A note on almost-Dedekind domains

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1. Introduction

In this section we state the necessary background results from [1]. In the remainder of this paper, R will denote a commutative integral domain with identity and quotient field K. $I(R)$ will denote the collection of non-zero fractionary ideals of R. A fractionary ideal of the form Rx, $x \neq 0$, $x \in K$, is called a principal fractionary ideal.

A relation $\prec$ is defined on $I(R)$ as follows: $A \prec B$ iff every principal fractionary ideal of R which contains A also contains B. The relation $\prec$ is reflexive and transitive. If we define $\equiv$ on $I(R)$ by $A \equiv B$ iff $A \prec B$, and $B \prec A$, for $A, B \in I(R)$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\mathrm{div}_R(A)$ denotes the equivalence class of A with respect to $\equiv$ and is called the divisor of A; $\mathscr{D}(R)$ denotes the set of all such equivalence classes.

For $A \in I(R)$, we put $\tilde{A} = \bigcap_{A \subseteq Rx} Rx$, $x \in K$, $x \neq 0$. A fractionary ideal B of R is said to be divisorial if $\tilde{B} = B$. It follows that for $A \in I(R)$, $\mathrm{div}_R(A) = \mathrm{div}_R(\tilde{A})$ and that $\tilde{A}$ is the unique divisorial fractionary ideal belonging to $\mathrm{div}_R(A)$. It also follows from the definition that $\widetilde{\tilde{A}\tilde{B}} = \widetilde{AB}$ for A, $B \in I(R)$, so that $\mathscr{D}(R)$, together with the operation $+$ defined by $\mathrm{div}_R(A) + \mathrm{div}_R(B) = \mathrm{div}_R(AB)$ is a commutative semigroup with identity $0 = \mathrm{div}_R(R)$. Furthermore, $\mathscr{D}(R)$ is a group iff R is completely integrally closed. If we define $\leqq$ on $\mathscr{D}(R)$ by $\mathrm{div}_R(A) \leqq \mathrm{div}_R(B)$ iff $A \prec B$, then $\mathscr{D}(R)$ is a lattice-ordered semigroup.

2. An application of $\mathscr{D}(R)$

It is shown in [1] that the proper prime ideals in a Dedekind domain are divisorial. We now apply the general theory of $\mathscr{D}(R)$ to show that this property characterizes. Dedekind domains in the class of one-dimensional Prüfer domains.

Theorem 2. 1. *Let R be a Prüfer domain in which proper prime ideals are maximal. Then R is a Dedekind domain iff every minimal pirme ideal of R is divisorial.*

PROOF. ($\Leftarrow$) Since R is a one-dimensional Prüfer domain, R_M is a rank one valuation ring for each maximal ideal M of R. By [5, page 94], $R = \bigcap R_M$, where M runs over the collection of maximal ideals of R. Since R_M is completely integrally closed for each maximal ideal M of R, it follows that R is completely integrally closed, so that $\mathscr{D}(R)$ is a group (see [1]). Let P be any non-zero maximal (hence minimal) prime of R. Then $P(R:P) \subseteqq R$. If $P(R:P) \neq R$, then $P(R:P) \subseteqq M$ for some maximal ideal M of R. Then $P^2 \subseteqq P(R:P) \subseteqq M$; i.e., $P^2 \subseteqq M$. Thus $P \subseteqq M$, and so $P = M$ since both M and P are minimal (maximal). So we have $P(R:P) \subseteqq P$ if $P(R:P) \neq R$. But then in $\mathscr{D}(R)$ we have $0 = \operatorname{div}_R(P(R:P)) \geqq \operatorname{div}_R(P) > 0$, a contradiction ($\operatorname{div}_R(P) > 0$ since $P \subseteqq R$, and $\tilde{P} = P < R$). So we must have $P(R:P) = R$; i.e., P is invertible and hence finitely generated. Thus R is a ring in which every prime ideal is finitely generated and so by Cohen's theorem, [6, page 8] (see also [7]), R is Noetherian. We already have that R is integrally closed and that prime ideals are maximal. It follows that R is a Dedekind domain.

($\Rightarrow$) See [1].

In [4], Gilmer defines an almost-Dedekind domain (AD-domain) to be a domain R such that R_M is a Dedekind domain for each maximal ideal M of R. It follows that if R is an almost-Dedekind domain then R is a Prüfer domain in which prime ideals are maximal. We can now state the following corollary.

Corollary 2.2. Let R be an almost-Dedekind domain. Then R is Dedekind iff proper prime ideals of R are divisorial.

3. families of valuations and the construction of $\mathscr{A}(R)$

In this section we obtain a characterization of an AD-domain R in terms of a family F of valuations on K. The family is used to construct a partially ordered semigroup $\mathscr{A}(R)$ of fractionary ideal classes and the relation between $\mathscr{A}(R)$ and $\mathscr{D}(R)$ is described.

Definition 3.1. Let v be a rank one, discrete valuation on K which is non-negative on R. For any fractionary ideal A of R, put $v(A) = \inf_{a \in A} v(A)$.

