On the integrability of functions defined by trigonometric series

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1. HEYWOOD ([1]) has proved that if λ_n is ultimately positive, if

$$\frac{1}{2}\lambda_0+\sum_{n=1}^{\infty}\lambda_n=0,$$

and if

$$f(x)=\frac{1}{2}\lambda_0+\sum_{n=1}^{\infty}\lambda_n\cos nx,$$

then $\dfrac{f(x)}{x}\in L(0,1)$, if and only if $\sum_1^{\infty}\lambda_n\log n$ is convergent.

The object of this note is to prove more general results of this type, in which we do not assume that λ_n is ultimately positive.

2. **Theorem 1.** *If $f(x)=\sum_{n=0}^{\infty}\lambda_n e^{nix}$, where the series is uniformly convergent in the closed interval $(0,\pi)$, if $\sum_{n=0}^{\infty}\lambda_n=0$, and if $\sum_{n=1}^{\infty}\lambda_n\cdot\log n$ is convergent, then $\lim_{\delta\to+0}\int_{\delta}^{\pi}\frac{f(x)}{x}dx$ exists and is finite. Conversely, if $\sum_{n=0}^{\infty}\lambda_n e^{nix}$ is uniformly convergent in the closed interval $(0,\pi)$, if $S_n\log n\to\alpha$, as $n\to\infty$, and if $I_\delta=\int_{\delta}^{\pi}\frac{f(x)}{x}dx\to I$, as $\delta\to0$, then $\sum_{n=1}^{\infty}\lambda_n\log n$ is convergent, where α and I are finite and $S_n=\sum_{r=0}^{n}\lambda_r$.*

PROOF. If $z=x+iy$, integrating $\dfrac{1-e^{iz}}{z}$ round the closed contour formed by $|z|=a$, $|z|=na$, $0\leqq\arg z\leqq\dfrac{\pi}{2}$, and the segments of the real and imaginary axes, we have

$$(1)\qquad \int_a^{na}\frac{1-e^{ix}}{x}dx=\int_a^{na}\frac{1-e^{-x}}{x}dx-i\int_0^{\pi/2}(1-e^{niae^{i\theta}})d\theta+i\int_0^{\pi/2}(1-e^{iae^{i\theta}})d\theta.$$

Now, if $0<\delta<\pi$, we have

$$\int_\delta^\pi \frac{f(x)}{x}\,dx = -\sum_{n=0}^{\infty}\lambda_n\left[\int_0^\pi - \int_0^\delta\right]\frac{1-e^{nix}}{x}\,dx = -\sum_{n=0}^{\infty}\lambda_n\left[\int_0^{n\pi}-\int_0^{n\delta}\right]\frac{1-e^{ix}}{x}\,dx =$$

$$= -\sum_{n=0}^{\infty}\lambda_n\int_\delta^\pi \frac{1-e^{ix}}{x}\,dx - \sum_{n=0}^{\infty}\lambda_n\left[\int_\pi^{n\pi}-\int_\delta^{n\delta}\right]\frac{1-e^{ix}}{x}\,dx = -\sum_{n=0}^{\infty}\lambda_n\left[\int_\pi^{n\pi}-\int_\delta^{n\delta}\right]\frac{1-e^{ix}}{x}\,dx.$$

Putting $a=\pi$ and $a=\delta$ in (1) and subtracting, we have

$$\left[\int_\pi^{n\pi}-\int_\delta^{n\delta}\right]\frac{1-e^{ix}}{x}\,dx = \left[\int_\pi^{n\pi}-\int_\delta^{n\delta}\right]\frac{1-e^{-x}}{x}\,dx -$$

$$-i\int_0^{\pi/2}\left(e^{ni\delta e^{i\theta}}-e^{ni\pi e^{i\theta}}\right)d\theta + i\int_0^{\pi/2}\left(e^{i\delta e^{i\theta}}-e^{i\pi e^{i\theta}}\right)d\theta.$$

