On Symmetric G-Differential and Convex Functionals in Banach Spaces

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The problems concerning the existence of extreme points of nonlinear functionals belong to the very important problems of functional analysis. Let us mention here, for example, the variational methods for solving equations with linear and non-linear operators ([4], [6], [10], [11], [12], [13], [14]), critical points and branching points, eigenvalues and eigenvectors of operators which are important for applications to the solutions of integral and differential equations, critical tension problems, eigenswings, optimal control problems etc. ([6], [10], [12], [14]). It is well-known that, under suitable hypotheses, convex functionals have extreme points. It would be therefore very useful to establish some criteria for functionals to be convex. Havings this purpose in mind we introduce a rather general notion of differentiability of operators in Banach spaces, the so-called „symmetric G- and F-differentiability", to deduce some new criteria for the convexity of functionals.

Let X, Y be real Banach spaces, let $L(X, Y)$ denote the space of all continuous linear operators from X to Y and let Y' denote the conjugate space to Y. The symbol $\langle y, e' \rangle$ denotes the value of the continuous linear functional $e' \in Y'$ at the point $y \in Y$.

1. Symmetris G- and F-differential and their properties

Definition 1. 1. We shall say that an operator $F(x)$ from X to Y has the symmetric G-differential $V_s F(x_0, h)$ at a point $x_0 \in X$ if there exists for any (but fixed) $h \in X$ the following

$$\lim_{t \to 0} \frac{F(x_0 + th) - F(x_0 - th)}{2t} = V_s F(x_0, h),$$

where t is a real number.

The operator $F(x)$ will be called symmetrically differentiable in an open set $M \subset X$ if $F(x)$ has the symmetric G-differential at every point $x \in M$. The G-differential (Gâteaux's differential) will be denoted by $VF(x_0, h)$

$$\left(\text{i.e., } VF(x_0, h) = \lim_{t \to 0} \frac{F(x_0 + th) - F(x_0)}{t} \right).$$

Definition 1.2. We shall say that an operator $F(x)$ from X to Y has the symmetric F-differential $d_s F(x_0, h)$ at a point $x_0 \in X$ if there exists a continuous linear operator $d_s F(x_0, .) \in L(X, Y)$ such that

$$\lim_{\|h\| \to 0} \frac{\|F(x_0+h) - F(x_0-h) - 2d_s F(x_0, h)\|}{\|h\|} = 0.$$

The F-differential (Fréchet's differential) will be denoted by $dF(x_0, h)$ i.e., $dF(x_0, h)$ is an operator $dF(x_0, .) \in L(X, Y)$ such that

$$\lim_{\|h\| \to 0} \frac{\|F(x_0+h) - F(x_0) - dF(x_0, h)\|}{\|h\|} = 0.$$

These definitions imply the following

Theorem 1. 1. *Let the operator $F(x)$ from X to Y have the symmetric F-differential $d_s F(x_0, h)$* (resp. *F-differential) at a point $x_0 \in X$. Then $F(x)$ has also the symmetric G-differential $V_s F(x_0, h) = d_s F(x_0, h)$ (resp. G-differential $VF(x_0, h) = dF(x_0, h)$ at this point.*

Proporisiton 1. 1. *The existence of the symmetric G-differential (resp. G-differential) does not imply the existence of the symmetric F-differential (resp. F-differential) as shown in the following example*

$F(x) = \dfrac{x_1 x_2^2}{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \neq (0, 0)$, $F(0) = 0$. Then $VF(0, h) = \dfrac{h_1 h_2^2}{h_1^2 + h_2^2}$, $h = (h_1, h_2)$, but $dF(0, h)$ does not exist.

Theorem 1. 2. *Let the operator $F(x)$ from X to Y have G-differential at the point $x_0 \in X$ (resp. F-differential). Then $F(x)$ has also the symmetric G-differential $V_s F(x_0, h) = VF(x_0, h)$ (resp. the symmetric F-differential $d_s F(x_0, h)$).*

PROOF. Consider the following identity

$$\frac{F(x_0+th) - F(x_0-th)}{2t} = \frac{1}{2}\left\{\frac{F(x_0+th) - F(x_0)}{t} + \frac{F(x_0-th) - F(x_0)}{-t}\right\}.$$

Then for $t \to 0$ we obtain $\frac{1}{2}\{VF(x_0, h) + VF(x_0, h)\} = V_s F(x_0, h)$. The assertion concerning the F-differential follows analogously.

Proposition 1. 2. *The opposite assertion does not hold in general, because, for example, the functional $f(x) = \|x\|$, $x \in X$ has $V_s f(0, h) = 0$, but $Vf(0, h)$ does not exist.*

Proposition 1. 3. *Let the G-differential of the right of $F(x)$ $V_+ F(x_0, h) = \lim\limits_{t \to 0_+} \dfrac{F(x_0+th) - F(x_0)}{t}$ and the G-differential on the left $V_- F(x_0, h) = \lim\limits_{t \to 0_-} \dfrac{F(x_0+th) - F(x_0)}{t}$ exist. Then there exits $V_s F(x_0, h)$ and $V_s F(x_0, h) = \dfrac{1}{2}\{V_+ F(x_0, h) + V_- F(x_0, h)\}$.*

Proposition 1.4. *$V_sF(x_0, h)$ is homogeneous in the variable $h \in X$ (i.e., $V_sF(x_0, \tau h) = \tau V_sF(x_0, h)$ for any real number τ), but $V_sF(x_0, h)$ is not additive in $h \in X$ in general.* Tn *the case it is additive, we denote it by $D_sF(x_0, h)$.*

Definition 1.3. Let the operator $F(x)$ from X to Y have the symmetric G-differential (G-differential) at the point $x_0 \in X$ and let $V_sF(x_0, .) \in L(X, Y)$ $(VF(x_0, .) \in L(X, Y))$. Then we denote $V_sF(x_0, h)$ by $F'_s(x_0)h$ $(VF(x_0, h)$ by $F'(x_0)h)$ and call the operator $F'_s(x_0)(F(x_0))$ the symmetric G-derivative (G-derivative) of the operator $F(x)$ at the point $x_0 \in X$. The symmetric F-derivative (F-derivative) is defined analogously.

Definition 1.5. An operator $F(x)$ from X to Y is called continuous on every segment of a convex set $M \subset X$ if the function $f(t) = F(x+th)$ is continuous abstract function of the real variable $t \in [0, 1]$ for any fixed $x, x+h \in M$.

The following "mean value theorem" for symmetrically differentiable operators will have the leading role in the next investigations.

