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# Investigation of relative increments of distributions functions

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**Summary.** The monotone properties of hazard rates and relative increments of probability distribution functions are investigated. The main results are formulated in Theorems 1 and 2 containing sufficient conditions under which the corresponding relative increment function and hazard rate increase (or decrease, or have two monotone phases). Our method enables us to avoid the incovenient term $1 - F$ appearing in the expressions of the relative increment function, the hazard rate and our auxiliary function in Lemma 1. Instead, it deals with the relatively simple expression $f/f'$. The results have been applied to logistic, extreme value, Fisher's $z$-, Pareto of the third kind, Weibull, trigonometric and many other distributions.

## Introduction

We shall use standard mathematical notations such as

$\mathbb{R}$ *means* the set of all real numbers;

iff *means* "if and only if".

The relative increment function was introduced and used first by PORTER and DUDMAN (1960) [they called it the relative increment of decay or RID index], and was further used and investigated by ADLER and SZABO (1972, 1974, 1979a, 1979b, 1984) and SZABO (1976, 1989).

In this paper we investigate the *hazard rate* and *relative increment functions* of some (cumulative) distribution functions.

Let $f$ be a (probability) density function. The corresponding distribution function is defined as usual:

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

*Key words and phrases*: distribution function, density function, hazard rate, relative increment function, (strictly) increasing/decreasing function.

By the *relative increment function* [briefly, RIF] of $F$ we mean the fraction

$$h(x) = [F(x+a) - F(x)]/[1 - F(x)],$$

where $a$ is a positive constant, and $F(x) < 1$ for all $x$.

Monotone properties of RIFs are important from the points of view of
(a) statistics, probability theory;
(b) applied statistics, e.g., in

(b.1) modelling bounded growth processes in biology, medicine and dental science [see Adler and Szabo (1972, 1974, 1979a, 1979b, 1984), Porter and Dudman (1960), and Szabo (1989)]

and in

(b.2) reliability and actuarial theories, where the probability that an individual, having survived to time $x$, will survive to time $x + a$ is $h(x)$; *"death rate per unit time"* in the time interval $[x, x+a]$ is $h(x)/a$, and the *hazard rate (failure rate or force of mortality)* is defined to be

$$\lim_{a \to 0} h(x)/a = f(x)/[1 - F(x)].$$

[See Sec. 7, Chap. 33. in Vol. 2 of Johnson and Kotz (1970), or §5.34 and §5.38 of Stuart and Ord (1987).]

In Sec. 7.2, Chap. 33 of Johnson and Kotz (1970), some distributions are classified by their increasing/decreasing hazard rates.

We will need the following

**Lemma 1.** *Let $F$ be a twice differentiable distribution function with $F(x) < 1$, $f(x) > 0$ for all $x$. We define the auxiliary function $\Psi$ as follows:*

$$\Psi(x) := [F(x) - 1] \cdot f'(x)/f^2(x).$$

*If $\Psi < (>)1$, then the function $h$, the* RIF *of $F$ strictly increases (strictly decreases).*

Proof. Suppose $\Psi < 1$. Let $G(x) := 1 - F(x)$. Since $G > 0$, the RIF $h$ can be written in the form $h(x) = 1 - G(x+a)/G(x)$. The function $h$ strictly increases, iff

$$G(x+a)/G(x) > G(x+2a)/G(x+a), \quad \text{i.e., iff}$$
$$\ln G(x+a) > [\ln G(x) + \ln G(x+2a)]/2, \quad \text{i.e., iff}$$

$G$ is strictly log concave.

From the condition $\Psi(x) < 1$, we obtain

$$G(x) \cdot G''(x) < G'^2(x), \quad \text{i.e., } [\ln G(x)]'' < 0.$$

Thus $G$ is strictly log concave. When $\Psi > 1$, the proof is the same. □

In a very similar way, one can prove the

**Lemma 2.** *If the conditions of Lemma 1 are fulfilled, then the following implications hold:*
*the* RIF $h$ *increases iff* $\Psi \le 1$*;*
*the* RIF $h$ *decreases iff* $\Psi \ge 1$.

*Remark 0.1.* There is an immediate connection with the theory of reliability. By the Mathematical Preliminaries of Barlow and Proschan (1967) Sec. 1. p. 549, a distribution function $F$ has IFR (increasing failure rate) iff $\ln[1-F(x)]$ is concave down i.e., iff $\Psi(x) \le 1$. Similarly, $F$ has DFR (decreasing failure rate) iff $\ln[1-F(x)]$ is concave up, i.e., $\Psi(x) \ge 1$.

*Remark 0.2.* Through the entire paper we investigate the auxiliary function $\Psi$. In order to get rid of the inconvenient term $(F-1)$ in $\Psi$, we reduce all problems to simple formulae containing the fraction $f/f'$ only.