Since the series $\sum_{n=0}^{\infty}\lambda_n e^{niz}$ converges uniformly to $f(z)$, when $0\leqq x\leqq\pi$, $I\geqq 0$, we have

$$(2)\quad \int_\delta^\pi \frac{f(x)}{x}dx = -\sum_{n=0}^{\infty}\lambda_n\left[\int_\pi^{n\pi}-\int_\delta^{n\delta}\right]\frac{1-e^{ix}}{x}\,dx = -\sum_{n=0}^{\infty}\lambda_n\left[\int_\pi^{n\pi}-\int_\delta^{n\delta}\right]\frac{1-e^{-x}}{x}\,dx -$$

$$-i\int_0^{\pi/2}\left(f(\delta e^{i\theta})-f(\pi e^{i\theta})\right)d\theta + i\int_0^{\pi/2}\left(e^{i\delta e^{i\theta}}-e^{i\pi e^{i\theta}}\right)d\theta\cdot\sum_{n=0}^{\infty}\lambda_n.$$

Since e^{-x} decreases steadily, when $x\geqq 0$, for every fixed a, $0\leqq a\leqq\pi$, $\frac{1}{\log n}\int_a^{na}\frac{e^{-x}}{x}\,dx$ is a non-increasing function of n, $n>1$; and since

$$\frac{1}{\log n}\int_a^{na}\frac{1-e^{-x}}{x}\,dx = \frac{1}{\log n}\int_a^{na}\frac{dx}{x} - \frac{1}{\log n}\int_a^{na}\frac{e^{-x}}{x}\,dx = \begin{cases} 0, \\ \text{or} \\ 1-\dfrac{1}{\log n}\displaystyle\int_a^{na}\frac{e^{-x}}{x}\,dx, \end{cases}$$

according as $a=0$ or $\neq 0$, it follows that, for every fixed a, $0\leqq a\leqq\pi$, $\frac{1}{\log n}\int_a^{na}\frac{1-e^{-x}}{x}\,dx$

is a non-decreasing function of n, $n>1$. Also, we have $0 \leqq \frac{x}{\log n} \int\limits_a^{na} \frac{1-e^{-x}}{x}\, dx \leqq 1$ and so $\frac{1}{\log n} \int\limits_a^{na} \frac{1-e^{-x}}{x}\, dx$ is uniformly bounded with respect to a and n, $0 \leqq a \leqq \pi$, $n>1$. Consequently, if $\sum\limits_{n=1}^{\infty} \lambda_n \cdot \log n$ is convergent, by Abel's theorem, it follows that the series $\sum\limits_{n=1}^{\infty} \lambda_n \int\limits_a^{na} \frac{1-e^{-x}}{x}\, dx$ is uniformly convergent, when $0 \leqq a \leqq \pi$; and so, by (2), it follows that $\lim\limits_{\delta \to +0} \int\limits_{\delta}^{\pi} \frac{f(x)}{x}\, dx$ exists and is finite. Moreover, if $\sum\limits_{n=1}^{n} \lambda_n \log n$ has bunded partial sums, by Dirichlet's theorem, it follows that $\int\limits_{\delta}^{\pi} \frac{f(x)}{x}\, dx$ is bounded, as $\delta \to +0$. We have, thus, proved more than what was contained in the statement of the theorem.

To prove the second part of the theorem, we observe that, if one of the two integrals

$$\int\limits_{\delta}^{\pi} \frac{f(x)}{x}\, dx \quad \text{and} \quad \int\limits_{\delta}^{\pi} \frac{f(x)\, d(e^{ix})}{e^{ix}-1}$$

is convergent or bounded, as $\delta \to +0$, so is the other, because the difference of the two integrals is absolutely integrable in $(0, \pi)$. Now integrating $\frac{e^{iz}(1-e^{niz})}{1-e^{iz}}$ round the closed contour formed by $|z|=\delta$, $|z|=\pi$, $0 \leqq \arg z \leqq \frac{\pi}{2}$, and the segments of the real and imaginary axes, we have

$$\int\limits_{\delta}^{\pi} \frac{1-e^{nix}}{1-e^{ix}}\, d(e^{ix}) = \int\limits_{\delta}^{\pi} \frac{1-e^{-nx}}{1-e^{-x}}\, d(e^{-x}) - i \int\limits_{0}^{\pi/2} \frac{(1-e^{ni\pi e^{i\theta}})}{1-e^{i\pi e^{i\theta}}}\, d(e^{\pi i e^{i\theta}}) +$$

