# Characterization of rings whose classical quotient rings are perfect rings

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## 1. Introduction

Rings with descending chain condition on principal right ideals have been studied by BASS ([3]), FAITH ([4]), KAPLANSKY ([10]) and SZÁSZ ([16], [17], [18]). Bass ([3]) studied rings $R$ such that every left $R$-module has a projective cover. He called these rings (left) perfect. He proved that this class of rings coincides with the class of rings having d. c. c. on principal right ideals. There exist at least two characterisations of rings that are orders in Artinian rings, see GUPTA and SAHA [9], ROBSON [14] and SMALL [15]. In this note orders in perfect rings and orders in semi-primary rings are characterised in sections 3 and 4. In section 5 it is shown that $R$ is an order in a perfect (semi-primary) ring if and only if $R_n$, the ring of $n\times n$ matrices, over $R$ is an order in a perfect (semi-primary) ring. Finally, it is shown by an example that if $R$ is an order in a semi-primary (perfect) ring, then $R[x]$ is in general not an order in a semi-primary (perfect) ring.

## 2. Definitions and basic results

Let $R$ be a ring containing regular elements. A ring $S$ with unit containing $R$ is said to be a right quotient ring of $R$ if $s\in S$ implies $s=am^{-1}$, with $a\in R$, $m$ a regular element of $R$.

ASANO [1] has shown that if $M$ is a multiplicatively closed set of regular elements of $R$ and $R$ has the common multiple property with respect to $M$:

If $a\in R$, $m\in M$, then there exist $a_1, m_1\in R$ with $m_1\in M$ such that $am_1=ma_1$, and then there exists a quotient ring $S$ of $R$ in which the elements of $M$ are invertible.

2. 2 A right quotient ring $Q$ of $R$ is said to be a classical right quotient ring of $R$ if all the regular elements of $R$ are invertible in $Q$. The classical right quotient ring of $R$ if it exists will be denoted by $Q(R)$. It is well known that $Q(R)$ exists if and only if $R$ satisfies the common multiple property with respect to the set $M$ of all regular elements of $R$.

2. 3 It has been proved by Bass [3] that $Q$ is perfect if and only if $Q/J(Q)$ is a semi-simple Artinian ring and $J(Q)$ is left $T$-nilpotent in the following sense:

If $\{a_n\}$ is any infinite sequence of elements of $J(Q)$, then there exists $N$ such that $a_1, a_2, \dots, a_n=0$, where $J(Q)$ denotes the Jacobson radical of $Q$.

2. 4 A ring $Q$ with unit is said to be semi-primary if $J(Q)$ is nilpotent and $Q/J(Q)$ is semi-simple Artinian. Clearly, a semi-primary ring is a perfect ring.
The following result follows from the definition:

**Proposition 2. 5.** *A homomorphic image of a perfect ring is a perfect ring.*

**Proposition 2. 6.** *If $Q$ is a perfect ring and a is a right regular element of $Q$, then $a$ is invertible in $Q$.*

PROOF. The chain $aQ \supset a^2Q \supset \dots \supset a^nQ \supset \dots$ stops giving $a^n = a^{n+1}b$ for some $n$ and some $b$. Hence $1 = ab$. Now, $a(ba-1) = (ab)a - a = 0$; so $(ba-1) = 0$ also.

Corollary 2.7. *If $R$ has a right quotient ring $Q$ which is perfect, then $Q$ is a classical right quotient ring of $R$.*

**Proposition 2. 7.** *If $R$ has a classical quotient ring $Q$ which is perfect, and $N$ is an ideal of $R$ such that whenever $a$ is regular in $R$, $a+N$ is regular in $R/N$, then $Q(R/N)$ exists and $Q(R/N) \cong Q/NQ$.*

PROOF. Let $\overline{M} = \{m+N | m$ regular in $R\}$. Clearly, $\overline{M}$ is a multiplicatively closed set of regular elements of $R/N$ and $R/N$ satisfies the common multiple property with respect to the set $\overline{M}$. Therefore $(R/N)_{\overline{M}}$ exists. It can be checked that $am^{-1} \to (a+N)(m+N)^{-1}$ is well defined map of $Q$ onto $(R/N)_{\overline{M}}$ and is a ring homomorphism. The kernel of this homomorphism is $NQ$. Consequently

$$Q/NQ \cong (R/N)_{\overline{M}}.$$

Now by 2. 7 and 2. 5 it follows that $(R/N)_{\overline{M}} = Q(R/N)$. Hence $Q(R/N) \cong Q/NQ$.