## The main results

**Theorem 1.** *Let $f$ be a probability density function and $F$ be the corresponding distribution function with the following properties.*

(1) $\begin{cases} I=(r,s)\subseteq\mathbb{R} \text{ is the possible largest finite or infinite open} \\ \text{interval in which } f>0 \text{ (i.e., } I \text{ is the open support of } f; \\ r \text{ and } s \text{ may belong to the extended real line} \\ \mathbb{R}^* = \mathbb{R}\cup\{-\infty,\infty\}); \end{cases}$

(2) *there exists an* $m \in I$ *at which* $f'$ *is continuous and* $f'(m)=0$;

(3) $f' > 0$ *in* $(r,m)$, *and* $f' < 0$ *in* $(m,s)$

(4) $f$ *is twice differentiable in* $(m,s)$

(5) $(f/f')' = d/dx[f(x)/f'(x)] > 0$ *in* $(m,s)$.

*Then the corresponding continuous RIF $h$ is either strictly increasing in $I$, or strictly increasing in $(r,y)$ and strictly decreasing in $(y,s)$ for some $y \in I$.*

*Moreover, if* $\Psi(s^-) = \lim_{x\to s^-}\Psi(x) \in \mathbb{R}^*$ *exists, then*
*(a)* $h$ *strictly increases in* $I$, *if* $\Psi(s^-) \le 1$;
*(b)* $h$ *strictly increases in* $(r,y)$ *and strictly decreases in* $(y,s)$ *for some* $y$ *in* $I$, *if* $\Psi(s^-) > 1$.

Prof. It is sufficient to show that $\Psi < 1$ in $(r,m)$ and, when $\Psi$ reaches the value one (say, at $x_0$), then it strictly increases in $(x_0,s)$.

It follows from (1) that

$$0 < F < 1 \quad \text{in } I. \tag{6}$$

By virtue of (1), (3) and (6), we have $\Psi = (F-1)\cdot f'/f^2 < 0$ in $(r,m)$. Thus, by Lemma 1, the RIF $h$ strictly increases in $(r,m)$. [If $m = s$, then $h$ strictly increases in the entire interval $I$.]

By (2) and (4), $f'$ is continuous in $[m,s)$, and so is $\Psi$. Thus, $\Psi < 1$ in $(r, m+p)$ for some $p > 0$.

We have two cases.

*Case 1:* $\Psi < 1$ in $I$. Then $h$ strictly increases in $I$. In this case, if $\Psi(s^-)$ exists, it will not exceed unity: $\Psi(s^-) \leq 1$.

*Case 2:* $\Psi(x_0) \geq 1$ for some $x_0 \in (m,s)$. Then

$$[F(x_0) - 1]\cdot f'(x_0) \geq f^2(x_0). \tag{7}$$

The conditions (1) and (4) allow us to form the derivative

$$\Psi' = [f^2 f' + (F-1)\cdot(ff'' - 2f'^2)]/f^3$$

in $(m,s)$. We shall prove that $\Psi'(x_0) > 0$. By contraposition, we assume that $\Psi'(x_0) \leq 0$, i.e.,

$$[f^2\cdot f' + (2f'^2 - f\cdot f'')\cdot(1-F)]\,|_{x=x_0} \leq 0. \tag{8}$$

We replace $f^2(x_0)$ in the first term by $[F(x_0)-1]\cdot f'(x_0)$. According to the relations (7) and $f'(x_0) < 0$, the left-hand side of (8) will decrease, and we get

$$(1-F)\cdot(f'^2 - f\cdot f'')\,|_{x=x_0} \leq 0.$$

By virtue of (3) and (6), this is equivalent to

$$f'^2(x_0) \leq f(x_0)\cdot f''(x_0), \quad \text{i.e.,}$$
$$1 - f(x_0)\cdot f''(x_0)/f'^2(x_0) = (f/f')'\,|_{x=x_0} \leq 0,$$

which contradicts (5). Hence, $\Psi'(x_0) > 0$, and $\Psi$, which is continuous in $(m,s)$, strictly increases in some neighborhood of $x_0$. If follows that, *once $\Psi$ reaches the value 1 [say , at $y \in (m,s)$], it will strictly increase in $(y,s)$.* Thus, the corresponding RIF $h$ strictly decreases in $(y,s)$. In this case, we have

$$1 < \Psi(s^-) \in \mathbb{R} \cup \{\infty\}. \qquad \square$$

(Roughly speaking, the "main idea" of the proof is that $\Psi$ *strictly increases at $x$ if* $\Psi(x) \geq 1$).

*Remark 1.1.* If $s$ is finite, one can consider the special case, when $m = s$, i.e. $(m, s) = \emptyset$. Then $\Psi < 0$ and $h$ strictly increases in $(r, m) = I$ and, of course, no inequality (5) is required.

*Remark 1.2.* If $r$ is finite, we can consider the case when $m = r$, i.e. $(r, m) = \emptyset$. Then it is enough to check the value of

$$\Psi(r^+) = \lim_{x\to r^+} \Psi(x) \quad \text{and} \quad \Psi(s^-),$$

provided the conditions (1), (3–5) are fulfilled.