$$+ i \int\limits_{2}^{\pi/2} \frac{(1-e^{ni\delta e^{i\theta}})}{1-e^{i\delta e^{i\theta}}}\, d(e^{i\delta e^{i\theta}}) = \int\limits_{e^{-\delta}}^{e^{-\pi}} \frac{1-\xi^n}{1-\xi}\, d\xi - i \int\limits_{0}^{\pi/2} \frac{(1-e^{ni\pi e^{i\theta}})}{1-e^{i\pi e^{i\theta}}}\, d(e^{i\pi e^{i\theta}}) +$$

$$+ i \int\limits_{1}^{\pi/2} \frac{(1-e^{ni\delta e^{i\theta}})}{1-e^{i\delta e^{i\theta}}}\, d(e^{i\delta e^{i\theta}}).$$

Therefore, we have

$$-\int_{\delta}^{\pi} \frac{f(x)\,d(e^{ix})}{1-e^{ix}} = \sum_{n=0}^{\infty} \lambda_n \int_{e^{-\delta}}^{e^{-\pi}} \frac{1-\xi^n}{1-\xi} d\xi - \sum_{n=0}^{\infty} \lambda_n i \int_0^{\pi/2} \frac{(1-e^{ni\pi e^{i\theta}})}{1-e^{i\pi e^{i\theta}}} d(e^{i\pi e^{i\theta}}) +$$

$$+ \sum_{n=0}^{\infty} \lambda_n i \int_0^{\pi/2} \frac{(1-e^{ni\delta e^{i\theta}})}{1-e^{i\delta e^{i\theta}}} d(e^{i\delta e^{i\theta}}) = \sum_{n=0}^{\infty} \lambda_n \int_0^{e^{-\pi}} \frac{1-\xi^n}{1-\xi} d\xi - \sum_{n=0}^{\infty} \lambda_n \int_0^{e^{-\delta}} \frac{1-\xi^n}{1-\xi} d\xi +$$

$$+ i \int_0^{\pi/2} \frac{f(\pi e^{i\theta})}{1-e^{i\pi e^{i\theta}}} d(e^{i\pi e^{i\theta}}) - i \int_0^{\pi/2} \frac{f(\delta e^{i\theta})}{1-e^{i\delta e^{i\theta}}} d(e^{i\delta e^{i\theta}}) =$$

(3)

$$= - \sum_{n=0}^{\infty} \int_0^{e^{-\pi}} \frac{\psi(\xi)}{1-\xi} d\xi - \sum_{n=0}^{\infty} \lambda_n \int_0^{e^{-\delta}} \frac{1-\xi^n}{1-\xi} d\xi +$$

$$+ i \int_0^{\pi/2} \frac{f(\pi e^{i\theta})}{1-e^{i\pi e^{i\theta}}} d(e^{i\pi e^{i\theta}}) - i \int_0^{\pi/2} \frac{f(\delta e^{i\theta})}{1-e^{i\delta e^{i\theta}}} d(e^{i\delta e^{i\theta}}),$$

where $\psi(\xi) = \sum_{n=0}^{\infty} \lambda_n \cdot \xi^n$, $0 \leqq \xi \leqq 1$. So, it follows that, if

(4)
$$I_\delta = \sum_{n=1}^{\infty} \lambda_n \cdot \int_0^{e^{-\delta}} \frac{1-\xi^n}{1-\xi} d\xi = \sum_{n=1}^{\infty} \lambda_n \left(e^{-\delta} + \frac{e^{-2\delta}}{2} + \ldots + \frac{e^{-n\delta}}{n} \right),$$

I_δ tends to a definite finite limit or oscillates finitely, as $\delta \to +0$, according as $\int_{\delta}^{\pi} \frac{f(x)}{1-e^{ix}} d(e^{ix})$ is convergent or bounded.