Theorem 1.3. *Let the operator $F(x)$ from X to Y be continuous on every segment of a convex set $M \subset X$ and let $F(x)$ be symmetrically differentiable in M. Then for any $x, x+h \in M$ there are real numbers $\tau_1, \tau_2 \in (0, 1)$ such that*

$$\langle V_sF(x+\tau_1 h, h), e'\rangle \leqq \langle F(x+h)-F(x), e'\rangle \leqq \langle V_sF(x+\tau_2 h, h), e'\rangle \quad \text{for any} \quad e' \in Y'.$$

PROOF. Consider the function $f(t) = F(x+th)$ of the real variable $t \in [0, 1]$ for fixed (but arbitrary) $x, x+h \in M$. Then we obtain

$$f'_s(t) = \lim_{u\to 0}\frac{f(t+u)-f(t-u)}{2u} = \lim_{u\to 0}\frac{F(x+th+uh)-F(x+th-uh)}{2u} = V_sF(x+th, h)$$

and the real function $g(t) = \langle f(t), e'\rangle$, $e' \in Y'$, of the real variable $t \in [0, 1]$ has the symmetric derivative $g'_s(t) = \langle f'_s(t), e'\rangle$ for any $t \in (0, 1)$. According to [1] (the mean value theorem) there exist real numbers $\tau_1, \tau_2 \in (0, 1)$ such that the following inequality holds

$$g'_s(\tau_1) \leqq g(1)-g(0) \leqq g'_s(\tau_2).$$

This inequality proves our theorem.

Corollary 1.1. Let the assumptions of Theorem 1.3 hold. Then for any $x, x+h \in M$ there exist real numbers $\tau_1, \tau_2 \in (0, 1)$ such that

$$\|V_sF(x+\tau_1 h, h)\| \leqq \|F(x+h)-F(x)\| \leqq \|V_sF(x+\tau_2 h, h)\|.$$

PROOF. Using the corollary of the well-known Hahn-Banach theorem (see, for example, [9], p. 177) we can choose $e', f' \in Y'$ such that $\|e'\| = \|f'\| = 1$ and

$$\langle V_sF(x+\tau_1 h, h), e'\rangle = \|V_sF(x+\tau_1 h, h)\| \leqq \langle F(x+h)-F(x), e'\rangle \leqq$$
$$\leqq \|F(x+h)-F(x)\|\cdot\|e'\| = \|F(x+h)-F(x)\|;$$
$$\langle F(x+h)-F(x), f'\rangle = \|F(x+h)-F(x)\| \leqq \langle V_sF(x+\tau_2 h, h), f'\rangle \leqq$$
$$\|V_sF(x+\tau_2 h, h)\|\cdot\|f'\| = \|V_sF(x+\tau_2 h, h)\|.$$

Corollary 1.2. Let Y be the space of real numbers and let the assumptions of Theorem 1. 3 hold. Then there exist real numbers $\tau_1, \tau_2 \in (0, 1)$ such that

$$V_s F(x+\tau_1 h, h) \leqq F(x+h) - F(x) \leqq V_s F(x+\tau_2 h, h)$$

for any $x, x+h \in M$.

Corollary 1. 3. If the operator $F(x)$ has the symmetric G-derivative at any point $x \in M$ and if $F(x)$ is continuous on every segment of M then $F(x)$ satisfies the Lipschitz condition on M, i.e., for any $x, x+h \in M$ there exists a real number $\tau \in (0, 1)$ such that

$$\| F(x+h) - F(x)\| \leqq \| F_s'(x+\tau h)\| \cdot \|h\|.$$

Theorem 1. 4. *Let the operator $F(x)$ from X to Y be continuous on every segment of a convex set $M \subset X$, let $F(x)$ be symmetrically differentiable on M and let $V_s F(x, h)$ be a continuous operator in the variable $x \in M$ on every segment of M for any but fixed $h \in X$. Then $F(x)$ has also a G-differential at every point $x \in M$ and $VF(x, h) = V_s F(x, h)$.*

Proof. Let $x, x+th \in M$, $t \neq 0$ be a real number. According to Theorem 1. 3 there exist real numbers $\tau_1, \tau_2 \in (0, 1)$ such that

$$\langle V_s F(x+\tau_1 th, th), e'\rangle \leqq \langle F(x+th) - F(x), e'\rangle \leqq \langle V_s F(x+\tau_2 th, th), e'\rangle$$

for any $e' \in Y'$. Using Proposition 1. 4 we obtain

$$\langle V_s F(x+\tau_1 th, h), e'\rangle \leqq \left\langle \frac{F(x+th)-F(x)}{t}, e'\right\rangle \leqq \langle V_s F(x+\tau_2 th, h), e'\rangle \quad \text{for} \quad t>0$$

$$\left(\langle V_s F(x+\tau_1 th, h), e'\rangle \geqq \left\langle \frac{F(x+th)-F(x)}{t}, e'\right\rangle \geqq \langle V_s F(x+\tau_2 th, h), e'\rangle \quad \text{for} \quad t<0\right)$$

The real function $\langle V_s F(x+\tau th, h), e'\rangle$ is a real continuous function of the variable $\tau \in [0,1]$ and therefore we can choose a number $\tau_3 \in [0,1]$ such that $\langle V_s F(x+\tau_3 th, h), e'\rangle = \left\langle \frac{F(x+th)-F(x)}{t}, e'\right\rangle$ and consequently we have $\lim_{t\to 0} \langle V_s F(x+\tau_3 th, h), e'\rangle = \lim_{t\to 0} \left\langle \frac{F(x+th)-F(x)}{t}, e'\right\rangle = \left\langle \lim_{t\to 0} \frac{F(x+th)-F(x)}{t}, e'\right\rangle$. It follows that $\langle V_s F(x, h), e'\rangle = \langle VF(x, h), e'\rangle$ for any $e' \in Y'$ and thus $V_s F(x, h) = VF(x, h)$.

Theorem 1. 5. *Let $F(x)$ be an operator from X to Y and let $U(x_0)$ be a convex neighbourhood of the point $x_0 \in X$ such that $F(x)$ is continuous on every segment of $U(x_0)$ and symmetrically G-differentiable on $U(x_0)$. Then the following assertion holds: If the symmetric G-differential $V_s F(x, h)$ is continuous at the point x_0 then $F(x)$ has also a G-differential $VF(x_0, h) = V_s F(x_0, h)$ at the point x_0.*

Proof. Let $x+th\ U(x_0)$, $t \neq 0$ be a real number. According to Theorem 1. 3. there exist real numbers $\tau_1, \tau_2 \in (0, 1)$ such that

$$\langle V_s F(x_0+\tau_1 th,\ th) - V_s F(x_0, th), e'\rangle \leqq \langle F(x_0+th) - F(x_0) - V_s F(x_0, th), e' \leqq$$
$$\leqq \langle V_s F(x_0+\tau_2 th, th) - V_s F(x_0, th), e'\rangle$$

for any $e' \in Y'$. Using Proposition 1. 4, and the same corollary of Hahn-Banach theorem as in the proof of Corollary 1. 1, we can write

$$\|V_s F(x_0+\tau_1 th, h)-V_s F(x_0, h)\| \leqq \left\| \frac{F(x_0+th)-F(x_0)}{t} - V_s F(x_0, h)\right\| \leqq$$

$$\leqq \|V_s F(x_0+\tau_2 th, h)-V_s F(x_0, h)\| \text{ for } t>0, \text{ or } \|V_s F(x_0+\tau_1 th, h)-V_s F(x_0, h)\| \geqq$$

$$\geqq \left\| \frac{F(x_0+th)-F(x_0)}{t} - V_s F(x_0, h)\right\| \geqq \|V_s F(x_0+\tau_2 th, h)-V_s F(x_0, h)\|$$

for $t<0$, and the continuity of $V_s F(x, h)$ at the point x_0 implies the assertion of the theorem.