If $\Psi(s^-) > 1$, then

(a) if $\Psi(r^+) > 1$ or [$\Psi(r^+) = 1$ and $\Psi > 1$ in some right neighborhood of $r$], then $\Psi > 1$ in the entire interval $I$ since the "main idea" of the proof of Thm. 1 applies. Thus, by our Lemma 1, the RIF $h$ strictly decreases in $I$;

(b) if $\Psi(r^+) < 1$ or [$\Psi(r^+) = 1$ and $\Psi < 1$ in some right neighborhood of $r$], then there exists $y$ in $I$ such that $\Psi < 1$ in $(r, y)$ and $\Psi > 1$ in $(y, s)$, since the "main idea" of the proof applies. So, the Lemma 1 gives that the RIF $h$ strictly increases in $(r, y)$ and strictly decreases in $(y, s)$.

If $\Psi(s^-) < 1$ or [$\Psi(s^-) = 1$ and $\Psi < 1$ in some left neighborhood of $s$], then $\Psi < 1$ in $I$, and $h$ strictly increases in $I$.

The event [$\Psi(s^-) = 1$ and $\Psi \geq 1$ in some left neighborhood of $s$] is impossible because of the "main idea".

*Remark 1.3.* If $f_\infty := \lim_{x\to\infty} x \cdot f(x) = 0$ then, by L'Hospital's rule, we have $\lim_{x\to\infty}\{[F(x) - 1]/[x \cdot f(x)]\} = \lim_{x\to\infty}[1 + x \cdot f'(x)/f(x)]^{-1}$.

*Remark 1.4.* If $\lim_{x\to s^-} f^2(x)/f'(x) = 0$, then L'Hospital's rule gives

$$\begin{aligned}\Psi(s^-) &= \lim_{x\to s^-} [F(x) - 1]/[f^2(x)/f'(x)] \\ &= \lim_{x\to s^-} f'^2(x)/[2f'^2(x) - f(x) \cdot f''(x)] = \lim_{x\to s^-} [1 + (f(x)/f'(x))']^{-1}.\end{aligned}$$

*Remark 1.5.* Since $(f/f')' = -(\ln f)''/[(\ln f)']^2$, then condition (5) can be formulated as follows:

$$(\ln f)'' < 0, \quad x \in (m, s). \tag{5'}$$

We define the functions $f$ and $g$ be $\overset{''}{\sim}$-equivalent (we write $f \overset{''}{\sim} g$), if $(\ln f(x))'' = (\ln g(x))''$. E.g., if $f$ has the form

$$f(x) = c \cdot \exp(Ax + B) \cdot g(x),$$

then $(\ln f)' = (\ln c + Ax + B + \ln g)' = A + (\ln g)'$, and $f \overset{''}{\sim} g$. We denote $(\ln f)''$ by $\ell''$.

*Remark 1.6.* If the conditions $\lim_{x\to s^-} f^2(x)/f'(x) = 0$ and (5) [or (5')] are fulfilled for some density $f$, then

$$0 < [1 + (f(x)/f'(x))']^{-1} < 1$$

and, by the Remark 1.4, we get

$$\Psi(s^-) \in [0,1].$$

*Remark 1.7.* The assertions of Theorem 1 do not remain true, if we replace the condition (5) by the weaker one

$$(f/f')' \geq 0, \quad x \in (m,s)$$

because, for the exponential distribution $f(x) = \lambda\cdot\exp(-\lambda\cdot(x-a))$, $x > a$, we have $f/f' = -\lambda^{-1}$ and $\Psi \equiv 1$. By Lemma 2, the RIF $h$ increases and decreases at the same time, thus $h = \text{const}$. [In other words, by the Remark 1.5, $f \overset{=}{\sim} g \equiv 1$, so $(\ln g)'' \equiv 0$ for each $x$.]

In addition, this feature of the exponential distribution is characteristic. [If $h(x) \equiv \text{const.}$, then $\Psi(x) \equiv 1$, i.e., $(G'^2 - G\cdot G'')/G'^2 = 0$, where $G := 1 - F(x)$. Hence $(G/G')' = 0$, $G'/G = c$, $\ln G = c\cdot(x-a)$ and $F = 1 - \exp(c\cdot(x-a))$. From the requirement $\lim_{x\to\infty} F(x) = 1$ it follows that $c = -\lambda$, $\lambda > 0$.]

*Remark 1.8.* The value of $\Psi(r^+)$ can be determined easily: $\Psi(r^+) = -\lim_{x\to r^+} f'(x)/f^2(x)$.

Let us apply our results to 11 distributions as follows.

*Example 1.* $F(x) = \sin(x)$; $I = (0, \pi/2)$; $(f/f')' = \sin^{-2} x > 0$ in $I$; $m = 0 \notin I$.