Also, if $e^{-\delta} = 1 - \frac{1}{N^2}$, where N is any positive integer, we have

$$\sum_{r=1}^{N+1} \frac{e^{-r\delta}}{r} = \sum_{r=1}^{N+1} \frac{1}{r} \left(1 - \frac{1}{N^2} \right)^r = \sum_{r=1}^{N+1} \frac{1}{r} \left(1 - \frac{r}{N^2} + O\left(\frac{1}{N^2} \right) \right) =$$

$$= \sum_{r=1}^{N+1} \frac{1}{r} + O\left(\frac{1}{N} \right) = \log N + \mathbf{r} + u_{N+1} + O\left(\frac{1}{N} \right),$$

where $\mathbf{r}$ is Euler's constant and u_n tends steadily to zero, as $n \to \infty$.
It follows easily that, if N is fixed,

$$\lim_{M \to \infty} \sup \frac{\sum_{r=1}^{M} \frac{e^{-r\delta}}{r}}{\log M} = 0.$$

Also we have

$$\sum_{n=N+2}^{\infty} \frac{e^{-n\delta}}{n\log n} = I_1+I_2+I_3,$$

$$I_1 = \sum_{n=N+2}^{\bar{N}} \frac{e^{-n\delta}}{n\log n}, \quad I_2 = \sum_{\bar{N}+1}^{N^2} \frac{e^{-n\delta}}{n\log n}, \quad I_3 = \sum_{n=N^2+1}^{\infty} \frac{e^{-n\delta}}{n\log n}, \quad \bar{N} = \left[\frac{N^2}{\log\lg N}\right],$$

where $[x]$ denotes the greatest integer not exceeding x. It follows easily that $I_1 \to \log 2$, $I_2 \to 0$ and $I_3 \to 0$, as $N \to \infty$.

Now, we have

$$I_\delta = \sum_{n=1}^{N} \lambda_n \left(\sum_{r=1}^{n} \frac{e^{-r\delta}}{r}\right) + \sum_{n=N+1}^{\infty} \lambda_n \left(\sum_{r=1}^{n} \frac{e^{-r\delta}}{r}\right) = I_{1,\delta}+I_{2,\delta},$$

$$I_{1,\delta} = \sum_{n=1}^{N} \lambda_n \left(\sum_{r=1}^{n} \frac{1}{r}\left(1-\frac{1}{N^2}\right)^r\right) = \sum_{n=1}^{N} \lambda_n \left(\sum_{r=1}^{n} \frac{1}{r}\left(1-\frac{r}{N^2}+O\left(\frac{1}{N^2}\right)\right)\right) =$$

$$= \sum_{n=1}^{N} \lambda_n \left(\sum_{r=1}^{n} \frac{1}{r}\right) - \frac{1}{N^2}\sum_{n=1}^{N} n\lambda_n + O\left(\frac{1}{N}\right) = \sum_{n=1}^{N} \lambda_n(\log n + \mathbf{r} + u_n) + O(1) =$$

$$= \sum_{n=1}^{N} \lambda_n \log n + \sum_{n=1}^{N} \lambda_n u_n - \mathbf{r}\lambda_0 + O(1) = \sum_{n=1}^{N} \lambda_n \log n + H + O(1),$$

where $H = -\mathbf{r}\lambda_0 + \sum_{n=1}^{\infty} \lambda_n u_n$, $\mathbf{r}$ and u_n having the same meaning as before. Also, since by hypothesis, $S_n \log n \to \alpha$, as $n \to \infty$, we have

$$I_{2,\delta} = \sum_{n=N+1}^{\infty} \lambda_n \left(\sum_{r=1}^{n} \frac{e^{-r\delta}}{r}\right) = \lim_{M\to\infty} \sum_{N+1}^{M} \lambda_n \left(\sum_{r=1}^{n} \frac{e^{-r\delta}}{r}\right) =$$

$$= \lim_{M\to\infty} \sum_{N+1}^{M} (S_n - S_{n-1}) \left(\sum_{r=1}^{n} \frac{e^{-r\delta}}{r}\right) = \lim_{M\to\infty} \left[S_M \sum_{r=1}^{M} \frac{e^{-r\delta}}{r}\right] -$$

$$- \lim_{M\to\infty} \sum_{N+2}^{M} S_{n-1} \frac{e^{-n\delta}}{n} - S_N \sum_{r=1}^{N+1} \frac{e^{-r\delta}}{r} = \lim_{M\to\infty} \left[(S_M \log M) \frac{\sum_{r=1}^{M} \frac{e^{-r\delta}}{r}}{\log M}\right] -$$

$$- \lim_{M\to\infty} \sum_{N+2}^{M} (S_{n-1}\log n) \frac{e^{-n\delta}}{n\log n} - (S_N \log N) \frac{\sum_{r=1}^{N+1} \frac{e^{-r\delta}}{r}}{\log N} =$$

$$= -\sum_{N+2}^{\infty} (S_{n-1}\log n - \alpha) \frac{e^{-n\delta}}{n\log n} + \alpha \sum_{N+2}^{\infty} \frac{e^{-n\delta}}{n\log n} - (S_N \log N)\big(1+O(1)\big).$$