It is obvious that we could assume in Theorem 1. 5 only that $\lim_{t\to 0} V_s F(x_0+th, h = = V_s F(x_0, h)$ for any fixed $h \in X$ in place of the continuity at x_0.

The following theorem gives a sufficient and necessary condition for the existence of the symmetric G-derivative.

Theorem 1. 6. *Let the operator $F(x)$ from X to Y have the symmetric G-differential $V_s F(x_0, h)$ at a point $x_0 \in X$. $F(x)$ has the symmetric G-derivative at the point x_0 (i.e., $V_s F(x_0, h) = F_s'(x_0)h$, $h \in X$, $F_s'(x_0) \in L(X, Y)$) if and only if the following conditions hold*

1. *For any $h \in X$, $\|h\|=1$ there exists a positive real number $\delta(h)>0$, such that for every real number t satisfying $|t|<\delta(h)$ the relation $\|F(x_0+th)-F(x_0-th)\| \leqq \leqq 2C\|th\|$ holds, where $C>0$ is a constant non-depending on h;*

2. $$\Delta^2_{s, th_1, th_2} F(x_0) = F(x_0+th_1+th_2)-F(x_0+th_1)-F(x_0+th_2)+$$
$$+F(x_0-th_1)+F(x_0-th_2)-F(x_0-th_1-th_2) = o(t)$$

for any fixed $h_1, h_2 \in X$.

Proof. Let $F_s'(x_0)$ exist. Then

$$\left\| \frac{1}{2t}\{F(x_0+th)-F(x_0-th)\}\right\| = \|F_s'(x_0)h+\alpha(x_0, th)\|,$$

where

$$\lim_{t\to 0}\left\| \frac{\alpha(x_0, th)}{t}\right\| = 0 \text{ for any } h \in X. \text{ Therefore}$$

$$\lim_{t\to 0}\left\| \frac{1}{2t}\{F(x_0+th)-F(x_0-th)\}\right\| \leqq \|F_s'(x_0)\|\,\|h\| < (\|F_s'(x_0)\|+1)\|h\|,$$

and we can choose $\delta(h)>0$ such that for $|t|<\delta(h)$ the relation

$$\|F(x_0+th)-F(x_0-th)\| < 2|t|(\|F_s'(x_0)\|+1)\|h\| = 2C\|th\|,$$

holds and 1) is satisfied. On the other hand, for every $\varepsilon > 0$ and any fixed $h_1, h_2 \in X$ there exists $\delta_0(\varepsilon) > 0$ such that for every real number t satisfying $|t| < \delta_1(\varepsilon)$ we have

$$F_s'(x_0)h_1 = \frac{1}{2t}\{F(x_0+th_1)-F(x_0-th_1)\}+\alpha_1(x_0, th_1)$$

$$F_s'(x_0)h_2 = \frac{1}{2t}\{F(x_0+th_2)-F(x_0-th_2)\}+\alpha_2(x_0, th_2)$$

$$F_s'(x_0)(h_1+h_2) = \frac{1}{2t}\{F(x_0+th_1+th_2)-F(x_0-th_1-th_2)\}+\alpha_3(x_0, th_1+th_2),$$

where $\|\alpha_i(x_0, h)\| < \frac{\varepsilon}{3}$; $i=1, 2, 3$. Using the additivity of $F_s'(x_0)$, we obtain $\frac{1}{2|t|} \Delta^2_{s,th_1,th_2} F(x_0)\| = \|\alpha_1+\alpha_2-\alpha_3\| < \varepsilon$. This shows that the condition 2) is satisfied.

On the contrary, let 1) and 2) hold. Then

$$V_s F(x_0, h)\| = \lim_{\to 0}\left\| \frac{F(x_0+th)-F(x_0-th)}{2t} \right\| \leqq \lim_{t\to 0} \frac{2C\|th\|}{2|t|} = C\|h\|$$

and thus $V_s F(x_0, h)$ is continuous in h at the point $h=0$. Using Proposition 1. 4, we get the boundedness of the operator $V_s F(x_0, h)$ on each ball $K_r = \{h \in X, \|h\| \leqq r; r > 0$. In fact, for some given constant $M > 0$ there is a $\delta > 0$ such that for $\|h\| < \delta$ we have $\|V_s F(x_0, h)\| < M$. Then for $\|h\| \leqq r$, $r > 0$ we obtain $\|V_s F(x_0, h)\| = \left\| V_s F\left(x_0, \frac{\delta h}{r}\right)\right\| \frac{r}{\delta} \leqq \frac{Mr}{\delta}$. Further more

$$\|V_s F(x_0, h_1+h_2)-V_s F(x_0, h_1)-V_s F(x_0, h_2)\| \leqq \frac{1}{2|t|}\|\Delta^2_{s,th_1,th_2} F(x_0)\|+\varepsilon$$

for every $\varepsilon > 0$ and each real number t sufficiently close to zero. According to 2) we obtain the additivity of the operator $V_s F(x_0, h)$ in h and, consequently, $V_s F(x_0, h)$ is a continuous linear operator in the variable $h \in X$, and our theorem is proved.

Theorem 1. 7. *Let the operator $F(x)$ from X to Y be continuous on every segment of a convex neighbourhood $U(x_0)$ X of the point $x_0 \in X$, symmetrically G-differentiable on $U(x_0)$ and let the abstract function $f(t) = V_s F(x_0+th, h)$ be continuous at the point $t=0$ for any fixed $h \in X$. Then $F(x)$ has G-derivative $F'(x_0)$ at the point x_0 if and only if the conditions 1) and 2) of Theorem 1. 6 are fulfilled.*

PROOF. It runs analogously as the proof of Theorem 1. 5 and 1. 6.

Theorem 1. 8. *Let the operator $F(x)$ from X to Y be continuous in a neighbourhood $U(x_0)$ of the point $x_0 \in X$ and let $F(x)$ have the symmetric G-differential $V_s F(x, h)$ which is continuous at the point x_0 for any fixed $h \in X$ and continuous at the point $h=0$ in the variable h. Then $F(x)$ has a G-derivative $F'(x_0)$ at the point x_0.*

PROOF. According to Theorem 1. 5 there exists $VF(x_0, h)$ and, using [14] (Theorem 3. 1, p. 56), we get the above assertion.

Theorem 1.9. *Let the operator $F(x)$ from X to Y satisfy the following assumptions*

a) *There exists a convex neighbourhood $U(x_0)$ of the point $x_0 \in X$ such that $F(x)$ is continuous on every segment of $U(x_0)$*

b) *there exists the symmetric G-derivative $F_s'(x_0)$ which is continuous at the point x_0.*

Then the operator $F(x)$ has the symmetric F-differential $d_s F(x_0, h) = F_s'(x_0)h$ at the point x_0.