Remark 1.4 applies:

$$\lim_{x\to\pi/2^-} (f^2/f') = \lim_{x\to\pi/2^-} (-\cos^2 x/\sin x) = 0,$$

thus

$$\Psi(\pi/2^-) = \lim_{x\to\pi/2^-} [1 + \sin^{-2} x]^{-1} = 1/2.$$

Remark 1.2 applies to give $\Psi < 1$ in $I$, and $h$ strictly increases in $I$.

*Example 2.* $F(x) = 2 - \operatorname{ch} x$; $I = (\ln(2 - 3^{1/2}), 0)$; $m = r$; $(f/f')' = 4 \cdot e^{2x} \cdot (1 + e^{2x})^{-2} > 0$, everywhere.

Remark 1.4 applies:

$$\lim_{x \to 0^-} (f^2/f') = \lim_{x \to 0^-} (-\operatorname{sh}^2 x / \operatorname{ch} x) = - \lim_{x \to 0^-} \operatorname{sh}^2 x = 0,$$

thus

$$\Psi(0^-) = \lim_{x \to 0^-} [1 + 4 \cdot e^{2x}/(1 + e^{2x})^2]^{-1} = 1/2.$$

Remark 1.2 applies to give that the RIF $h$ strictly increases in $I$.

*Example 3.* $F(x) = (1 - e^{-\lambda x})^k$, $\lambda > 0$, $k > 1$.

Conditions of Theorem 1 are fulfilled with $I = (0, \infty)$;

$$f'(x) = 0 \text{ iff } e^{\lambda \cdot x} = k, \text{ so } m = (\ln k)/\lambda \in I;$$

$$f'(x) > 0 \text{ iff } k > e^{\lambda \cdot x}, \quad \text{i.e. } x \in (0, m).$$

Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$; now we have $f \overset{\shortparallel}{\sim} g = (1 - e^{-\lambda x})^{k-1}$, so $\ell'' = -\lambda^2 \cdot (k-1) e^{\lambda x} \cdot (e^{\lambda x} - 1)^{-2} < 0$, $x \in I$.

Remark 1.6 applies:

$$\lim_{x \to \infty} (f^2/f') = k \cdot \lim_{x \to \infty} e^{-\lambda x} \cdot (1 - e^{-\lambda x})^k \cdot (k \cdot e^{-\lambda x} - 1)^{-1} = 0,$$

so $\Psi(\infty) \in [0, 1]$, and the RIF $h$ strictly increases in $I$.

*Example 4.* $F(x) = 1 - \exp(-\lambda \cdot e^x)$, $\lambda > 0$.

Theorem 1 applies with $I = \mathbb{R}$; $m = -\ln \lambda (\in I)$;

$$f'(x) > 0 \text{ iff } 1 > \lambda \cdot e^x \quad \text{i.e. } x \in (-\infty, m).$$

Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$.

Remark 1.5 gives $f \overset{\shortparallel}{\sim} g = \exp(-\lambda \cdot e^x)$, and $\ell'' = (-\lambda \cdot e^x)'' = -\lambda \cdot e^x < 0$ in $\mathbb{R}$.

We apply Remark 1.6: $\lim_{x \to \infty} (f^2/f') = \lambda \cdot \lim_{x \to \infty} \exp(-\lambda \cdot e^x) \cdot (e^{-x} - \lambda)^{-1} = 0$, so we have $\Psi(\infty) \leq 1$, and the RIF $h$ strictly increases in $I$. (Actually, $\Psi(x) \equiv 1 - e^{-x}/\lambda < 1$).

*Example 5.* $F(x) = (1 + e^{-x})^{-k}$, $k > 0$. [9, Chap. 12, Sec. 4.5].

Theorem 1 applies: $I = \mathbb{R}$; $m = \ln k (\in I)$;

$$f'(x) > 0 \text{ iff } k \cdot e^{-x} > 1, \text{ i.e. } x \in (-\infty, m).$$

Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$; $(f/f')' = [(1 + e^x)/(k - e^x)]' = (k+1) \cdot e^x \cdot (k - e^x)^{-2} > 0$, $x \in \mathbb{R} \setminus \{m\}$.

Remark 1.6 applies: $\lim_{x \to \infty} (f^2/f') = k \cdot \lim_{x \to \infty} (1 + e^{-x})^{-k} \cdot (k - e^x)^{-1} = 0$, thus $\Psi(\infty) \in [0, 1]$, and the RIF $h$ strictly increases in $I = \mathbb{R}$.

(Actually, $\Psi(\infty) = \lim_{x \to \infty} [1 + (k+1) \cdot e^x \cdot (k - e^x)^{-2}]^{-1} = 1$.)

*Example 6.* $F(x) = 2^{-k}(1 + \operatorname{th} x)^k$, $k > 0$. [9, Chap. 12, Sec. 4.5].

Theorem 1 applies: $I = \mathbb{R}$; $m = (\ln k)/2 (\in I)$; $f'(x) > 0$ iff $e^{2x} < k$, i.e. $x \in (-\infty, m)$. Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$.