Since we have proved thaat

$$\sum_{N+2}^{\infty} \frac{e^{-n\delta}}{n\log n} \to \log 2, \quad \text{as} \quad N \to \infty,$$

by (3) and (4), it follows that $\sum_{n=1}^{\infty} \lambda_n \log n$ converges or oscillates finitely, according as $\int_{\delta}^{\pi} \frac{f(x)}{x}\, dx$ converges or occilates finitely, as $\delta \to +0$; and this completes the proof of the theorem.

Remarks. (i) Since, by hypothesis, $S_n \to 0$, as $n \to \infty$, it follows easily that

$$\lim_{N\to\infty} \left(\sum_{n=N^2+1}^{\infty} \frac{S_n e^{-n\delta}}{n} \right) = 0.$$

Also, we have

$$\sum_{n=N+1}^{N^2} \frac{1-e^{-n\delta}}{n} < \sum_{n=1}^{N^2} \frac{1-e^{-n\delta}}{n} = \sum_{n=1}^{N^2} \frac{1}{n}\left[1-\left(1-\frac{1}{N^2}\right)^n\right] =$$

$$= \sum_{n=1}^{N^2} \frac{1}{n}\left[1-\left(1-\frac{n}{N^2}+\frac{n(n-1)}{2!}\cdot\frac{1}{N^4}+\ldots\right)\right] < e-1.$$

Consequently, we have

$$\lim_{N\to\infty} \sum_{n=N+1}^{N^2} S_{n-1}\left(\frac{1-e^{-n\delta}}{n}\right) = 0.$$

Now, we have

$$I_{2,\delta} = -\sum_{n=N+1}^{\infty} S_n \frac{e^{-n\delta}}{n} - S_N \sum_{r=1}^{N} \frac{e^{-r\delta}}{r} = -\sum_{n=N+1}^{N^2} \frac{S_{n-1}}{n} - S_N \sum_{r=1}^{N} \frac{1}{r} + O(1).$$

Also, we have

$$I_{1,\delta} = \sum_{n=1}^{N} \lambda_n \left(\sum_{r=1}^{n} \frac{1}{r} \right) + O(1) = \sum_{n=1}^{N} (S_n - S_{n-1}) \sum_{r=1}^{n} \frac{1}{r} + O(1) =$$

$$= -\sum_{n=1}^{N} \frac{S_{n-1}}{n} + S_N \sum_{r=1}^{N} \frac{1}{r} + O(1).$$

Therefore, we have

$$I_\delta = I_{1,\delta} + I_{2,\delta} = -\sum_{n=1}^{N^2} \frac{S_{n-1}}{n} + O(1).$$

Consequently, by (3) and (4) it follows that

$$\int_{\delta}^{\pi} \frac{f(x)}{x}\, dx = \sum_{n=1}^{N^2} \frac{S_{n-1}}{n} + H + O(1),$$

where H is a fixed number.

(ii) An analogue of Theorem 1 for power series can be stated as follows:

$$\textit{If } \sigma_n = \sum_{r=0}^{n} \frac{a_r}{r+1} \to 0, \quad \textit{as} \quad n \to \infty,$$

any two of

(I) $$f(x)=\sum_{n=0}^{\infty} a_n x^n \to l, \quad as \quad x \to 1-0,$$

(II) $n\sigma_n \to k$, *as* $n \to \infty$,

(III) $\sum_{n=0}^{\infty} a_n$ *is convergent, where* l *and* k *are finite, imply the third.*