PROOF. Let $x_0 + h \in U(x_0)$ and let us denote $\langle \omega(x_0, h), e' \rangle$ by $\langle F(x_0+h) - F(x_0-h) - 2F_s'(x_0)h, e' \rangle$ for an arbitrary $e' \in Y'$. According to Theorem 1. 3 there exists a real number $\tau \in (0, 1)$, such that

$$\langle \omega(x_0, h), e' \rangle \leqq \langle F_s'(x_0 - h + 2\tau h, 2h), e' \rangle - \langle F_s'(x_0)(2h), e' \rangle.$$

Using the corollary of the Hahn—Banach theorem mentioned above, we can choose $e' \in Y'$, $\|e'\| = 1$ such that

$$\langle \omega(x_0, h), e' \rangle = \|\omega(x_0, h)\| \leqq \|F_s'(x_0 - h + 2\tau h)(2h) - F_s'(x_0)(2h)\| \cdot \|e'\| =$$

$$= 2\|F_s'(x_0 - h + 2\tau h)h - F_s'(x_0)h\| \leqq 2\|F_s'(x_0 - h + 2\tau h) - F_s'(x_0)\| \cdot \|h\|.$$

Therefore (using the continuity of $F_s'(x)$ at x_0) we obtain

$$\lim_{\|h\| \to 0} \frac{\|\omega(x_0, h)\|}{\|h\|} = \lim_{\|h\| \to 0} 2\|F_s'(x_0 - h + 2\tau h) - F_s'(x_0)\| = 0.$$

Thus $d_s F(x_0, h)$ exists and by the application of Theorem 1. 1 we obtain $F_s'(x_0)h = d_s F(x_0, h)$.

2. Convex functionals

In this section we mention some properties of convex functionals and show a certain generalization of the well-known criteria of the convexity [5] using the symmetric differential.

Definition 2. 1. A real functional $f(x)$ defined on a convex set $M \subset X$ will be called convex on M (quasi-convex on M) if for arbitrary $x, y \in M$ and for any real numbers $\lambda \geqq 0$, $\mu \geqq 0$, $\lambda + \mu = 1$ the following inequality holds

$$f(\lambda x + \mu y) \leqq \lambda f(x) + \mu f(y) \tag{1}$$

$$\left(f(\lambda x + \mu y) \leqq \max\,[f(x), f(y)] \right). \tag{1'}$$

We shall say that $f(x)$ is strictly convex (strictly quasi-convex) on M if in (1) (in (1')) "$<$" holds for any $\lambda > 0$, $\mu > 0$ and $x \neq y$. The functional $f(x)$ will be called convex on an open set $D \subset X$ if $f(x)$ is convex on every convex neighbourhood in D.

Definition 2. 2. Let $F(x)$ be an operator from X to X'. $F(x)$ is said to be monotone (strictly monotone) on a set $M \subset X$ if $\langle F(x+h) - F(x), h \rangle \geqq 0$ ($\langle F(x+h) - F(x), h \rangle > 0$ for $h \neq 0$) holds for any $x, x+h \in M$.

Definition 2. 3. A functional $f(x)$ defined on X will be called lower (upper) semi-continuous at a point $x_0 \in X$ if $\underline{\lim}_{n\to 0} f(x_n) \geqq f(x_0)$ $\left(\overline{\lim}_{n\to 0} f(x_n) \leqq f(x_0)\right)$ for any sequence $\{x_n\} \in X$, $x_n \to x_0$. The functional $f(x)$ will be called weakly lower (upper) semi-continuous at x_0 if these inequalities hold for any sequence $\{x_n\} \in X$ converging weakly to x_0.

Lemma 2. 1. *A real functional $f(x)$ on X is convex on a convex set $M \subset X$ if and only if the function $\varphi(t) = f(x+th)$ is a convex function of the real variable $t \in [0, 1]$ for any fixed $x, x+h \in M$.*

PROOF. Let $\varphi(t)$ be a convex function. Then

$\varphi(\alpha t+\beta s) \leqq \alpha\varphi(t)+\beta\varphi(s)$ for any $t, s \in [0, 1]$ and each $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta = 1$.

Hence

$$(*) \quad f(x+\alpha th+\beta sh) = f\big(\alpha(x+th)+\beta(x+sh)\big) \leqq \alpha f(x+th)+\beta f(x+sh),$$

and for $t=0$, $s=1$ we obtain $f\big(\alpha x+\beta(x+h)\big) \leqq \alpha f(x)+\beta f(x+h)$ for arbitrary $x, x+h \in M$. Conversely, if $f(x)$ is a convex functional, then the inequality $(*)$ gives the convexity of the function $\varphi(t)$.

Lemma 2. 2. A real functional $f(x)$ on X, which is upper semi-continuous on every segment of a convex set $M \subset X$, is a convex functional on M if and only if

$$(2) \qquad f\left(\frac{x+y}{2}\right) \leqq \frac{1}{2}f(x)+\frac{1}{2}f(y) \text{ for any } x, y \in M.$$

PROOF. If $f(x)$ is convex then (2) is satisfied evidently. Assume that (2) holds and (1) does not hold. Then the function $\psi(\alpha) = f(\alpha x+(1-\alpha)y)-\alpha f(x)-(1-)\alpha f(y)$ is an upper semi-continuous function of the real variable $\alpha \in [0, 1]$. Denote $M_0 = \max_{\alpha \in [0,1]} \psi(\alpha)$ and let α_0 be the smallest α such that $\psi(\alpha) = M_0$. Then $M_0 > 0$ and $\alpha_0 \in (0, 1)$. We can find a real number $\delta > 0$ such that $(\alpha_0-\delta, \alpha_0+\delta) \subset [0, 1]$. Then for $x^* = (\alpha_0-\delta)x+(1-\alpha_0+\delta)y$, $y^* = (\alpha_0+\delta)x+(1-\alpha_0-\delta)y$ we obtain

$$f\left(\frac{x^*+y^*}{2}\right) = f(\alpha_0 x+(1-\alpha_0)y) \leqq \frac{1}{2}f((\alpha_0-\delta)x+(1-\alpha_0+\delta)y)+$$
$$+\frac{1}{2}f((\alpha_0+\delta)x+(1-\alpha_0-\delta)y).$$

according to (2). Therefore $\psi(\alpha_0) \leqq \dfrac{\psi(\alpha_0-\delta)+\psi(\alpha_0+\delta)}{2} < M_0$ and we arrive to a contradiction which proves our lemma.

Theorem 2. 1. *Let $f(x)$ be a real functional on X, let $f(x)$ be continuous on every segment of a convex set $M \subset X$ and let $f(x)$ have the symmetric G-derivative $f_s'(x)$ at any point $x \in M$. Such an $f(x)$ is a convex functional on M if and only if the operator $f_s'(x)$ is monotone on M.*

PROOF. Let $x, y, z \in M$, $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta = 1$ and let $f_s'(x)$ be motonone on M. According to Theorem 1. 3 (Corollary 1. 2) there exists a real number $\tau \in (0, 1)$ such that

$$f(x)-f(z) \geqq \langle f_s'(z+\tau(x-z)), x-z\rangle = \frac{1}{\tau}\langle f_s'(z+\tau(x-z))-f_s'(z), \tau(x-z)\rangle +$$

$$+\langle f_s'(z), x-z\rangle. \text{ Hence } f(x)-f(z) \geqq \langle f_s'(z), x-z\rangle.$$