Remark 1.5 applies: $f \overset{||}{\sim} g = (1 + \operatorname{th} x)^{k-1}/(e^x + e^{-x})^2$, thus $\ell'' = 2(k + 1) \cdot [(e^{2x} + 1)^{-1}]' = -4 \cdot (k + 1) \cdot e^{2x} \cdot (e^{2x} + 1)^{-2} < 0$, $x \neq 0$.

Remark 1.6 applies: $\lim_{x\to\infty}(f^2/f') = k \cdot 2^{-k} \lim_{x\to\infty}[(1+\operatorname{th} x)^k/(k - e^{2x})] = 0$, thus $\Psi(\infty) \in [0, 1]$, and the RIF $h$ strictly increases in $\mathbb{R}$.

*Example 7.* Logistic distribution $F(x) = (1 + e^{-\lambda x})^{-1}$, $\lambda > 0$.

The conditions of Theorem 1 are fulfilled: $I = \mathbb{R}$; $m = 0 \in I$;

$$f'(x) > 0 \text{ iff } e^{-\lambda x} > 1 \text{ iff } x < 0 \text{ i.e. } x \in (-\infty, 0); \quad f'(x) < 0 \text{ iff } x \in (0, \infty);$$

$f''$ exists in $(-\infty, 0) \cup (0, \infty)$.

Remark 1.5 applies, since $f \overset{||}{\sim} g = (1 + e^{-\lambda x})^{-2}$, and $\ell'' = 2\lambda \cdot [e^{-\lambda x} \cdot (1 + e^{-\lambda x})^{-1}]' = -2\lambda \cdot e^{\lambda x} \cdot (e^{\lambda x} + 1)^{-2} < 0$ if $x \in (0, \infty)$.

Remark 1.6 applies, since $\lim_{x\to\infty}(f^2/f') = \lim_{x\to\infty}(e^{-\lambda x} - e^{\lambda x})^{-1} = 0$, so $\Psi(\infty) \leq 1$, thus the RIF $h$ strictly increases in $I$. (Actually, $\Psi(x) = 1 - \exp(-\lambda x) < 1$ in $\mathbb{R}$.)

On the other hand, the compression of the $x$-axis does not change asymptotic behavior of $\Psi$ and monotonic properties of the RIF, so one can consider the logistic distribution as a special case of that in Example 5.

*Example 8.* Fisher's $z$-distribution

$$f(x) = C \cdot e^{nx} \cdot (1 + k \cdot e^{2x})^{-\alpha}, \text{ where } k := n/n', \ \alpha := (n + n')/2 \ (> 0),$$

$(0 <)C := 2 \cdot k^{n/2} \cdot \Gamma(\alpha) \cdot [\Gamma(n/2) \cdot \Gamma(n'/2)]^{-1}$ and $n, n'$ are positive integers. Theorem 1 applies:

$I = \mathbb{R}$; $m = 0 \in I$;

$f'(x) > 0$ iff $e^{2x} < 1$ i.e. $x \in (-\infty, 0)$. Similarly, $f'(x) < 0$ iff $x \in (0, \infty)$.

Remark 1.5 applies to give

$$f \overset{||}{\sim} g = (1 + k \cdot e^{2x})^{-\alpha}, \quad \text{and}$$
$$\ell'' = -2\alpha k \cdot [e^{2x}/(1 + k \cdot e^{2x})]' = -4\alpha k \cdot e^{2x}(1 + k \cdot e^{2x})^{-2} < 0, \quad x \in (0, \infty).$$

Remark 1.6 applies as well, since

$$\lim_{x\to\infty}(f^2/f') = C/n \cdot \lim_{x\to\infty}(e^{-2x} + k) \cdot e^{nx} \cdot (e^{-2x} - 1)^{-1} \cdot (1 + k \cdot e^{2x})^{-\alpha}$$
$$= -k \cdot C/n \cdot \lim_{x\to\infty} e^{nx} \cdot (1 + k \cdot e^{2x})^{-\alpha} = 0, \text{ thus } \Psi(\infty) \leq 1,$$

and the RIF $h$ strictly increases in $I$. (Actually, it can be shown that $\Psi(\infty) = 1$.)

*Example 9.* Weilbull distribution when $\alpha > 1$; $F(x) = 1-\exp(-\lambda \cdot x^\alpha)$, $\lambda > 0$.

Theorem 1 applies: $I = (0, \infty)$; $f'(x) = 0$ iff $x^\alpha = (\alpha - 1)/(\lambda\alpha)$, so $m = [(\alpha-1)/(\lambda\alpha)]^{1/\alpha} \in I$; $f'(x) > 0$ iff $\alpha - 1 > \lambda\alpha \cdot x^\alpha$, i.e. $x \in (0, m)$. Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$.