PROOF. Let $x = 1-\dfrac{c}{N}$, where N is a positive integer and c is a fixed positive number $(c<N)$. We have

$$\sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} a_n \sum_{r=0}^{n} (-1)^r \binom{n}{r}\left(\frac{c}{N}\right)^r = \sum_{n=0}^{N} a_n + M,$$

where

$$M = \sum_{n=0}^{N} a_n \sum_{r=1}^{n} (-1)^r \binom{n}{r}\left(\frac{c}{N}\right)^r.$$

Also, for every fixed r, $r \geqq 1$, we have

$$v_r = \frac{1}{N^r}\sum_{n=0}^{N}\binom{n}{r} a_n = \frac{1}{N^r}\sum_{n=1}^{N}\binom{n}{r}(n+1)(\sigma_n-\sigma_{n-1}) =$$

$$= \frac{-2\beta}{N^r} + \frac{\binom{N}{r}(N+1)\sigma_N}{N^r} - \frac{1}{N^r}\sum_{n=2}^{N}\left[\binom{n}{r}-\binom{n-1}{r}\right] n\sigma_{n-1} - \frac{1}{N^r}\sum_{n=2}^{N}\frac{\binom{n}{r} n\sigma_{n-1}}{n},$$

where $\beta=\sigma_0$ or 0, according as $r=1$ or >1. Now, it is easy to see that, if $n\sigma_n \to k$, as $n\to\infty$, $v_r \to -\dfrac{k}{r!}$, as $N\to\infty$, r being fixed, and that $v_r \cdot r!$ is uniformly bounded with respect to r, as $N\to\infty$. Consequently, it follows that M tends to a definite finite limit, as $N\to\infty$.

Also, we have

$$\sum_{N+1}^{\infty} a_n x^n = \lim_{M\to\infty}\sum_{N+1}^{M}(\sigma_n-\sigma_{n-1})(n+1)x^n =$$

$$= -(N+2)x^{N+1}\cdot\sigma_N - \sum_{N+2}^{\infty} n\sigma_{n-1}\left[\frac{(n+1)x^n - nx^{n-1}}{n}\right].$$

It follows easily that $\sum_{N+2}^{\infty} n\sigma_{n-1}(x^n-x^{n-1})$ tends to a definite finite limit, as $N\to\infty$. To prove that $\sum_{N+2}^{\infty}\sigma_{n-1}x^n$ tends to a definite finite limit, as $N\to\infty$, we have

$$\sum_{N+2}^{\infty}\sigma_{n-1}x^n = \sum_{N+2}^{\infty}(n\sigma_{n-1}-k)\frac{x^n}{n} + k\sum_{N+2}^{\infty}\frac{x^n}{n},$$

$$\sum_{N+2}^{\infty}\frac{x^n}{n} \leqq \frac{1}{N+2}\sum_{0}^{\infty}x^n = \frac{N}{c(N+2)}$$

and

$$\sum_{N+2}^{\infty} \frac{x^n}{n} = -\log(1-x) - \sum_{n=1}^{N+1} \frac{x^n}{n} = \log N - \log c -$$

$$-\sum_{n=1}^{N+1} \frac{1}{n}\left[1 - \frac{nc}{N} + \frac{n(n-1)c^2}{N^2 2!} - \dots\right] = k_1 + o(1) + \left[c - \frac{(N+1)c^2}{2N \cdot 2!} + \dots\right] \to p,$$

as $N \to \infty$, k_1 and p being finite. Moreover, if $S_n = \sum_{r=0} a_r$ and if $\sum_{n=0} a_n$ is convergent, $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent in (0, 1). Since, by hypothesis, $\sigma_n \to 0$, as $n \to \infty$, we have

$$n\sigma_n = -n \sum_{n+1}^{\infty} \frac{a_r}{r+1} = -n \sum_{n+1}^{\infty} \frac{S_r - S_{r-1}}{r+1} = \frac{n}{n+2} S_n - n \sum_{n+2}^{\infty} \frac{S_{r-1}}{r(r+1)},$$

and so $n\sigma_n$ tends to a definite finite limit, as $n \to \infty$

Theorem 2. *If* $f(x) = \sum_{n=0}^{\infty} \lambda_n e^{nix}$, λ_n *being real and the series being uniformly convergent in the closed interval* $(0, \pi)$, *if* $\sum_{r=0}^{n} \lambda_n = 0$ *and if* $n^{r-1} S_n \to 0$, *as* $n \to \infty$, *where* $S_n = \sum_{r=0}^{n} \lambda_r$ *and r is any fixed number such that* $1 < r < 2$, *then* $\lim_{\delta \to +0} \int_{\delta}^{\pi} \frac{\operatorname{Re}\left(e^{\frac{r\pi i}{2}} f(x)\right)}{x^r} dx$ *exists and is finite.*