**Proposition 2. 8.** *If $R$ has a classical right quotient ring $Q$ which is perfect, $N'$ is an ideal of $Q'$ and $N = N' \cap R$, then whenever $a$ is regular in $R$, $a+N$ is regular in $R/N$. Also, if $N' \subset J(Q)$, then $a+N$ regular in $R/N$ implies $a$ regular in $R$.*

PROOF. Suppose $a$ is regular in $R$. $ax \in N$, $x$ in $R$, implies $ax \in N'$. $a^{-1}ax \in N'$. $x \in N' \cap R = N$. Similarly $xa \in N$, $x$ in $R$, implies $x \in N$. Now, assume that $N' \subset J(Q)$ and $a+N$ is regular in $R/N$. By 2. 7 it follows that $Q(R/N) \cong Q/N'$ because $(N' \cap R)Q = N'$. Now $a+N$ is invertible in $Q(R/N)$. Consequently by its image $a+N'$ is invertible in $Q/N'$. There exists $b$ in $Q$ such that $ab-1$, $ba-1 \in N' \subset J(Q)$. Consequently $ab$ and $ba$ are invertible in $Q$. Therefore $a$ is invertible in $Q$. Hence $a$ is regular in $R$.

**Proposition 2. 9.** *If $0 \neq N$ is a left T-nilpotent ideal of a ring $R$, then there exists $0 \neq x \in N$ such that $xN = 0$.*

PROOF. Assume that there exists no such $x$. Let $0 \neq x_1 \in N$. Assume there exist $x_1, x_2, \dots, x_n$ in $N$ such that $x_1x_2x_3 \dots x_n \neq 0$. Assumption implies that there exists $x_{n+1} \in N$ such that $x_1x_2x_3 \dots x_nx_{n+1} \neq 0$. That contradicts the left $T$-nilpotency of $N$.

**Proposition 2. 10.** (Mark L. Teply) *A simple left T-nilpotent ring $R$ is a trivial ring.*

PROOF. Let $I = \{x : xR = 0\}$. $I$ is a two-sided ideal of $R$ and $I \neq 0$ by 2. 9. Hence $I = R$. $R^2 = 0$.

## 3. Perfect quotient rings

Definition 3.0. For any ring $R$, we define $T_\alpha(R)$, an ideal of $R$ as follows: $T_0(R)=(0)$. $T_{\alpha+1}(R) = \{x : x \in N(R),\ xN(R) \subset T_\alpha(R)\}$ for an ordinal number of the type $\alpha+1$, and $T_\alpha(R) = \bigcup_{\beta<\alpha} T_\beta(R)$ for a limit ordinal $\alpha$, where $N(R)$ denotes the upper nil radical of $R$.

**Theorem 3. 1.** *A ring $R$ has a classical right quotient ring $Q$ which is perfect if and only if*

(i) *$N(R)$, the upper nil radical of $R$, is left $T$-nilpotent.*

(ii) *$R/N(R)$ is a right Goldie ring.*

(iii) *$a+N(R)$ regular in $R$ implies $a$ is regular in $R$.*

(iv) *$a_\alpha R_\alpha$ is an essential right ideal of $R_\alpha$ for every right regular element $a_\alpha$ in $R_\alpha$ and for every ordinal number $\alpha$ where $R_\alpha = R/T_\alpha(R)$.*

Proof. Necessity. Assume that $R$ has a classical right quotient ring $Q$ which is perfect. Notice that $J(Q)$ is nil and therefore $N(Q)=J(Q)$. It is proved in [8, Theorem 4. 1] that $N(R) = N(Q) \cap R$. Therefore $N(R)$ is left $T$-nilpotent. Also (iii) follows by using 2. 8. Also by using 2. 8 and 2. 7 it follows that $Q(R/N(R))$ exists and $Q(R/N(R)) \cong Q/N(Q)$. As $Q/N(Q)$ is semi-simple Artinian, $R/N(R)$ is a right Goldie ring [6]).

In order to prove (iii) for $\alpha=0$ we have to show that $aR$ is an essential right ideal of $R$ every right regular element $a$ in $R$. Now a being right regular in $R$ $a$ is also right regular in $Q$. Therefore $a^{-1} \in Q$ by 2. 6. It is sufficient to prove that $aR \cap bR \neq 0$ for every $0 \neq b$ in $R$. Now $a^{-1}b \in Q \Rightarrow a^{-1}b = cd^{-1}$, $c$, $d$ in $R$. $bd = ac \neq 0$. Hence $aR \cap bR \neq 0$.