Remark 1.5 applies: $f \overset{!!}{\sim} g = x^{\alpha-1} \cdot \exp(-\lambda \cdot x^\alpha)$, and $\ell'' = (1-\alpha) \cdot x^{-2} \cdot (1 + \lambda\alpha \cdot x^\alpha) < 0$.

Remark 1.6 applies:

$$\lim_{x\to\infty} (f^2/f') = \lambda\alpha \cdot \lim_{x\to\infty} \exp(-\lambda \cdot x^\alpha)/[(\alpha-1) \cdot x^{-\alpha} - \lambda\alpha] = 0,$$

thus $\Psi(\infty) \in [0, 1]$, and the RIF $h$ strictly increases in $I$.

(Actually, $\Psi(x) = 1 - (\alpha-1)/(\lambda\alpha \cdot x^\alpha) < 1$ in $I$.)

*Example 10.* (Extreme value distribution)
$f(x) = \exp(-x - e^{-x})$. [11, §5.47, p. 192]

Theorem 1 applies: $I = \mathbb{R}$; $m = 0 (\in I)$;

Remark 1.5 applies: $f \overset{!!}{\sim} g = \exp(-e^{-x})$, and $\ell'' = -e^{-x} < 0$.

Remark 1.6 applies: $\lim_{x\to\infty} f^2/f' = \lim_{x\to\infty} (e^{-x} - 1)^{-1} \cdot \exp(-x - e^{-x}) = 0$, we have $\Psi(\infty) \in [0, 1]$, and the RIF $h$ strictly increases in $I$.

*Example 11.* $F(x) = 1 - 2 \cdot [c \cdot (1 + e^x)^k - c + 2]^{-1}$; $c, k > 0$ (Sec. 4.5, Chap. 12 in [9]).

Theorem 1 applies if $k = 1$ or ($k = 2$ and $0.0122 \le c \le 0.3125$).

*Example 11.1.* $k = 1$. $I = \mathbb{R}$; $m = \ln 2 - \ln c \in I$; $f'(x) > 0$ iff $2 > c \cdot e^x$ i.e. $x \in (-\infty, m)$; similarly, $f' < 0$ iff $x \in (m, \infty)$.

Remark 1.5 applies: $f \overset{!!}{\sim} g = (2 + c \cdot e^x)^{-2}$, thus

$$\ell'' = 4 \cdot [(2 + c \cdot e^x)^{-1}]' = -4c \cdot e^x \cdot (2 + c \cdot e^x)^{-2} < 0.$$

Remark 1.6 applies: $\lim_{x\to\infty} (f^2/f') = 2c \cdot \lim_{x\to\infty} e^x \cdot (2 - c \cdot e^x)^{-1} \cdot (2 + c \cdot e^x)^{-1} = 0$, thus $\Psi(\infty) \in [0, 1]$ and the RIF $h$ strictly increases in $I$.

*Example 11.2.* Let $k = 2$ and $0.0122 \le c \le 0.3125$. Theorem 1 applies.

(1): $I = \mathbb{R}$;

(2): $f'(x) = 0$ iff $t(x) = 0$, where $t(x) := T(e^x)$ and $T(y) := 2 + 2 \cdot (2 - c) \cdot y - 3c \cdot y^2 - 2c \cdot y^3$, $y = e^x > 0$. We have $T(2) = 10 - 32c \ge 0$, $T(\infty) = -\infty$. $T'(y)$ has the zeros $y_1$, $y_2$, with

$$y_1 < 0 < y_2 = ([(8-c)/(3c)]^{\frac{1}{2}} - 1)/2.$$

The point of inflexion of $T$ is at $-\frac{1}{2}$, and $T\left(-\frac{1}{2}\right) = c/2 > 0$. It follows that $T$ is concave down in $\left(-\frac{1}{2}, \infty\right)$ and has a unique positive zero $y_0$ in $(2, \infty)$. Furthermore, $m = \ln y_0 > \ln 2$ is the mode of $f$, since $f'(m) = T(y_0) = 0$.

(3): We have $t(x) > 0$ and $f'(x) > 0$ in $(-\infty, m)$, since $T(y) > 0$ for $y \in (0, y_0)$. Similarly, $f'(x) < 0$ in $(m, \infty)$, since $T < 0$ in $(y_0, \infty)$.

Remark 1.5 applies: $f \overset{\shortparallel}{\sim} g = (1 + e^x) \cdot [v(x)]^{-2}$, where $v(x) := 2 + 2c \cdot e^x + c \cdot e^{2x}$. Hence,

$$\ell'' = [e^x/(1 + e^x) - 4c \cdot (e^x + e^{2x})/v(x)]' < 0$$

iff $e^x/(1 + e^x)^2 < 4c \cdot e^x \cdot (2 + 4 \cdot e^x + c \cdot e^{2x})/[v(x)]^2$, i.e. $0 < W$, where $W := 4 \cdot (2c - 1) + 24c \cdot e^x + 36c \cdot e^{2x} + 4c \cdot (4 + c) \cdot e^{3x} + 3c^2 \cdot e^{4x}$.
If we replace $c$ by 0.0122 and $e^x$ by 2, then $W$ decreases since $e^x > e^m = y_0 > 2$, and we get

$$W > 4 \cdot (20c^2 + 82c - 1) > 4 \cdot 0.0034 > 0,$$

which means that (5') is fulfilled in $(m, \infty)$.