PROOF. Integrating round the closed contour formed by the semi circles $|z| = \delta$, $|z| = \pi$, $0 < \delta < \pi$, in the upper half plane, and the two segments of the real axis, we have

$$(5) \quad \int_{\delta}^{\pi} \frac{f(-x)\,dx}{(-x)^r} + \int_{\delta}^{\pi} \frac{f(x)\,dx}{x^r} = -i \int_{0}^{\pi} \frac{f(\pi e^{i\theta})\, e^{-(r-1)i\theta}}{\pi^{r-1}} d\theta + i \int_{0}^{\pi} \frac{f(\delta e^{i\theta})\, e^{-(r-1)i\theta}}{\delta^{r-1}} d\theta,$$

because the series $\sum_{n=0}^{\infty} \lambda_n e^{niz}$ is uniformly convergent, when $-\pi \leqq x \leqq \pi$, $y \geqq 0$. Let $\delta = \frac{1}{N}$, where N is any positive integer, and let denote the semicircle $|z| = \delta$, $0 \leqq \arg z \leqq \pi$. We have

$$i \int_{0}^{\pi} \frac{f(\delta e^{i\theta})\, e^{-(r-1)i\theta}\, d\theta}{\delta^{r-1}} = \int_{\Delta} \frac{f(z)\,dz}{z^r} = \sum_{n=0}^{\infty} \lambda_n \int_{\Delta} \frac{e^{niz}\,dz}{z^r} =$$

$$= \sum_{n=0}^{N} \lambda_n \int_{\Delta} \frac{e^{niz}\,dz}{z^r} + \sum_{N+1}^{\infty} \lambda_n \int_{\Delta} \frac{e^{niz}\,dz}{z^r} = I_1 + I_2.$$

But, we have

$$I_1 = iN^{r-1}\sum_{n=0}^{N}\lambda_n\int_0^{\pi} e^{ni\delta e^{i\theta}}e^{-(r-1)i\theta}d\theta = iN^{r-1}\sum_{n=0}^{N-1}S_n\int_0^{\pi}(1-e^{i\delta e^{i\theta}})e^{ni\delta e^{i\theta}}e^{-(r-1)i\theta}d\theta +$$

$$+ iN^{r-1}S_N\int_0^{\pi} e^{ie^{i\theta}}e^{-(r-1)i\theta}d\theta = iN^{r-1}\sum_{k=0}^{\infty}\sum_{n=0}^{N-1}\frac{S_n(ni\delta)^k}{k!}\int_0^{\pi} e^{ki\theta}(1-e^{i\delta e^{i\theta}})e^{-(r-1)i\theta}d\theta +$$

$$+ iN^{r-1}S_N\int_0^{\pi} e^{ie^{i\theta}}\cdot e^{-(r-1)i\theta}d\theta.$$

Since $1-e^{i\delta e^{i\theta}} = -i\delta e^{i\theta}+o(\delta^2)$, and $n^{r-1}S_n\to 0$ as $n\to\infty$, it is easy to see that, if

$$w_k(N) = N^{r-1}\sum_{n=0}^{N-1}\frac{S_n(ni\delta)^k}{k!}\cdot\int_0^{\pi} e^{ki\theta}(1-e^{i\delta e^{i\theta}})e^{-(r-1)i\theta}d\theta,$$

$w_k(N)$. $(k!)$ tends to zero uniformly with respect to k, as $N\to\infty$, Consequently, it follows that $I_1\to 0$, as $N\to\infty$.