Now to prove (iii) for every ordinal number $\alpha$ we note in the next lemma that for every ordinal $\alpha$ $T_\alpha(R) = T_\alpha(Q) \cap R$. By using 2. 8 and 2. 7 it follows that $R/T_\alpha(R)$ has a classical right quotient ring isomorphic to $Q/T_\alpha(Q)$ which is perfect. The case $\alpha=0$, therefore, implies (iii) for all $\alpha$.

**Lemma 3. 2.** *If $R$ has a classical right quotient ring $Q$ which is perfect,*

$$T_\alpha(Q) \cap R = T_\alpha(R)$$

*$\forall$ ordinal $\alpha$.*

Proof. Trivial for $\alpha=0$. Assume $T_\beta(Q) \cap R = T_\beta(R)$ for $\beta<\alpha$. If $\alpha$ is a limit ordinal, then

$$T_\alpha(R) = \bigcup_{\beta<\alpha} (T_\beta(R)) = \bigcup_{\beta<\alpha} (T_\beta(Q) \cap R) = (\bigcup_{\beta<\alpha} T_\beta(Q)) \cap R = T_\alpha(Q) \cap R.$$

Assume now that $\alpha = \beta+1$ for some $\beta$. Then $T_\beta(Q) \cap R = T_\beta(R)$. Let $x \in T_{\beta+1}(R)$. Then $x \in N(R) \subset N(Q)$ and $xN(R) \subset T_\beta(R)$. Therefore $xN(R)Q \subset T_\beta(R)Q$. $xN(Q) \subset \subset T_\beta(Q)$, because $N(R)Q = N(Q)$ and $T_\beta(R)Q = T_\beta(Q)$. Hence $x \in T_{\beta+1}(Q) \cap R$. Conversely, suppose $x \in T_{\beta+1}(Q) \cap R$. Then $x \in N(Q) \cap R = N(R)$ and $xN(Q) \subset \subset T_\beta(Q)$. $xN(R) \subset T_\beta(Q) \cap R = T_\beta(R)$, $x \in T_{\beta+1}(Q)$. Hence $T_\alpha(Q) \cap R = T_\alpha(R)$ for every ordinal number $\alpha$.

Before proving the sufficiency part of 3. 1, we prove first a lemma.

**Lemma 3. 3.** *Let $R$ be a ring satisfying conditions* (i), (ii), (iii) *and* (iv). *Let*

$$M = \{a \in R | a + N(R) \text{ regular in } R/N(R)\}.$$

*Then $M$ is a non-empty multiplicatively closed set of regular elements of $R$. If $a \in M$ then $a + T_\alpha(R)$ is right regular in $R/T_\alpha(R)$ for every ordinal number $\alpha$.*

PROOF. $M$ is non empty because $R/N(R)$ contains regular elements [6, Theorem 3. 9]. Also $M$ is a closed set of regular elements in $R$, because of (iii) and because the set of regular elements in $R/N(R)$ is closed. Now assume $a \in M$. $a$ being regular in $R$ is right regular in $R$. That proves the assertion for $\alpha = 0$. Now assume that $a_\beta = a + T_\beta(R)$ is right regular in $R_\beta$ for every $\beta < \alpha$. If $\alpha$ is a limit ordinal, then $ax \in T_\alpha(R)$, $x \in R$ implies $ax \in T_\beta(R)$ for some $\beta < \alpha$. Therefore by the induction hypothesis, $x \in T_\beta(R)$. $x \in T_\alpha(R)$. Now let $\alpha = \beta + 1$ for some $\beta$. Suppose $ax \in T_{\beta+1}(R)$ for some $x$ in $R$. $axN(R) \subset T_\beta(R)$. By the induction hypothesis $xN(R) \subset T_\alpha(R)$. Also $ax \in T_{\beta+1}(R) \Rightarrow ax \in N(R)$, which implies $x \in N(R)$, as $a + N(R)$ is regular in $R/N(R)$. Hence $x \in T_{\beta+1}(R)$. Hence $a + T_\alpha(R)$ is right regular in $R/T_\alpha(R)$ for every ordinal number $\alpha$.

We are now ready to prove the sufficiency part of 3. 1. We prove below a result which, in view of lemma 3. 3, implies the sufficiency part of theorem 3. 1.