Remark 1.6 applies as well:

$$\lim_{x \to -\infty} (f^2/f') = 4c \cdot \lim_{x \to -\infty} e^x \cdot (1 + e^x)^2 \cdot [v(x) \cdot t(x)]^{-1} = 0,$$

thus $\Psi(\infty) \in [0, 1]$, and the RIF $h$ strictly increases in $I$.

**Theorem 2.** *Let $f$ be a density function with* (1), (3–4), $m = r$ *and*

(9) $$(f/f')' < 0 \quad \textit{in } (m, s).$$

*Then $r$ is finite, and*

(10) *if* $\Psi(r^+) < 1$ *or*

$[\Psi(r^+) = 1$ *and* $\Psi < 1$ *in some right neighborhood of* $r]$,

*then $\Psi < 1$ in $I$, and the corresponding* RIF *strictly increases in $I$;*

(11) *if* $\Psi(r^+) > 1$,

*then*

(11.1) *if* $\Psi(s^-) \geq 1$,
*then $\Psi > 1$ and the* RIF *strictly decreases in $I$;*

(11.2) *if* $\Psi(s^-) < 1$,
*then $\Psi > 1$ in $(r, y)$ and $\Psi < 1$ in $(y, s)$ for some $y \in I$,*

*thus the* RIF *strictly decreases first and, after reaching its local minimum, strictly increases.*

Proof. By (4), $f'$ is continuous in $(m, s) = I$, and so is $\Psi$. Since $f' < 0$ in $I$, $f$ decreases and the value of $r$ must be finite.

Suppose (10) holds. Then $\Psi < 1$ in $(r, m+p)$ for some $p > 0$. Let $x_0 \in (m, m+p)$. Then $\Psi(x_0) < 1$, i.e.

$$[F(x_0) - 1] \cdot f'(x_0) < f^2(x_0). \tag{12}$$

We will prove that $\Psi'(x_0) < 0$. We assume that $\Psi'(x_0) \geq 0$, i.e.,

$$[f^2 \cdot f' + (2f'^2 - f \cdot f'') \cdot (1 - F)] \mid_{x=x_0} \geq 0 \tag{13}$$

By virtue of (12), the left-hand side in (13) will increase, if we replace $f^2$ in the first term by $[F(x_0) - 1] \cdot f'(x_0)$:

$$(1 - F) \cdot (f'^2 - f \cdot f'') \mid_{x=x_0} \geq 0,$$

which is equivalent to $f'^2(x_0) \geq f(x_0) \cdot f''(x_0)$, i.e., $(1 - f \cdot f''/f'^2) \mid_{x=x_0} = (f/f')' \mid_{x=x_0} \geq 0$, which contradicts (9). Thus, $\Psi'(x_0) < 0$, i.e. $\Psi$ strictly decreases at $x_0$. Consequently, $\Psi$ will be a strictly decreasing function in $(m, s)$, since it is continuous there.

Hence, $\Psi < 1$ in the entire interval $I$, and the corresponding RIF $h$ strictly increases in $I$.

Roughly speaking, the main idea of the proof has been the following: *if* $\Psi(x_0) < 1$, *then* $\Psi$ *strictly decreases at each* $x \in [x_0, s)$.

From this it follows that, if (11) and (11.1) are fulfilled, then there is no $x_0 \in I$ with $\Psi(x_0) \leq 1$. Furthermore, if (11) and (11.2) hold, then there is a unique $y \in (r, s)$ with $\Psi(y) = 1$. Thus $\Psi > 1$ in $(r, y)$ and $\Psi < 1$ in $(y, s)$. Then the corresponding RIF strictly decreases in $(r, y)$ and, after taking its local minimum at $y$, strictly increases. □

*Remark 2.1.* There is no density function $f$ with (1–4) and (9) because, for some $v > 0$, we have $f'' < 0$ in $U = (m, m+v)$, so $f \cdot f''/f'^2 < 1$ and $(f/f')' > 0$ in $U$.

*Remark 2.2.* Similarly as in Remark 1.5, the condition (9) can be formulated as follows:

$$\ell'' := (\ln f)'' > 0 \quad \text{in } (m, s). \tag{9'}$$

Let us check a few examples in which the Theorem 2 and Remark 2.2 apply.