Also, integrating by parts, we have

$$I_2 = \lim_{N\to\infty}\sum_{n=N+1}^{M}\lambda_n\cdot\int_{\Delta}\frac{e^{niz}dz}{z^r} = -S_N\int_{\Delta}\frac{e^{(N+1)iz}dz}{z^r} + \sum_{n=N+1}^{\infty}S_n\int_{\Delta}\frac{(1-e^{iz})e^{niz}dz}{z^r} =$$

$$= -N^{r-1}S_N\int_0^{\pi} e^{\frac{(N+1)ie^{i\theta}}{N}}e^{-(r-1)i\theta}d\theta + \sum_{N+1}^{\infty}S_n\left(\frac{e^{niz}}{niz^r}-\frac{e^{(n+1)iz}}{(n+1)iz^r}\right)_{\Delta} +$$

$$+ \sum_{N+1}^{\infty}S_n\frac{r}{ni}\int_{\Delta}\left(e^{niz}-\frac{ne^{(n+1)iz}}{n+1}\right)\frac{dz}{z^{r+1}} = -N^{r-1}S_N\int_0^{\pi} e^{\frac{(N+1)ie^{i\theta}}{N}}e^{-(r-1)i\theta}d\theta +$$

$$+ \sum_{N+1}^{\infty}S_n\left(\frac{e^{niz}}{n(n+1)iz^r}\right)_{\Delta} + \sum_{N+1}^{\infty}S_n\left[\frac{e^{niz}(1-e^{iz})}{(n+1)iz^r}\right]_{\Delta} + \sum_{N+1}^{\infty}S_n\cdot\frac{r}{n\cdot i}\int_{\Delta}\frac{e^{niz}(1-e^{iz})}{z^{r+1}}dz +$$

$$+ \sum_{N+1}^{\infty}S_n\cdot\frac{r}{n(n+1)i}\int_{\Delta}\frac{e^{(n+1)iz}dz}{z^{r+1}}.$$

Therefore

$$|I_2| \leqq \pi N^{r-1}|S_N| + 2N^r\sum_{N+1}^{\infty}\frac{|n^{r-1}S_n|}{n^{r+1}} + 4N^{r-1}\sum_{N+1}^{\infty}\frac{|n^{r-1}S_n|}{n^r} +$$

$$+ 2\pi rN^{r-1}\sum_{N+1}^{\infty}\frac{|n^{r-1}S_n|}{n^r} + \pi rN^r\sum_{N+1}^{\infty}\frac{|n^{r-1}S_n|}{n^{r+1}} \leqq$$

$$\leqq A_N\left[\pi + (4+2\pi r)N^{r-1}\cdot\sum_{N+1}^{\infty}\frac{1}{n^r} + (2+\pi r)N^r\cdot\sum_{N+1}^{\infty}\frac{1}{n^{r+1}}\right] \leqq$$

$$\leqq A_N\left[\pi + (4+2\pi r)N^{r-1}\int_N^{\infty}\frac{dx}{x^r} + (2+\pi r)N^r\int_N^{\infty}\frac{dx}{x^{r+1}}\right] \leqq A_N\left[\frac{\pi+4+2\pi r}{r-1} + \frac{(2+\pi r)}{r}\right],$$

where $A_N = \max |n^{r-1} S_n|$ $(n = N, N+1, \ldots)$. We have, thus, proved that $I_2 \to 0$, as $N \to \infty$. Now, the theorem follows easily, if we take the real part of each side of (5).

Remark. By the method of proof of Theorem 2, it is easy to prove that, if $f(x) = \sum_{n=0}^{\infty} \lambda_n e^{nix}$, λ_n being real and the series being uniformly convergent in $(0, \pi)$, and if nS_n tends to a definite finite limit, as $n \to \infty$, each of the limits

$$\lim_{\delta \to +0} \int_{\delta}^{\pi} \frac{\operatorname{Re} f(x)\, dx}{x^2 \cdot \log x} \quad \text{and} \quad \lim_{\delta \to +0} \int_{\delta}^{\pi} \frac{\operatorname{Im} f(x)\, dx}{x^2((\log x)^2 + \pi^2)}$$

exists and is finite; further, if λ_n is ultimately of constant sign, then $\frac{\operatorname{Re} f(x)}{x^2 \cdot \log x} \in L(0, \pi)$.

Reference

[1] P. Heywood, On the integrability of functions defined by trigonometric series, (II), *Quart. Jour. Math. Oxford,* (2), **6** (1955), 77—79.

(Received August 26, 1968.)