**Proposition 3. 5.** *Let $R$ be a ring such that*

(i) *$N(R)$ the upper nil radical of $R$ is left T-nilpotent.*

(ii) *$R/N(R)$ is a right Goldie ring.*

(iii) *There exists a multiplicatively closed set $M$ of regular elements of $R$ such that the elements of $\overline{M} = \{m + N(R) | m \in M\}$ are regular in $R/N(R)$ and $(R/N(R))_{\overline{M}}$ exists and coincides with $Q(R/N(R))$.*

(iv) *For every $m \in M$ and every ordinal number $\alpha$, $m_\alpha R_\alpha$ is an essential right ideal of $R_\alpha$, where $m_\alpha = m + T_\alpha(R)$. Then $R_M$ exists and is perfect.*

Before proving this result we prove two lemmas.

**Lemma 3. 6.** *Let $A$ be an arbitrary ring and $B$, an ideal of $A$. Let $a, b \in A$ be such that $aA$ is an essential right ideal of $A$ and $bB = 0$, then $\overline{b^{-1}(aA)} = b^{-1}(aA) + B$ is an essential right ideal of $R/B$, where $b^{-1}(aA) = \{x \in R | bx \in aA\}$.*

PROOF. It is sufficient to prove that $I \cap b^{-1}(aA)$ contains an element of $B$, whenever $I$ is *a* right ideal of $R$ not contained in $B$. Let $I$ be a right ideal of $R$ not contained in $B$. Either $bI = 0$, in which case $I \cap b^{-1}(aA) = I \subsetneqq B$. If $bI \neq 0$, then $bI \cap aA \neq 0$, because $aA$ is an essential right ideal of $A$. Therefore there exists $x \in I$, $y \in A$ such that $bx = ay \neq 0$. $x \in I \cap b^{-1}(aA)$. $x \in B$, because $x \in B$ gives $bx = 0$, which is a contradiction.

**Lemma 3. 7.** *If $R$ is a semiprime right Goldie ring, and $M$ a multiplicatively closed set of regular elements of $R$ such that $R_M$ exists and coincides with $Q(R)$, then every essential right ideal of $R$ contains an element of $M$.*

PROOF. Let $I$ be an essential right ideal of $R$. Then $I$ contains a regular element [6, Theorem 3. 9]. Therefore $IQ = Q$. But $IQ = IRM = \{rm^{-1} | r \in I, m \in M\}$ [15, Cor. 1. 5]. Hence $1 = rm^{-1}$ for some $r \in I$, $m \in M$. Therefore $m(=r) \in I$.

PROOF OF 3. 5. Assume $N(R)\neq 0$. By 2. 9 it is immediate that $T_1(R)\neq 0$. In general if $T_\alpha(R)\subsetneqq N(R)$, then $T_\alpha(R)\subsetneqq T_{\alpha+1}(R)$. From the definition of $T_{\alpha+1}(R)$

$$T_{\alpha+1}(R)/T_\alpha(R)=\{x+T_\alpha(R)\in N(R)/T_\alpha(R)\mid (x+T_\alpha(R))N(R)/T_\alpha(R)=0\}$$

As $N(R)/T_\alpha(R)$ is left $T$-nilpotent $T_{\alpha+1}(R)/T_\alpha(R)\neq 0$ by 2. 9. Hence $T_\alpha(R)\subsetneqq T_{\alpha+1}(R)$. So $T_v(R)=N(R)$ for some ordinal number $v$. Now we prove that for every $a$, $ma$ in $N(R)$ $m$ in $M$ there exist $a_1$, $m_1a_1$ in $Rm_1$ in $M$ such that $am_1=ma_1$. This is trivial if $a\in T_0(R)=(0)$. Assume this is true for all $a\in T_\beta(R)$ where $\beta<\alpha$. If $\alpha$ is a limit ordinal and $a\in T_\alpha(R)$, then $a\in T_\beta(R)$ for some $\beta<\alpha$ and again the result follows. If, however, $\alpha=\beta+1$ and $a\in T_\alpha(R)$, $m\in M$ then denoting $a+T_\beta(R)$ by $a_\beta$, $a_\beta(N(R)/T_\beta(R))=0$, and $m_\beta R_\beta$ is an essential right ideal of $R_\beta$ by condition (iv). By 3. 6

$$\overline{a_\beta^{-1}(m_\beta R_\beta)}=a_\beta^{-1}(m_\beta R_\beta)+N(R)/T_\beta$$

