On J_n-groups

By F. FENYVES (Debrecen)

1. Introduction. We start with the following definition. Let G be a group, and let n be a positive integer. An endomorphism α of G is said to be a J_n-endomorphism if α satisfies the J_n-identity

$$((g_1g_2 \cdots g_n)^\alpha g_{n+1})^\alpha((g_2g_3 \cdots g_ng_{n+1})^\alpha g_1)^\alpha \cdots ((g_ig_{i+1} \cdots g_ng_{n+1}g_1 \cdots g_{i-2})^\alpha g_{i-1})^\alpha \cdots$$
$$\cdots ((g_ng_{n+1}g_1 \cdots g_{n-2})^\alpha g_{n-1})^\alpha((g_{n+1}g_1 \cdots g_{n-1})^\alpha g_n)^\alpha = e$$

for all $g_1, g_2, \ldots, g_{n+1}$ in G, where g_i^α denotes the image of g_i under α and e is the identity of G. (A J_n-automorphism is defined in the obvious way.) It is easy to see that the trivial endomorphism ϑ defined by $g^\vartheta = e$ for all g in G is a J_n-endomorphism. Also, it is easy to check that the mapping $\alpha: g \to g^{-1}$ is a J_1-endomorphism of G if (and only if) G is an abelian group.

A group G which admits a nontrivial J_n-endomorphism is said to be a J_n-group. For example, every abelian group is a J_1-group.

The J_2-groups were first considered by B. M. PUTTASWAMAIAH in [2]. In [1] J. MORGADO proved that the group G is a J_2-group if and only if G is a semidirect product of a proper normal subgroup and an abelian group with the unique square root property.

The purpose of this article is to extend the above theorem to J_n-groups. We also extend the other results of [1] and [2] to the general case.

2. Preliminary lemmas. Recall that a group G has the unique nth root property, if for every g in G the equation $x^n = g$ has a unique solution in G.

Lemma 1. *The following properties of a torsion group are equivalent:*

(i) *G has the unique nth root property;*

(ii) (ord $g, n) = 1$ *for every g in G.*

PROOF. (i) implies (ii). Suppose, to obtain a contradiction, that there exists an element g in G such that $(\text{ord}\, g, n) > 1$. Then we can choose a prime p for which $\text{ord}\, g = pk$ and $n = pl$ where k, l are positive integers. Hence we must have $g^k \neq e$ as an element of order p. Since $p|n$, it results in $e = (g^k)^p = (g^k)^n = e^n$, which yields, by (i), $g^k = e$. This contradiction completes the proof of (i) implies (ii).

(ii) implies (i). Consider g in G, since $(\text{ord}\, g, n) = 1$ there exist integers k and l such that $kn = l \cdot \text{ord}\, g + 1$. If there exists an element x in G such that $x^n = g$, then $x^{n \cdot \text{ord}\, g} = (x^n)^{\text{ord}\, g} = e$ which implies, by $(\text{ord}\, x, n) = 1$, $\text{ord}\, x | \text{ord}\, g$ i.e. $x^{\text{ord}\, g} = e$.

Therefore $x = x^{l \cdot \mathrm{ord}\, g}\, x = x^{nk} = g^k$, that is we must have $x = g^k$. Conversely, $x = g^k$ is a solution of the equation $x^n = g$. In fact, one obtains $(g^k)^n = g^{l \cdot \mathrm{ord}\, g + 1} = g$.

Notation. If α and β are mappings of G into G, then $g^{\alpha+\beta}$ denotes $g^\alpha g^\beta$ and $g^{\alpha\beta}$ denotes $(g^\alpha)^\beta$.

Lemma 2. *Let α be an endomorphism of the group G. Then the following conditions are necessary and sufficient in order that α be a J_n-endomorphism of G:*

(1) $$g_1^{n\alpha^2+\alpha} = e \qquad \textit{for all} \quad g_1 \ \textit{in} \ G;$$

(2) $$g_1^{\alpha^2} g_2^{\alpha^2} = g_2^{\alpha^2} g_1^{\alpha^2} \quad \textit{for all} \quad g_1, g_2 \ \textit{in} \ G.$$

PROOF. Let us begin by supposing that α is a J_n-endomorphism of G. The J_n-identity can be rewritten in the following equivalent form

(∗) $$(g_1^{\alpha^2} g_2^{\alpha^2} \dots g_n^{\alpha^2} g_{n+1}^{\alpha})(g_2^{\alpha^2} g_3^{\alpha^2} \dots g_{n+1}^{\alpha^2} g_1^{\alpha}) \dots$$
$$\dots (g_i^{\alpha^2} g_{i+1}^{\alpha^2} \dots g_n^{\alpha^2} g_{n+1}^{\alpha^2} g_1^{\alpha^2} \dots g_{i-2}^{\alpha^2} g_{i-1}^{\alpha}) \dots (g_{n+1}^{\alpha^2} g_1^{\alpha^2} \dots g_{n-1}^{\alpha^2} g_n^{\alpha}) = e$$

for all $g_1, g_2, \dots, g_{n+1}$ in G.

