Cohen's theorem for a class of Noetherian semirings

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1. Introduction. I. S. COHEN [1] has proven that a commutative ring with an identity is Noetherian if and only if every prime ideal is finitely generated. As well known, prime ideals of rings are very important for the theory of radicals (cf. DIVINSKY [2]) and for classical ideal theory. In this paper, it will be shown that the above Cohen's theorem is valid for a large class of semirings.

2. Fundamentals. There are many different definitions of a semiring appearing in the literature. Throughout this paper, a semiring will be defined as follows:

Definition 1. A set R together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) will be called a *semiring* provided:

(i) addition is a commutative operation,

(ii) $\exists\, 0 \in R$ such that $x+0 = x$ and $x0=0x=0$ for each $x \in R$, and

(iii) multiplication distributes over addition both from the left and from the right.

A semiring is said to be *commutative* if multiplication is a commutative operation. The semiring R is said to have an *identity* if $\exists\, 1 \in R$ such that $x1=1x=x$ for each $x \in R$.

Definition 2. A subset I of a semiring R will be called an *ideal* if $a, b \in I$ and $r \in R$ implies $a+b \in I$, $ra \in I$ and $ar \in I$.

Definition 3. An ideal I in a semiring R will be called a *k-ideal* if the following condition is satisfied: if $a \in I$, $b \in R$ and $a+b \in I$, then $b \in I$.

It is clear that every ideal in a ring is a k-ideal.

Definition 4. An ideal I in a commutative semiring R will be called *prime* if $a, b \in R$ and $ab \in I$ implies $a \in I$ or $b \in I$.

Definition 5. Let I be an ideal in the semiring R. A subset B of I will be called a *basis* for I if every element in I can be written in the form $\sum_{i=1}^{n} r_i b_i$ where $r_i \in R$ and $b_i \in B$.

3. Noetherian semirings. A commutative ring with an identity is called Noetherian if it satisfies the ascending chain condition for ideals. A Noetherian semiring is defined in an analogous manner as follows:

Definition 6. A commutative semiring R with an identity will be called a *Noetherian* semiring if $I_1 \subset I_2 \subset \cdots \subset I_n \subset I_{n+1} \subset \cdots$ is an ascending chain of ideals in R implies there exists a positive integer N such that $I_n = I_N$ for $n > N$.

It is an easy matter to obtain the following characterization of Noetherian semirings:

Proposition 7. If R is a commutative semiring with an identity, then R is Noetherian if and only if every ideal in R has a finite basis.

Let R be a commutative semiring with an identity and let I be an ideal in R that does not have a finite basis. Moreover, suppose $\{B_q\}_{q\in Q}$ is the collection of all ideals in R such that $I \subset B_q$ and B_q is maximal with respect to not having a finite basis. One can use Zorn's lemma to show the collection $\{B_q\}_{q\in Q}$ is nonempty. In general, it is not true that $\exists\, q_0 \in Q$ such that B_{q_0} is a k-ideal. Consequently, the following classification seems natural:

Definition 8. A commutative semiring R with an identity will be called an *M-semiring* provided the following condition is valid: if I is an ideal in R that does not have a finite basis, then there exists a k-ideal B in R such that $I \subset B$ and B is maximal with respect to not having a finite basis.

Let R be a commutative ring with an identity. If I is an ideal in R that does not have a finite basis, the above argument implies there exists an ideal B in R such that $I \subset B$ and B is maximal with respect to not having a finite basis. Since B is a k-ideal, it follows that every commutative ring with an identity is an M-semiring.

Examples of M-semirings other than commutative rings with an identity will now be given.

Example 9. $Z^+ = \{0, 1, 2, \ldots\}$ together with the usual operations of addition and multiplication of integers is a commutative semiring with an identity. Since every ideal in Z^+ has a finite basis, it is clear that Z^+ is an M-semiring.

Example 10. Let $R = \{x \mid x \text{ is real and } 0 \leqq x \leqq 1\} \cup \{2, 3\}$. R is a fully ordered set under the usual ordering of the real numbers. Define $a+b = \max(a, b)$ and $ab = \min(a, b)$. It can be shown that R is a commutative semiring with an identity. If I is an ideal in R that does not have a finite basis, then $I = \{x \in R \mid 0 \leqq x < r\}$ for some r such that $0 < r \leqq 1$. Thus, I is contained in the ideal $B = \{x \in R \mid 0 \leqq x < 1\}$ and B is maximal with respect to not having a finite basis. Moreover, B is a k-ideal in R, and it follows that R is an M-semiring. It is easy to see that R can not be imbedded in a ring since R does not satisfy the additive cancellation law.