is an essential right ideal of $(R/T_\beta(R))/(N(R)/T_\beta(R))$. As $(R/T_\beta(R))/(N(R)/T_\beta(R))\cong$ $\cong R/N(R)$, it follows by conditions (ii), (iii) and Lemma 3. 6 that there exists $d\in R$ such that $a_\beta d_\beta\in m_\beta R_\beta$ and $d+N(R)\in\overline{M}$. Therefore $d+N(R)=m'+N(R)$ for some $m'\in M$. $a_\beta m'_\beta=a_\beta(m'-d)_\beta+a_\beta d_\beta\in m_\beta R_\beta$ because $a_\beta(m'-d)_\beta=0$. There exists $a'\in R$ such that $am'-ma'\in T_\beta(R)$. By the induction hypothesis there exist $a''$, $m''$, $a''$ in $R$ $m''$ in $M$ such that $(am'-ma')m''=ma''$. $a(m'm'')=m(a'm''+a'')$. The result is now proved by taking $d=v$.

Now to show that $R_M$ exists let $a\in R$ $m\in M$. As $(R/N(R))_{\overline{M}}$ exists, there exist $a_1$, $m_1a_1$ in $R\,m_1$ in $M$ such that $\bar a\bar m_1=\bar m\bar a_1$. $am_1-ma_1\in N(R)$. There exist $a_2$, $m_2$ $a_2$ in $R$, $m_2$ in $M$ such that $(am_1-ma_1)m_2=ma_2$. $a(m_1m_2)=m(a_1m_2+a_2)$. Hence $R_M$ exists.

To prove now that $R_M$ is left perfect it is easy to check that the map $am^{-1}\to$ $\to(a+N(R))(m+N(R))^{-1}$ is well defined and is a homomorphism of $R_M$ onto $(R/N(R))_{\overline{M}}=Q(R/N(R))$. The kernel of this homomorphism is $N(R)R_M$. Consequently, $N(R)R_M$ is a two sided ideal of $R_M$ and $R_M/N(R)R_M$ being isomorphic to $Q(R/N(R))$ is a semi-simple Artinian ring. It is sufficient to show that $N(R)R_M$ is left $T$-nilpotent. Now $N(R)R_M=\{nm^{-1}\mid n\in N(R),\ m\in M\}$ [15, Cor 1. 5]. Let $\{n_im_i^{-1}\}$ be any sequence of elements of $N(R)R_M$: We show that there exist sequences $\{n'_i\}$ and $\{m'_i\}$ of elements of $N(R)$ and $M$ respectively such that

$$n_1m_1^{-1}n_2m_2^{-1}\dots n_rm_r^{-1}=(n'_1\dots n'_r)m_r'^{-1}\ \forall r.$$

And this gives that the given sequence is left vanishing. We can take $n'_1=n_1$ and $m'_1=m_1$. Assume that there exist $n'_1, n'_2, \dots, n'_r\in N(R)$ and $m'_1,\dots,m'_r\in M$ such that

$$n_1m_1^{-1}n_2m_2^{-1}\dots n_rm_r^{-1}=(n'_1n'_2\dots n'_r)m_r'^{-1}.$$

Post multiplying by $n_{r+1}m_{r+1}^{-1}$,

$$n_1m_1^{-1}n_2m_2^{-1}\dots n_rm_r^{-1}n_{r+1}m_{r+1}^{-1}=(n'_1n'_2\dots n'_r)m_r'^{-1}n_{r+1}m_{r+1}^{-1}.$$

Now $m_r'^{-1}n_{r+1}=am^{-1}$, $a\in R$, $m\in M$. So that we can take $n'_{r+1}=a$ and $m'_{r+1}=$ $=m_{r+1}m$ if we know that $a\in N(R)$. We know that $n_{r+1}m=m'_ra$. So $m'_r\ a\in N(R)$. Therefore $a\in N(R)$, because $m'_r+N(R)$ is regular in $R/N(R)$. Hence $R_M$ is left perfect.

## 4. Semiprimary quotient rings

It is easy to see that $T_k = l(N(R)^k) \cap N(R)$ for all integers $k$. If $N(R)$ be nilpotent of index $n$, then $T_{n-1} = l(N(R)^{n-1}) \cap N(R) = N(R)$. In the "only if" part of 3. 1 if $J(Q)$ is nilpotent then so is $N(R)$ because $N(R) = J(Q) \cap R$. Conversely, if $N(R)$ is nilpotent, then $N(R) R_M$ is also nilpotent is clear by going through the argument used. Consequently we have the following

**Theorem 4. 1.** *A ring $R$ has a classical right quotient ring $Q$ which is semiprimary iff*
(i) *$N(R)$, the upper nil radical of $R$ is nilpotent,*
(ii) *$R/N(R)$ is a right Goldie ring,*
(iii) *If $a+N(R)$ is regular in $R/N(R)$, then $a$ is regular in $R$,*
(iv) *$a_k R_k$ is an essential right ideal of $R_k$ for every integer $k$ and every right regular element $a_k$ in $R_k$.*

Stronger version of the 'if' part can be phrased easily.

