \times (L \cdot B)$ so R satisfies (b).

Conversely, let G be a counterexample of smallest order. Let M_0 be the product of every Sylow subgroup of G each subgroup of which is normal in G. Clearly M_0 is a nilpotent normal Hall subgroup of G. By the Theorem of Zassenhaus there is a subgroup N_0 such that $M_0 \cdot N_0 = G$ and $M_0 \cap N_0 = 1$. Clearly N_0 is a Hall subgroup in G and it satisfies the conditions of our theorem. By the minimality of G N_0 is a solvable T^*-group. Using Theorem 2 of [1] $N_0 = A \cdot B$ where A is a nilpotent normal Hall subgroup of G, B is a nilpotent Hall subgroup of G and $A \cap B = 1$ furthermore for arbitrary $a \in A$, $b \in B$ there is a natural number i such that $a^b = a^i$. Assume $A \neq 1$. Let P be a Sylow subgroup of A. Using $N_{N_0}(P) = N_0$, $B \leq C_G(P)$ follows by the conditions whence $B \leq C_G(A)$. Thus $N_0 = A \times B$ and N_0 is a nilpotent normal Hall subgroup of G. Using Theorem 2 of [1] it follows that G is a solvable T^*-group.

References

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