Analogously, we obtain $f(y)-f(z) \geqq \langle f_s'(z), y-z\rangle$. From these inequalities it follows

$$\alpha f(x)+\beta f(y) \geqq \alpha f(z)+\beta f(z)+\langle f_s'(z), \alpha(x-z)\rangle+\langle f_s'(z), \beta(y-z)\rangle.$$

Let us put $z = \alpha x+\beta y$. Then

$$\alpha f(x)+\beta f(y) \geqq (\alpha+\beta) f(\alpha x+\beta y)+\langle f_s'(z), \alpha x\rangle-\langle f_s'(z), \alpha x+\beta y\rangle\alpha +$$

$$+\langle f_s'(z), \beta y\rangle-\langle f_s'(z), \alpha x+\beta y\rangle\beta = f(\alpha x+\beta y)+$$

$$+\langle f_s'(z), \alpha x+\beta y\rangle-\langle f_s'(z), \alpha x+\beta y\rangle = f(\alpha x+\beta y).$$

Hence, $f(x)$ is a convex functional on M. On the other hand, if (1) is satisfied, then, according to Lemma 2. 1, the function $\varphi(t) = f(x+th)$ is a convex function of the real variable $t \in [0, 1]$ for any $x, x+h \in M$. Using [7] (Lemma 1. 1, p. 13), we can show that $\varphi(t)$ has at any point $t \in [0, 1]$ left- and right-hand non-decreasing derivative and, following Proposition 1. 3, there exists $\varphi_s'(t) = \langle f_s'(x+th), h\rangle$ and it is non-decreasing in the interval $[0, 1]$. Hence $\varphi_s'(1)-\varphi_s'(0) = \langle f_s'(x+h), h\rangle-\langle f_s'(x), h\rangle = \langle f_s'(x+h)-f_s'(x), h\rangle \geqq 0$, and therefore $f_s'(x)$ is a monotone operator on M.

Theorem 2. 2. *Let $f(x)$ be a convex functional on an open set $\omega \subset X$. Then $f(x)$ is symmetrically G-differentiable in ω and, moreover, the following assertions hold:*

a) *If $V_s f(x, h)$ is continuous at the point $x_0 \in \omega$ in the variable x for any $h \in X$ and if it is continuous at the point $h=0$ in the variable h, then there exists the G-derivative $f'(x_0)$ at the point x_0.*

b) *If $f(x)$ is continuous at x_0 then there exists the symmetric G-derivative $f_s'(x_0)$ at x_0 and $f_s'(x_0)h$ is weakly lower semicontinuous in h on each bounded open convex subset E of X.*

PROOF. It follows from the second part of the proof of Theorem 2. 1 that $V_s f(x, h) = \dfrac{1}{2}\{V_+ f(x, h)+V_- f(x, h)\}$. To prove a) we use Theorem 1. 5 and the convexity of $V_+ f(x_0, h) = Vf(x_0, h)$ as well as that of $-V_- f(x_0, h) = -Vf(x_0, h)$. Then $Vf(x_0, h)$ must be an additive operator in $h \in X$. Indeed, according to Lemma 2. 2 we obtain

$$V_+ f\left(x_0, \frac{h+k}{2}\right) = \frac{1}{2} Vf(x_0, h+k) \leqq \frac{1}{2}\{Vf(x_0, h)+Vf(x_0, k)\}$$

$$V_- f\left(x_0, \frac{h+k}{2}\right) = \frac{1}{2} Vf(x_0, h+k) \geqq \frac{1}{2}\{Vf(x_0, h)+Vf(x_0, k)\}.$$

Hence $Vf(x_0, h+k) = Vf(x_0, h) + Vf(x_0, k)$. Using [9] (Theorem 1, p. 125), we deduce that $Vf(x_0, h)$ is a continuous linear operator in .h Let $f(x)$ be continuous at x_0. Then according to [8] (Theorem 8, p. 749), and using the inequality $V_s f(x_0, h) \leqq V_+ f(x_0, h)$, we can easily deduce the assertion b).

Proposition 2. 1. *The functional* $f(x) = \|x\|$ *has (according to Theorem 2. 2) the symmetric G-derivative at any point* $x \in X$. *If* X *is a Hilbert space, then* $V_s f(x, h) = = Vf(x, h) = f'(x)h = \frac{x}{\|x\|}, h$ for $x \neq 0$; $V_s f(0, h) = 0$, *where* $(.\,,.)$ *denotes the inner product in* X.

Lemma 2. 3. *Let* $\varphi(t)$ *be a real function of the real variable* $t \in [a, b]$. $\varphi(t)$ *is a convex function on* $[a, b]$ *if and only if the function* $\psi_\varepsilon(t) = \varphi(t) + \varepsilon t^2$ *is convex for any real number* $\varepsilon > 0$.

Proof. Let $\varphi(t)$ be convex and let $t+s, t-s \in [a, b]$. Using Lemma 2. 2, we obtain

$$\psi_\varepsilon(t+s) + \psi_\varepsilon(t-s) - 2\psi_\varepsilon(t) = \varphi(t+s) + \varphi(t-s) - 2\varphi(t) + 2\varepsilon s^2 \geqq 0.$$

Since $\varphi(t)$ is convex, it is continuous on $[a, b]$ and thus $\psi_\varepsilon(t)$ is also continuous on $[a, b]$. According to Lemma 2. 2 $\psi_\varepsilon(t)$ is convex. On the contrary, let $\varphi_\varepsilon(t)$ be convex for any $\varepsilon > 0$. Put $\varepsilon = 1/n$, $n = 1, 2, \ldots$. Then $\varphi(t) = \lim_{n\to\infty} \psi_{\frac{1}{n}}(t)$ also is a convex function on $[a, b]$ as a limit of convex functions.

Definition 2. 4. We say that an operator $F(x)$ from X to Y has at $x_0 \in X$ upper (lower) symmetric G-differential of the second order $\overline{V}_s^2 F(x_0, h)$ $(\underline{V}_s^2 F(x_0, h))$ if

$$\overline{V}_s^2 F(x_0, h) = \overline{\lim_{t\to 0}} \frac{F(x_0+th) + F(x_0-th) - 2F(x_0)}{t^2}$$

$$\left(\underline{V}_s^2 F(x_0, h) = \underline{\lim_{t\to 0}} \frac{F(x_0+th) + F(x_0-th) - 2F(x_0)}{t^2}\right)$$

We say that $F(x)$ has at $x_0 \in X$ a symmetric G-differential of the second order $V_s^2 F(x_0, h)$ if $\overline{V}_s^2 F(x_0, h) = \underline{V}_s^2 F(x_0, h) = V_s^2 F(x_0, h)$. If $V_s^2 F(x_0, h)$ is a homogeneous polynomial operator of the second order in $h \in X$, then we put $V_s^2 F(x_0, h) = = F_s''(x_0)(h)^2$ and the operator $F_s''(x_0)$ wil be called the second symmetric G-derivative of the operator $F(x)$ at the point x_0.

Definition 2. 5. We say that an operator $F(x)$ from X to Y has at $x_0 \in X$ the symmetric F-differential of the second order $d_s^2 F(x_0, h)$ if

$$\lim_{\|h\|\to 0} \frac{\|F(x_0+h) + F(x_0-h) - 2F(x_0) - d_s^2 F(x_0, h)\|}{\|h\|^2} = 0,$$

where $d_s^2 F(x_0, h)$ is a homogeneous polynomial operator of the second order in $h \in X$.