## 5. Matrix rings

If $Q$ is *a* left perfect ring, then $Q_n$, the ring of all $n \times n$ matrices over $Q$ is also left perfect, see Bass [3, page 475]. If $Q$ be semi-primary, then $J(Q_n) = (J(Q))_n$ which is nilpotent and therefore $Q_n$ is semiprimary. ($R$ is said to be an order in $Q$ if $Q$ is the classical right quotient ring of $R$.)

**Theorem 5. 1.** *A ring $R$ is an order in a perfect (semiprimary) ring iff $R_n$ is an order in a perfect (semi-primary) ring.*

PROOF. Assume that $R$ is an order in a perceft (semi-primary) ring $Q$. Let $[q_{ij}] \in Q_n$. There exists $m$ regular in $R$ such that $q_{ij} m \in R$ $\forall i, j = 1, 2, \ldots, n$ [15, Lemma 1. 4]. Then

$$[q_{ij}] = [q_{ij} m][\operatorname{diag}(m, m, \ldots, m)]^{-1}.$$

Hence $R_n$ is an order in $Q_n$.

Conversely, suppose $R_n$ is an order in a perfect (semiprimary) ring, then $N(R_n)$, the upper nil radical of $R_n = l(R_n)$, the lower nil radical of $R_n = J(Q_n) \cap R_n$ [8, Theorem 4. 1]. But $l(R_n) = (l(R))_n$ [13, Cor. page 101]. Hence $l(R)$ is left $T$-nilpotent (nilpotent). Now $R_n/N(R_n) = R_n/l(R_n) = R_n/(l(R))_n \cong (R/l(R))_n$. Therefore $(R/l(R))_n$ is a right Goldie ring. Consequently $R/l(R)$ is a right Goldie ring. Hence $N(R) = l(R)$ [8, Cor 3. 3, page 92]. Therefore $N(R)$ is left $T$-nilpotent (nilpotent) and $R/N(R)$ is right Goldie ring.

Now it is easy to check that $(T_\alpha(R))_n = T_\alpha(R_n)$ $\forall$ ordinal $\alpha$. Therefore $(R/T_\alpha(R))_n \cong R_n/T_\alpha(R_n) = T_\alpha(R_n)$. If $a_\alpha$ is a right regular element in $R_\alpha$, then $[\operatorname{diag}(a_\alpha, a_\alpha, \ldots, a_\alpha)]$ is right regular in $(R_\alpha)_n$ and therefore $[\operatorname{diag}(a_\alpha, a_\alpha, \ldots, a_\alpha)]\,(R_\alpha)_n$ is an essential right ideal of $(R_\alpha)_n$. Hence $a_\alpha R_\alpha$ is an essential right ideal of $R_\alpha$. Now to prove (iii) suppose that $a + N(R)$ be regular in $R/N(R)$. Then $\operatorname{diag}(a, a, \ldots, a) + (N(R))_n$ is regular in $R_n/(N(R))_n$. But

$$R_n/(N(R))_n = R_n/N(R_n).$$

Therefore $\operatorname{diag}(a, a, \ldots, a)$ is regular in $R_n$. Hence $a$ is regular in $R$.

## 6. Polynomial rings

It is well known that if $R$ be an order in a right Artinian ring, then $R[x]$ is an order in an Artinian ring. An example of ring $R$ such that $R$ is an order in a semi -primary ring, but $R[x]$ is not an order in a semi-primary ring. In fact the ring $R$ is semi-primary, but $R[x]$ has no classical right quotient ring.

Let $R=\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a,\ b \text{ real},\ c \text{ rational}\right\}$.

Let $f(x) = x-\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where $a$ is a transcendental real number, $b$ any real number and $c$ any rational number. Then it can be verified that $f(x)\,R[x]\cap \alpha R[x] = (0)$, where $\alpha=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence $f(x)\ R[x])]$ is not an essential right ideal of $R[x]$. However, $f(x)$ is a regular element of $R[x]$, because $f(x)$ has regular leading coefficient.

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