Then taking $g_2 = g_3 = \dots = g_{n+1} = e$ in (∗) we see that $g_1^{\alpha^2} g_1^{\alpha} g_1^{(n-1)\alpha^2} = e$ whence follows

(1) $$g_1^{n\alpha^2+\alpha} = e \text{ for all } g_1 \text{ in } G.$$

This of course implies

(3) $$g_1^{\alpha^2} g_1^{\alpha} = g_1^{\alpha} g_1^{\alpha^2} \text{ for all } g_1 \text{ in } G.$$

Also, it follows from (∗), by $g_3 = g_4 = \dots = g_{n+1} = e$,

$$g_1^{\alpha^2} g_2^{\alpha^2} g_2^{\alpha^2} g_1^{\alpha} g_1^{\alpha^2} g_2^{\alpha} (g_1^{\alpha^2} g_2^{\alpha^2})^{n-2} = e.$$

But this can be written as

$$(g_1 g_2)^{\alpha^2} (g_2 g_1)^{\alpha^2} (g_1 g_2)^{\alpha} (g_1 g_2)^{(n-2)\alpha^2} = e$$

in view of (3). Thus

$$(g_1 g_2)^{\alpha^2} (g_2 g_1)^{\alpha^2} = (g_1 g_2)^{-(n-2)\alpha^2 - \alpha}.$$

Using (1), we express this relation as

$$(g_1 g_2)^{\alpha^2} (g_2 g_1)^{\alpha^2} = (g_1 g_2)^{2\alpha^2}$$

and this implies that $(g_2 g_1)^{\alpha^2} = (g_1 g_2)^{\alpha^2}$. Whence one gets

(2) $$g_1^{\alpha^2} g_2^{\alpha^2} = g_2^{\alpha^2} g_1^{\alpha^2} \text{ for all } g_1, g_2 \ G.$$

Conversely, suppose that (1) and (2) hold. First of all, we show that

(4) $$g_1^{\alpha} g_2^{\alpha^2} = g_2^{\alpha^2} g_1^{\alpha} \text{ for all } g_1, g_2 \text{ in } G.$$

In fact, it follows from (1) that $g_1^{n\alpha^2+\alpha} g_2^{n\alpha^2+\alpha} = e$. Using (3) and (2) we express this relation as

$$g_1^{\alpha} g_2^{\alpha^2} g_1^{n\alpha^2} g_2^{(n-1)\alpha^2+\alpha} = e,$$

which implies, since $g_1^{n\alpha^2}=g_1^{-\alpha}$ and $g_2^{(n-1)\alpha^2+\alpha}=g_2^{-\alpha^2}$,

$$g_1^{\alpha} g_2^{\alpha^2} g_1^{-\alpha} g_2^{-\alpha^2} = e$$

that is (4) holds.

Then by repeated application of (4), (2) and (1) we have

$$(g_1^{\alpha^2} g_2^{\alpha^2} \dots g_n^{\alpha^2} g_{n+1}^{\alpha})(g_2^{\alpha^2} g_3^{\alpha^2} \dots g_{n+1}^{\alpha^2} g_1^{\alpha}) \dots (g_{n+1}^{\alpha^2} g_1^{\alpha^2} \dots g_{n-1}^{\alpha^2} g_n^{\alpha}) =$$

$$= (g_1^{\alpha^2} g_1^{\alpha} g_1^{(n-1)\alpha^2})(g_2^{2\alpha^2} g_2^{\alpha} g_2^{(n-2)\alpha^2}) \dots (g_k^{k\alpha^2} g_k^{\alpha} g_k^{(n-k)\alpha^2}) \dots$$

$$(g_n^{n\alpha^2} g_n^{\alpha})(g_{n+1}^{\alpha} g_{n+1}^{n\alpha^2}) = \prod_{i=1}^{n+1} g_i^{n\alpha^2+\alpha} = e \text{ for all } g_1, g_2, \dots, g_{n+1} \text{ in } G.$$

Consequently we have proved that α is a J_n-endomorphism of G.