Proposition 2. 2. *The existence of the symmetric F-differential of of the second order implies the existence of the symmetric G-different-of the second order. The opposite assertion is not true generally.*

Proposition 2. 3. *The existence of $V_s^2F(x_0, h)(d_s^2F(x_0, h))$ does not imply the existence of $V_sF(x_0, h)(d_sF(x_0, h))$ generally as shown by the following example* $F(t)=t\sin\frac{1}{t^2}$, $t\neq 0$; $F(0)=0$; *where t is a real variable. Then $V_s^2F(0,h)=0$, but $V_sF(0, h)$ does not exist.*

Theorem 2. 3. *A real functional $f(x)$ on X which is upper semi-continuous on every segment of a convex set $M\subset X$, is convex on M if and only if $\overline{V}_s^2f(x, h)\geqq 0$ for any $x\in M$, $h\in X$.*

PROOF. If $f(x)$ is convex, then, according to Lemma 2. 1, the function $f(x+th)$ is a convex function of the real variable $t\in[0, 1]$. From Lemma 2. 2 we obtain

$$f(x+th)+f(x-th)-2f(x)\geqq 0.$$

Hence $\overline{V}_s^2f(x, h)\geqq 0$. Let us assume that $\overline{V}_s^2f(x, h)\geqq 0$. It is sufficient to prove that the assertion holds only if $\overline{V}_s^2f(x, h)>0$. Indeed, if it is really so, then for the function $\psi_\varepsilon(t)=f(x+th)+\varepsilon t^2$ we obtain $\overline{\psi}''_{\varepsilon s}(0)=\overline{V}_s^2f(x, h)+2\varepsilon>0$ for any $\varepsilon>0$. According to Lemma 2. 2 the function $f(x+th)$ must be a convex function. Let $\overline{V}_s^2f(x, h)>0$. If the functional $f(x)$ is not convex on M, then there exist $x_0, x_0+h_0\in M$ such that the function $\varphi_0(t)=f(x_0+th_0)$ is not convex on the interval $[0, 1]$ and, therefore, the function

$$g(t)=\varphi_0(t)+(1-t)(\varphi_0(1)-\varphi_0(0))-\varphi_0(1)=\varphi_0(t)-t\varphi_0(1)-(1-t)\varphi_0(0)$$

takes its positive maximum on the interval $[0, 1]$. Let $\max\limits_{t\in[0,1]} g(t)=g(t_0)$. Then $t_0\in(0, 1)$ because $g(0)=g(1)=0$. It is obvious that $\bar{g}''_s(t_0)\leqq 0$, but $\bar{g}''_s(t_0)=\overline{\varphi}''_{0s}(t_0)$. Hence, for $x_1=x_0+t_0h_0$, we obtain

$$\overline{\lim_{t\to 0}}\frac{f(x_1+th_0)+f(x_1-th_0)-2f(x_1)}{t^2}\leqq 0.$$

Following the Definition 2. 4 $\overline{V}_s^2f(x_1, h_0)\leqq 0$, and this contradiction proves the assertion of our theorem.

Corollary 2. 1. Let $f(x)$ be a continuous real functional having non-negative upper symmetric F-differential of the second order at any point x of the open set $\omega\subset X$. Then $f(x)$ is a convex functional on ω.

Definition 2. 6. An operator $F(x)$ from X to Y will be called "F-smooth" at he point $x_0\in X$ if

$$\lim_{\|h\|\to 0}\frac{\|F(x_0+h)+F(x_0-h)-2F(x_0)\|}{\|h\|}=0.$$

The operator $F(x)$ will be called F-smoth in an open set $D\subset X$ if it is continuous on D and F-smooth at any point $x\in D$.

Definition 2. 7. An operator $F(x)$ from X to Y will be called "G-smooth" at the point $x_0\in X$ if

$$\lim_{t\to 0}\frac{F(x_0+th)+F(x_0-th)-2F(x_0)}{t}=0 \text{ for every } h\in X.$$

The operator $F(x)$ will be called G-smooth on an open set $D \subset X$ if $F(x)$ is continuous on every segment in each convex neighbourhood of any point $x \in D$ and G-smooth at any point $x \in D$.

Theorem 2. 4. *If the operator $F(x)$ from X to Y has the second symmetric G-(F-)derivative at the point $x_0 \in X$, then $F(x)$ is G-(F-) smooth at x_0.*

PROOF. Following the Definition 2. 5, we can write

$$F(x_0+h)+F(x_0-h)-2F(x_0) = F_s''(x_0)(h)^2+\omega(x_0, h),$$

where

$$\lim_{\|h\|\to 0} \frac{\|\omega(x_0, h)\|}{\|h\|^2} = 0. \text{ Hence}$$

$$\lim_{\|h\|\to 0} \frac{\|F(x_0+h)+F(x_0-h)-2F(x_0)\|}{\|h\|} \leqq \lim_{\|h\|\to 0} \frac{\|F_s''(x_0)(h)^2\|+\|\omega(x_0, h)\|}{\|h\|} \leqq$$

$$\leqq \lim_{\|h\|\to 0}\left(\|F_s''(x_0)\|\cdot\|h\|+\frac{\|\omega(x_0, h)\|\cdot\|h\|}{\|h\|^2}\right) = 0.$$

The assertion on the "G-smooth" property can be proved analogously.

Theorem 2. 5. *If the operator $F(x)$ from X to Y has G-(F-)differential at any point of an open set $D \subset X$, then $F(x)$ is G-(F-)smooth on D.*

PROOF. Let $F(x)$ be F-differentiable in D. Then $F(x)$ is continuous on D and, following the Definition 1. 2, we can write

$$\lim_{\|h\|\to 0} \frac{\|F(x+h)+F(x+h)-2F(x)\|}{\|h\|} \leqq \lim_{\|h\|\to 0} \frac{\|F(x+h)-F(x)-dF(x, h)\|}{h} +$$

$$+\lim_{\|h\|\to 0} \frac{\|F(x-h)-F(x)-dF(x, -h)\|}{\|h\|} = 0 \text{ for } x \in D.$$

Thus $F(x)$ is F-smooth on D. The proof of the "G-smooth" property is analogous.

It is obvious that the G-(F-)smooth operator $F(x)$ need not be G-(F-)differentiable but the following proposition was recently proved for the author by J. KOLOMÝ.

Propositon 2. 5. *Let $f(x)$ be a convex functional on a reflexive Banach space X and let $f(x)$ be G-differentiable and F-smooth at a point $x_0 \in X$. Then $f(x)$ is F-differentiable at x_0.*

Theorem 2. 6. *Let $f(x)$ be a real functional on X and let $f(x)$ be G-smooth in an open set $D \subset X$. Let, furthermore, $f(x)$ have a local extreme at a point $x_0 \in D$. Then the following assertions hold:*

a) *The functional $f(x)$ is G-differentiable at x_0 and $Vf(x_0, h)=0$ for any $h \in X$.*

b) *If the functional $f(x)$ is F-smooth at x_0, then $f(x)$ is F-differentiable at x_0 and $f'(x_0)=0$.*

PROOF. The functions $\varphi(t) = f(x_0+th)-f(x_0)$, $\psi(t) = f(x_0-th)-f(x_0)$ have the same sign (i.e., both functions are positive or negative) for all t sufficiently close to zero. But

$$\lim_{t\to 0}\frac{\varphi(t)+\psi(t)}{t} = 0, \quad \text{and thus} \quad \lim_{t\to 0}\frac{f(x_0+th)+f(x_0)}{t} = 0;$$

hence $Vf(x_0, h) = 0$. If $f(x)$ is F-smooth, then the proof is analogous.