*Example 12.* $F(x) = 1 + \text{sh } x$, $I = (\ln(2^{\frac{1}{2}} - 1), 0)$, $m = r = \ln(2^{\frac{1}{2}} - 1)$;

$$\ell'' = (\ln(\text{ch } x))'' = (\text{th } x)' = \text{sech}^2 x > 0 \text{ in } I;$$

(10): $\Psi(r^+) = \lim_{x \to r^+} \text{th}^2 x = [\text{th}(\ln(2^{\frac{1}{2}} - 1))]^2 = \frac{1}{2} < 1$, thus the RIF strictly increases in $I$. (Actually, $\Psi(x) = \text{th}^2 x < 1$ in $I$.)

*Example 13.* $F(x) = 1 + \tan x$, $I = (-\pi/4, 0)$, $m = r = -\pi/4$;

$$\ell'' = -2 \cdot (\ln \cos x)'' = 2/\cos^2 x > 0 \text{ in } I;$$

(10): $\Psi(-\pi/4^+) = \lim_{x \to -\pi/4^+} 2 \sin^2 x = 1$ and $\Psi(x) = 2 \cdot \sin^2 x < 1$ in $I$, so the RIF strictly increases in $I$. (The inequality $\Psi(x) < 1$ itself implies that $h$ strictly increases.)

*Example 14.* $F(x) = 1 - (\ln x)^{-\lambda}$, $\lambda > 0$, $I = (e, \infty)$, $m = e(= r)$;

$$f \overset{\shortparallel}{\sim} g = (\ln x)^{-\lambda-1}/x, \ \ell'' = -[\ln x + (\lambda + 1) \cdot \ln(\ln x)]''$$
$$= [1 + (\lambda + 1) \cdot (1 + \ln x)/(\ln x)^2]/x^2 > 0, \quad x \in I;$$

(11): $\Psi(e^+) = 1 + 2/\lambda > 1$;
(11.1): $\Psi(\infty) = \infty > 1$, and the RIF strictly decreases in $I$.

*Example 15.* $F(x) = 1 - \exp(-\lambda \cdot x^\alpha)$, $\lambda > 0$, $\alpha \in (0, 1)$, Weilbull distribution.

$$I = (0, \infty); \ m = 0(= r); \ f \overset{\shortparallel}{\sim} g = x^{\alpha-1} \cdot \exp(-\lambda \cdot x^\alpha),$$
$$\ell'' = [(\alpha - 1) \cdot \ln x - \lambda \cdot x^\alpha]'' = (1 - \alpha) \cdot (x^{-2} + \lambda\alpha \cdot x^{\alpha-2}) > 0, \quad x \in I;$$

(11): $\Psi(0^+) = \infty > 1$;
(11.1): $\Psi(\infty) = 1$, thus the RIF strictly decreases in $I$.

*Example 16.* $F(x) = 1 - a \cdot \exp(-bx) - c \cdot \exp(-dx)$; $a, b, c, d > 0$;

$$b \neq d; \quad a + c = 1, \quad I = (0, \infty), \quad m = 0(= r);$$
$$\ell'' = (\ln[ab \cdot \exp(-bx) + cd \cdot \exp(-dx)])''$$
$$= abcd \cdot (b - d)^2 \cdot \exp(-bx - dx)/[ab \cdot \exp(-bx)$$
$$+ cd \cdot \exp(-dx)]^2 > 0 \text{ for all } x \in \mathbb{R};$$

(11): $\Psi(0^+) = [A + B \cdot (b^2 + d^2)]/[A + B \cdot (2bd)] > 1$ where $A = a^2b^2 + c^2d^2$, $B = ac$, since $(b - d)^2 > 0$ and $b^2 + d^2 > 2bd$;
(11.1): $\Psi(\infty) = 1$, thus the RIF strictly decreases in $I$.

*Example 17.* $F(x) = 1-k\cdot\exp(-bx)\cdot x^{-a}$; $a,b,k>0$ (Pareto distribution of the third kind in Sec. 2, Chap. 19 of [9]). $I=(k,\infty)$; $m=k(=r)$;

$$f \overset{\shortparallel}{\sim} g = (a+bx)\cdot x^{-a-1}, \quad \ell'' = [\ln(a+bx)-(a+1)\cdot\ln x]''$$
$$= a\cdot x^{-2}\cdot(a+bx)^{-2}\cdot[(a+1)\cdot(a+2bx)+b^2x^2] > 0, \quad x\in I;$$

(11): $\Psi(k^+) = 1 + a/(a+bk)^2 > 1$;
(11.1): $\Psi(\infty)=1$, and the RIF $h$ strictly decreases in $I$.

*Remark 2.3.* By the Remark 0.1 we can say that the distributions in Examples 1–13 are IHR (increasing hazard rate), while the ones in Examples 14–17 are DHR (decreasing hazard rate) distributions.

*Remark 2.4.* We wish to emphasize that the method applied in this paper, particularly in Theorems 1 and 2, enables us to eliminate the inconvenient term
$$1-F(x) = \int_x^\infty f(t)dt,$$
appearing in each of the relative increment functions, the hazard rate (or failure rate) and the auxiliary function $\Psi$. Instead, our method deals with the relatively simple expression $f/f'$.

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