3. Results.

Theorem 1. *If α is a J_n-endomorphism of a group G then*

(i) *G^α is an abelian group,*

(ii) *the restrection of α to G^α is an automorphism of G^α,*

(iii) *G^α has the unique nth root property.*

PROOF. (i) In fact, if $a, b \in G^\alpha$ then there exist g, h in G such that $g^\alpha = a$ and $h^\alpha = b$. Thus using (1) and (2) we get

$$ab = g^\alpha h^\alpha = g^\alpha (h^{-n})^{\alpha^2} = (h^{-n})^{\alpha^2} g^\alpha = h^\alpha g^\alpha = ba.$$

(ii) First, it is clear that $G^{\alpha^2} \subseteqq G^\alpha$, because $g^{\alpha^2} = (g^\alpha)^\alpha$ for all g in G. Secondly, if a is an arbitrary element of G^α then exists $g \in G$ such that $g^\alpha = a$. Then it follows from (1) that $a = g^\alpha = (g^{-n})^{\alpha^2}$, and this implies $G^\alpha \subseteqq G^{\alpha^2}$. Hence the restrection of α to G^α is an epimorphism. Next, let $a(=g^\alpha)$, $b(=h^\alpha) \in G$. Then it follows from $a^\alpha = b^\alpha$ that $g^{\alpha^2} = h^{\alpha^2}$ i.e. $(g^{-1}h)^{\alpha^2} = e$. This combined with (1) implies $e = \left((g^{-1}h)^{\alpha^2}\right)^{-n} = (g^{-1}h)^{-n\alpha^2} = (g^{-1}h)^\alpha = (g^{-1})^\alpha h^\alpha$, and so $g^\alpha = h^\alpha$ that is $a = b$. In other words, the restriction of α to G^α is also a monomorphism, and thus it is an automorphism.

(iii) Consider $a = g^\alpha$ in G^α. Then it follows from (1) that $a = g^\alpha = g^{-n\alpha^2} = \left((g^{-\alpha})^\alpha\right)^n$. This means that $x = g^{-\alpha^2}$ is a solution of equation $x^n = a$ in G^α. This solution is unique. Indeed, if x_1 and x_2 are solutions of the equation $x^n = a$ in G^α then $x_1^{-1} x_2$ is a solution of the equation $x^n = e$ in G^α. But the equation $x^n = e$ has only the solution $x = e$ in G^α. In fact, if $(y^\alpha)^n = e$ for some $y \in G$, then using (1) we obtain $e = \left((y^\alpha)^n\right)^\alpha = y^{n\alpha^2} = y^{-\alpha}$. Hence $y^\alpha = e$ and so $x_1 = x_2$. With this the validity of (iii) is proved.

Theorem 2. *The following three assertions concerning a group G are equivalent.*

(A) *G has exactly one J_n-automorphism.*

(B) *G has a J_n-automorphism.*

(C) *G is an abelian group and has the unique nth root property.*

PROOF. That (A) implies (B) is trivial; that (B) implies (C) has been proved already by Theorem 1. The proof will be complete when we show that (C) implies (A).

Next we suppose that (C) holds. Let $\alpha: G \to G$ be the mapping defined by $g^\alpha = (g^{-1})^{\frac{1}{n}}$ for all g in G, where $g^{\frac{1}{n}}$ denote the unique nth root of g. We shall prove that α is a J_n-automorphism of G. First, α is onto; because if $g \in G$, then $(g^{-n})^\alpha = (g^n)^{\frac{1}{n}} = g$. Secondly, α is one to one; for if $g_1^\alpha = g_2^\alpha$, then $g_1^{-1} = ((g_1^{-1})^{\frac{1}{n}})^n = ((g_2^{-1})^{\frac{1}{n}})^n = g_2^{-1}$, and so $g_1 = g_2$. Next, α is an automorphism of G. Let $g_1, g_2, \in G$. Suppose $(g_1^{-1})^{\frac{1}{n}} = h_1, (g_2^{-1})^{\frac{1}{n}} = h_2$ i.e. $g_1^{-1} = h_1^n, g_2^{-1} = h_2^n$. Then $(g_1 g_2)^{-1} = g_1^{-1} g_2^{-1} = h_1^n h_2^n = (h_1 h_2)^n$, hence

$$(g_1 g_2)^\alpha = ((g_1 g_2)^{-1})^{\frac{1}{n}} = h_1 h_2 = (g_1^{-1})^{\frac{1}{n}} (g_2^{-1})^{\frac{1}{n}} = g_1^\alpha g_2^\alpha.$$