Theorem 2. 7. *Let $f(x)$ be a real functional on X and let $f(x)$ be G-smooth on an open set $D\subset X$. Then the following assertion holds: If $\overline{V}_s^2 f(x, h)\geqq 0$ for each $x\in D$, except, maybe, a countable subset $E\subset D$ and for any $h\in X$, then the functional $f(x)$ is convex on D.*

PROOF. Following the proof of Theorem 2. 3, we can assume that $\overline{V}_s^2 f(x, h)>0$ for each $x\in D-E$ and $h\in X$, $h\neq 0$. If $f(x)$ is not convex on D, then there is a convex neighbourhood $U(x_0)$ of a point $x_0\in D$ such that $f(x)$ is not convex on $U(x_0)$. Then there exists a $h_0\in X$ such that the function $\varphi_0(t) = f(x_0+th_0)$ is not convex on the interval [0, 1]. Hence, the function $\psi_0(t) = \varphi_0(t)-t\varphi_0(1)-(1-t)\varphi_0(0)$ has its positive maximum on the interval [0, 1]. Since $\psi_0(0)=\psi_0(1)=0$, we find that $\max_{t\in[0,1]}\psi_0(t)=\psi_0(t_0)$ where $t_0\in(0, 1)$. The same goes for the functions $\psi_\alpha(t) = \varphi_0(t)-\alpha t-\varphi_0(0)$ if α is a real number sufficiently close to $\varphi_0(1)-\varphi_0(0)$. Let $\psi_\alpha(t_\alpha)=\max_{t\in[0,1]}\psi_\alpha(t)$. Then $\overline{\psi}''_{\alpha s}(t_\alpha)=\overline{\varphi}''_{0s}(t_\alpha)\leqq 0$ and thus $x_\alpha = x_0+t_\alpha h_0\in E$, since $\overline{\varphi}''_{0s}(t_\alpha)=\overline{V}_s^2 f(x_\alpha, h_0)$. Using Theorem 2. 6, we obtain $\psi'_\alpha(t_\alpha) = \varphi'_\alpha(t_\alpha)-\alpha = 0$ for any real number α sufficiently close to $\varphi_0(1)-\varphi_0(0)$. Then the number of t_α is not countable, and thus E is no countable set. We have come to a contradiction which proves our theorem.

3. Extreme points of real functionals and solutions of operator equations

In this section we show some applications of results mentioned above to the problem of extreme points of real functionals on reflexive Banach spaces.

Lemma 3. 1. *Let $f(x)$ be a real quasi-convex functional on X. Let $f(x)$ has a strong local minimum at a point $x_0\in X$. Then $f(x)$ has its absolute minimum at x_0.*

PROOF. Let $\varepsilon>0$ be a real number such that for $x\in X$ satisfying $\|x-x_0\| \leqq \varepsilon$ the relation $f(x)>f(x_0)$, $x\neq x_0$ holds. Let us assume that there exists an $x_1\in X$ such that $\|x_1-x_0\| > \varepsilon$ and $f(x_1)<f(x_0)$. Then for $\beta = \dfrac{\varepsilon}{\|x_1-x_0\|}$ and $\alpha = 1-\beta$ we obtain $f(\alpha x_0+\beta x_1) \leqq \max[f(x_0), f(x_1)] = f(x_0)$, but $\|\alpha x_0+\beta x_1-x_0\| = \beta\|x_1-x_0\| = \varepsilon$, and we arrive to a contradiction.

Theorem 3. 1. *Let X be a real reflexive Banach space, let $M\subset X$ be a closed convex bounded set and let $f(x)$ be a continuous real functional on M. Moreover let, at least one of the following conditions hold.*

a) *$f(x)$ has a symmetric G-derivative $f'_s(x)$, which is monotone on M and continuous on every segment in M.*

b) *$f(x)$ is G-smooth in M and $\overline{V}_s^2 f(x, h) \geqq 0$ for any $h \in X$ and for each $x \in M$ except, maybe, a countable subset $E \subset M$.*

Then there exists a point $x_0 \in M$ such that

1) *$f(x_0) = \inf_{x \in M} f(x)$ and there are no strong local extremes of $f(x)$ in M different from $f(x_0)$.*

2) *$f(x)$ has the G-derivative $f'(x_0)$ at x_0 and $f'(x_0) = grad\ f(x_0) = 0$, provided $x_0 \in$ Int M.*

PROOF. According to Theorem 2. 1 (resp. Theorem 2. 7) $f(x)$ is a convex functional on M. Being continuous, $f(x)$ is also weakly lower semi-continuous in M. Being closed and convex, M is weakly closed, and, according to [14] (Theorem 9. 2), there is $x_0 \in M$ such that $f(x_0) = \inf_{x \in M} f(x)$ (the greatest lower bound of $f(x)$ in M). Using Lemma 3. 1, we get 1). To prove 2), we consider at first that (according to Theorem 1. 3) there exist real numbers $\tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2 \in (0, 1)$ such that

$$\langle f_s'(x_0 + \tau_1 th), th \rangle + \langle f_s'(x_0 + \bar{\tau}_1 th), -th \rangle \leqq f(x_0 + th) + f(x_0 - th) - 2f(x_0) \leqq$$
$$\leqq \langle f_s'(x_0 + \tau_2 th), th \rangle + \langle f_s'(x_0 + \bar{\tau}_2 th), -th \rangle$$

and using the continuity of $f_s'(x)$ on every segment in M, we obtain

$$\lim_{t \to 0} \frac{f(x_0 + th) + f(x_0 - th) - 2f(x_0)}{t} = 0.$$

Hence, $f(x)$ is G-smooth at x_0. According to Theorem 2. 6 we get $Vf(x_0, h) = 0$ and, using Theorem 2. 2, we obtain $Vf(x_0, h) = f'(x_0)h = 0$ for any $h \in X$, and thus the assertion 2) holds if the condition a) or b) are satisfied.

Theorem 3. 1′. *Let X be a reflexive Banach space, and let $f(x)$ be a real functional on X which is F-smooth on a closed convex bounded set $M \subset X$ and let at least one of the following conditions hold.*

a) *$f(x)$ has a monotone symmetric F-derivative $f_s'(x)$ in M.*

b) *$V_s f(x, h) \geqq 0$ for any $h \in X$, and for each $x \in M$ except, maybe, a countable subset $E \subset M$.*

Then there is a point $x_0 \in M$ such that

1) *$f(x_0) = \inf_{x \in M} f(x)$ and there are no strong local extremes of $f(x)$ in M different from $f(x_0)$*

2) *$f(x)$ is F-differentiable at x_0 and $f'(x_0) = \text{grad}\, f(x_0) = 0$, provided $x_0 \in$ Int M.*

The PROOF of this theorem is analogous to the proof of Theorem 3. 1, but to prove the assertion 2) we can use Proposition 2. 5.