For the class of M-semirings, one can obtain the following generalization of Cohen's theorem:

Theorem 11. *An M-semiring R is Noetherian if and only if every prime ideal in R has a finite basis.*

PROOF. If R is Noetherian, Proposition 7 implies every ideal in R has a finite basis; consequently, every prime ideal in R has a finite basis.

Conversely, suppose that every prime ideal in R has a finite basis. Let F denote the collection of all ideals in R that do not have a finite basis. If it can be shown that $F = \emptyset$, then every ideal in R has a finite basis and it will follow from Proposition 7 that R is Noetherian. Assume $F \neq \emptyset$. Let $I \in F$. Since R is an M-semiring, there exists a k-ideal B in R such that $I \subset B$ and B is maximal with respect to not having a finite basis. Since B is not prime, $\exists a, b \in R$ such that $a \notin B$, $b \notin B$ and $ab \in B$. It is clear that the ideals $Ra + B = \{ra + x \mid r \in R \text{ and } x \in B\}$ and $B:aR = \{x \in R \mid xaR \subset B\}$ both properly contain B ($a \in Ra + B$ and $b \in B:aR$). Since B is maximal with respect to not having a finite basis, it follows that $Ra + B$ and $B:aR$ have finite basis, say $\{a_i'\}_{i=1}^n$ and $\{b_j\}_{j=1}^m$, respectively.

It is clear that $a_i' = 1a_i' \in \sum_{i=1}^n Ra_i' = Ra + B$. Thus, $\exists r_i \in R$ and $a_i \in B$ such that $a_i' = r_i a + a_i$. If $x \in Ra + B$, then

$$x = \sum_{i=1}^n r_i' a_i' = \sum_{i=1}^n r_i'(r_i a + a_i) = \sum_{i=1}^n r_i' r_i a + \sum_{i=1}^n r_i' a_i.$$

Therefore, $x \in Ra + B$ implies $x \in Ra + \sum_{i=1}^n Ra_i$. If $x \in Ra + \sum_{i=1}^n Ra_i$, then $x = ra + \sum_{i=1}^n r_i a_i \in Ra + B$ since $a_i \in B$ and B is an ideal. Consequently, $\exists \{a_i\}_{i=1}^n$ such that each $a_i \in B$ and $Ra + B = Ra + \sum_{i=1}^n Ra_i$. Let B^* be the ideal generated by a_i, ab_j where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$; that is,

$$B^* = \sum_{i=1}^n Ra_i + \sum_{j=1}^m Rab_j.$$

Since $a_i \in B$ and B is an ideal, it is clear that $\sum_{i=1}^n Ra_i \subset B$. Since $b_j \in B:aR$, it follows that $\sum_{j=1}^m Rab_j \subset B$. Hence, $B^* \subset B$. Let $x \in B$. Thus, $x = 0a + x \in Ra + B = = Ra + \sum_{i=1}^n Ra_i \subset Ra + B^*$; since $\sum_{i=1}^n Ra_i \subset B^*$. Therefore, $\exists r \in R$ and $b^* \in B^*$ such that $x = ra + b^*$. Since $x, b^* \in B$ and B is a k-ideal, it follows that $ra \in B$. Thus, $r \in B:aR$ and it follows that $r = \sum_{j=1}^m r_j b_j$. Therefore, $ra = \sum_{j=1}^m r_j ab_j \in B^*$. Hence $x \in B^*$ and it follows that $B \subset B^*$. It has now been shown that $B = B^*$ has a finite basis, a contradiction.

Problem 1. (F. Szász) Investigate semirings for an analogue of Theorem 11, which are not M-semirings.

Problem 2. (F. Szász) Investigate the ideals of a semiring that are not k-ideals.

References

[1] I. S. Cohen, Commutative rings with restricted minimum condition, *Duke Math. J.* **17** (1950), 27—42.
[2] N. Divinsky, Rings and Radicals, *London* 1965.

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