Lastly, it results $g^{n\alpha^2+\alpha} = ((g^\alpha)^\alpha)^n g^\alpha = (((g^\alpha)^{-1})^{\frac{1}{n}})^n g^\alpha = (g^\alpha)^{-1} g^\alpha = e$ for all g in G, and so, by Lemma 2, α is a J_n-automorphism of G. On the other hand, if β were a second J_n-automorphism of G, then we prove that $\alpha = \beta$. Indeed, one has clearly

$$e = (((g^{-1})^\beta g^\beta)^\beta)^n = g^{-n\beta^2} (g^\beta)^{n\beta} = g^\beta (g^\beta)^{n\beta} = (g(g^\beta)^n)^\beta.$$

So we find that $g(g^\beta)^n = e$, hence $g^\beta = (g^{-1})^{\frac{1}{n}} = g^\alpha$ for all g in G, which means that $\alpha = \beta$. Consequently (C) implies (A).

Using Lemma 1, we now establish the following.

Corollary 1. *If G is a torsion group, then there exists a J_n-automorphism of G if and only if G is abelian and* $(\operatorname{ord} g, n) = 1$ *for all g in G.*

In particular, if G is an abelian group of exponent k then there exists a J_n-automorphism of G if and only if $(k, n) = 1$.

Corollary 2. *The following are equivalent conditions on a group G.*

(A) *G is a J_n-group under an inner automorphism of G.*

(B) *G is a J_n-group under the identity mapping of G.*

(C) *G is an abelian group of exponent $n+1$.*

We now can formulate a characterization of J_n-groups.

Theorem 3. *For any group G the following two statements are equivalent.*

(A) *G is a J_n-group.*

(B) *G is the semi-direct product of a proper normal subgroup B by an abelian subgroup A having the unique nth root property.*

Proof. Assuming (A), let α be a non-trivial J_n-endomorphism of G. Then it holds that the quotient group $G/\operatorname{Ker}\alpha$ is isomorphic to G^α. Moreover for all g in G we have $(g^{n\alpha} g)^\alpha = g^{n\alpha^2+\alpha} = e$. Hence $g^{n\alpha} g \in \operatorname{Ker}\alpha$, that is $g \operatorname{Ker}\alpha = g^{-n\alpha} \operatorname{Ker}\alpha$ and here $g^{-n\alpha} \in G^\alpha$. On the other hand, if h^α were a second element of G^α for which $g \operatorname{Ker}\alpha = h^\alpha \operatorname{Ker}\alpha$, then $h^{-\alpha} g \in \operatorname{Ker}\alpha$. Therefore $h^{-\alpha^2} g^\alpha = e$, hence $e = h^{-n\alpha^2} g^{n\alpha} = h^\alpha g^{n\alpha}$, which implies $h^\alpha = g^{-n\alpha}$. Consequently $(A) \Rightarrow (B)$.

Conversely, assume (B). Define a mapping α of G into G by $\alpha = \beta\gamma\delta$, where

(i) β is the natural homomorphism of G onto G/B;

(ii) γ is the isomorphism of G/B onto A defined by $(gB)^\gamma = g^*$, where g^* is the unique element of A for which $gB = g^*B$; and

(iii) δ is the unique J_n-automorphism of A.

Then the next facts are obvious: α is a nontrivial endomorphism of G; $G^\alpha = A$; and $\operatorname{Ker}\alpha = B$.

Moreover, one has

$$g^{n\alpha} = (g^\alpha)^n = (g^{\beta\gamma\delta})^n = \big((gB)^{\gamma\delta}\big)^n = (g^{*\delta})^n = (g^*)^{-1}.$$

This implies, by (ii), that $gB = g^{-n\alpha}B$. Hence it follows that $g^{n\alpha}g \in B$. Therefore we have

(1) $$e = (g^{n\alpha}g)^\alpha = g^{n\alpha^2+\alpha} \qquad \text{for all } g \text{ in } G.$$

Lastly, recalling that $\operatorname{Ker}\alpha$ is an abelian group, we also have

(2) $$g^{\alpha^2}h^{\alpha^2} = h^{\alpha^2}g^{\alpha^2} \text{ for all } g, h \text{ in } G.$$

Thus we have verified (see Lemma 2) that α is a nontrivial J_n-endomorphism of G, proving that (B) implies (A).

References

[1] J. Morgado, Note on Jacobi endomorphisms, *Gaz. Mat. (Lisboa)*. N. **105—108** (1967), 48—52.
[2] B. M. Puttaswamaiah, Jacobi endomorphisms, *Amer. Math. Monthly*. **73** (1966), 741—744.

(Received July 2, 1968; in revised form, June 4, 1969.)