Definition 3. 1. An operator $F(x)$ from X to X' will be called a weakly (strongly) S-potential operator if there exists a G-(F-)smooth functional $f(x)$ on X such that $V_s f(x, h) = \langle F(x), h \rangle$ $(d_s f(x, h) = \langle F(x), h \rangle)$. The functional $f(x)$ will be called the weak (strong) S-potential of the operator $F(x)$.

Theorem 3. 2. *Let X be a reflexive real Banach space, $M \subset X$ a bounded closed convex set and let $F(x)$ be a weakly S-potential operator from X onto X: with its S-potential $f(x)$ satisfying $f(x) - f(\bar{x}) > \langle x - \bar{x}, y \rangle$ for some $\bar{x} \in$ Int M, $y \in X'$ and any*

$x\in\partial M$ (the boundary of M). Furthermore, let at least one of the following conditions hold:
a) $F(x)$ is monotone and continuous on every segment in M.
b) $\nabla_s^2 f(x, h)\geqq 0$ for any $x\in M$, $h\in X$ of $F(x)$. Then the equation

$$F(x)=y'$$

has at least one solution $x_0\in M$. If $F(x)$ is a strictly monotone operator or, if $\nabla_s^2 f(x, h)>0$ for each $x\in M$, $h\in X$, $h\neq 0$, then the solution x_0 is unique.

PROOF. Consider the functional $g(x) = f(x)-\langle x, y'\rangle$. Then $g(x)$ is G-smooth on M and $g_s'(x) = F(x)-y'$ is a monotone operator on M provided, that the condition a) is satisfied. Since $\nabla_s^2 f(x, h)=\nabla_s^2 g(x, h)$, then using Theorem 3. 1, we can find an $x_0\in M$ such that $g'(x_0) = F(x_0)-y' = 0$. The assertion on uniqueness is obvious.

Theorem 3. 3. *Let X be a real reflexive Banach space, $M\subset X$ a bounded closed convex set. Let $F(x)$ be a weakly S-potential operator from X onto X' whose weak S-potential $f(x)$ satisfies $\nabla_s^2 f(x, h) \geqq C\|A\|\cdot\|h\|^2$ for any $h\in X$ and for each $x\in M$ except a countable subset $E\subset M$, where A is a bounded linear symmetric positive operator from X onto X', and $C>0$ is a positive constant. Then the equation*

$$F(x)=\lambda Ax$$

has the solution $x_\lambda\in M$ for any $\lambda\in[0, C]$, provided $0\in \operatorname{Int} M$ and $f(x)-f(0)>$ $>\frac{C}{2}\langle Ax, x\rangle$ for $x\in\partial M$.

PROOF. Consider the functional $g(x) = f(x)-\lambda\frac{1}{2}\langle Ax, x\rangle$. Then $\nabla_s^2 g(x, h) =$ $= \nabla_s^2 f(x, h)-\lambda\langle Ah, h\rangle \geqq \nabla_s^2 f(x, h)-C\|A\|\cdot\|h\|^2 \geqq 0$. Since $f(x)$ is G-smooth, then $g(x)$ is also G-smooth and, according to Theorem 3. 1, there exists a $x_\lambda\in M$ such that $g_s'(x_\lambda) = g'(x_\lambda) = F(x_\lambda)-\lambda Ax_\lambda = 0$.

Corollary 3. 1. Let H be a Hilbert space and let $F(x)$ be a strongly S-potential operator from a ball $D_r=\{x\in H, \|x\|\leqq r, r>0\}$ to H whose S-potential $f(x)$ satisfies $\nabla_s^2 f(x, h)\geqq C\|h\|^2$ for any $x\in D_r$, $h\in H$, where $C>0$ is a positive constant. Then the equation

$$F(x)=\lambda x$$

has the solution $x_\lambda\in D_r$ for each $\lambda\in[0, C]$, provided $f(x)-f(0)>\frac{C}{2}V^2$ for $x\in H$, $\|x\| = r$.

Finally, we show some examples of convex functionals:

1) Let A be a positive symmetric linear operator from a Hilbert space H to H. Then the functional $f(x)=\frac{1}{2}(Ax, x)$ is the convex potential of the operator A. Indeed, $\nabla_s^2 f(x, h)=(Ah, h)\geqq 0$.

2) Let $P(x)$ be a homogeneous symmetric polynomial operator of the degree n from H to H (i. e., let $(P^*(x_1, \ldots, x_n), x_{n+1})$ be symmetric in $x_1, \ldots, x_{n+1}$, where $P^*(x_1, \ldots, x_n)$ is the polar n-linear form: $P^*(x, \ldots x)=P(x)$). Then $P(x)$ is the potential operator and $f(x)=\frac{1}{n+1}(P(x), x)$ is its potential. The functional $f(x)$

is convex if $P(x)$ is a positive operator of odd degree $n = 2k-1$. As example for such an operator can serve the integral operator

$$P(x) \equiv y(t) = \int_0^1\int_0^1 \dots \int_0^1 K(t, s_1, \dots, s_{2k-1}) x(s_1) \dots x(s_{2k-1}) ds_1 \dots ds_{2k-1},$$

where $K(t, s_1, \dots, s_{2k-1})$ is a positive symmetric and on the $2k$-dimensional unit cube quadratically integrable function, and $x(t) \in L^2[0, 1]$.

3) Let $B \subset E_n$ be a measurable set of the n-dimensional Euclidean space, $g(u, x)$ be a real function continuous in the variable $u \in (+\infty, -\infty)$ and measurable in $x \in B$ for each u, satisfying $g'_u(u, x) \geqq 0$,

$$|g(u, x)| \leqq a(x) + b|u|^{p-1}, \text{ where } a(x) \in L^q(B), \ b > 0, \ \frac{1}{p} + \frac{1}{q} = 1, \ p \geqq 2.$$

Then the Nemyckij's operator

$$h(u) \equiv g(u(x), x)$$

has the convex continuous (and thus weakly lower semi-continuous) potential

$$f(u) = \int_B dx \int_0^{u(x)} g(v, x) dv.$$

4) The functionals of the theory of plasticity:

Let $D \subset E_n$ be an open bounded set of the Euclidean n-dimensional space E_n, and B a subspace of the Sobolev space $W_p^1(D)$. Let $\tau_j(u)$, $j = 1, 2, \dots, m$ be strictly convex functionals on B and $g_j(\xi)$, $j = 1, 2, \dots, m$ non-negative real functions defined almost everywhere and locally integrable on the interval $[0, +\infty)$. Let k, $1 \leqq k \leqq m$ be an index such that $g_k(\xi) \geqq a > 0$, where a is a positive constant. If, further, $g'_j(\xi) \geqq 0$, $j = 1, 2, \dots, m$ then the functional

$$f(u) = \int_D dx \sum_{j=1}^m \int_0^{\tau_j(u)} g_j(\xi) d\xi$$

is a strictly convex functional on B. This functionals occur, for example, in the problems of the elastically plastic deformations of plates (see [11]).

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