Theorem 3.2. *R is AD iff there is a family F of valuations on K such that.*

(i) *Each $v \in F$ has rank one and is discrete.*
(ii) $R = \bigcap_{v \in F} R_v$.
(iii) *$R_v = R_{P(v)}$, where $P(v)$ denotes the center of v on R for each $v \in F$.*
(iv) *R is the only ideal A of R such that $v(A) = 0$ for all $v \in F$.*

Proof. Suppose F is a family of valuations on K which satisfies (i), (ii), (iii), (iv) above, and let P be any proper prime ideal of R. By (ii), $v(P) \geqq 0$ for all $v \in F$. By (iv) there is $v \in F$ such that $v(P) > 0$. Then $(0) < P \subseteqq P(v) < R$, where $P(v)$ denotes the center of v on R. Since $P(v)$ is the center of a rank one valuation, $P(v)$ is a minimal prime in R. Thus $P = P(v)$ and $R_P = R_{P(v)} = R_v$ is a rank one, discrete valuation ring. It follows that R is AD.

Suppose R is AD. Let F be the family of valuations on K induced by the family of proper primes of R. It is easy to see that F satisfies (i), (ii), (iii). To see that F satisfies (iv), let A be any proper ideal of R. Then $A \subseteqq M$ for some maximal (and hence minimal prime) ideal M of R. If v denotes the valuation on K induced by

M, then $v \in F$. Furthermore, $v(A) \geqq v(M) > 0$. Thus R must be the only ideal A of R such that $v(A) = 0$ for all $v \in F$, and (iv) holds.

We note that condition (iv) insures that proper prime ideals are maximal and that valuation rings R_v are identical with the quotient rings R_P where P runs over all minimal primes of R. We call F the family of essential valuations of R. In the remainder of this section, R denotes an AD-domain, F the family of essential valuations of R.

We state the following lemmas:

Lemma 3. 3. *If $A, B \in I(R)$, $v \in F$, then $v(AB) = v(A) + v(B)$.*

Lemma 3. 4. *For any principal fractionary ideal Rx and any $v \in F$, $v(Rx) = v(x)$.*

Now, for $A, B \in I(R)$, define $A \sim B$ iff $v(A) = v(B)$ for all $v \in F$. Then $\sim$ is an equivalence relation on $I(R)$. For $A \in I(R)$ we let $[A]$ denote the equivalence class of A with respect to $\sim$ and we let $\mathscr{A}(R)$ denote the set of all such equivalence classes. We define $+$ on $\mathscr{A}(R)$ by $[A] + [B] = [AB]$, and we define $\leqq$ on $\mathscr{A}(R)$ by $[A] \leqq [B]$ iff $v(A) \leqq v(B)$ for each $v \in F$. With these definitions $\mathscr{A}(R)$ is a commutative, partially ordered semigroup with identity $0 = [R]$.

Lemma 3. 5. *Let $A \in I(R)$. Then, considering $[A]$ and $\operatorname{div}_R(A)$ as subsets of $I(R)$, $[A] \subseteqq \operatorname{div}_R(A)$.*

Proof. Let $B \in [A]$. Then $v(B) = v(A)$ for each $v \in F$. If $A \subseteqq Rx$, then $v(B) = v(A) \geqq v(Rx) = v(x)$ for each $v \in F$. Thus if $b \in B$, $v(b) - v(x) \geqq 0$ for all $v \in F$; i.e., $v\left(\frac{b}{x}\right) \geqq 0$ for all $v \in F$, so $\frac{b}{x} \in \bigcap_{v \in F} R_v = R$, and hence $B \subseteqq Rx$. Similarly, if $B \subseteqq Ry$ then $A \subseteqq Ry$. Then $\tilde{A} = \tilde{B}$ and $\operatorname{div}_R(A) = \operatorname{div}_R(B)$, But then $B \in \operatorname{div}_R(A)$.

Proposition 3. 6. *The map $g: \mathscr{A}(R) \to \mathscr{D}(R)$, defined by $g([A]) = \operatorname{div}_R(A)$, is an order-preserving homomorphism of the partially ordered semigroup $\mathscr{A}(R)$ onto the lattice-ordered group $\mathscr{D}(R)$.*

Proof. Lemma 3. 5 shows that g is well-defined and onto. It is easy to check that g is a homomorphism. To see that g preserves order, recall that $\operatorname{div}_R(A) \leqq \operatorname{div}_R(B)$ iff $A \prec B$. We shall show that if $[A] \leqq [B]$, then $A \prec B$. Thus suppose $[A] \leqq [B]$ and let $A \subseteqq Rx$. Then $v(B) \geqq v(A) \geqq v(x)$ for each $v \in F$ since $[A] \leqq [B]$. As in the proof of 3. 5 we have $B \subseteqq Rx$ so that $A \prec B$.

We can now prove the following theorem.

Theorem 3. 7. *Let R be an AD-domain with family F of essential valuations. The following statements are equivalent.*

(1) *Every proper prime ideal of R is divisorial.*

(2) *Every fractionary ideal of R is divisorial.*

(3) *R is a Dedekind domain.*

(4) *$\mathscr{A}(R)$ is a group.*

(5) *The map g of $\mathscr{A}(R)$ onto $\mathscr{D}(R)$ is an isomorphism.*

Proof. (1)$\Leftrightarrow$(3). This is corollary 2. 2.

(2)$\Rightarrow$(5) If every fractionary ideal is divisorial, then for $A \in I(R)$ we have $\operatorname{div}_R(A) = \{A\}$. But $A \in [A] \subseteqq \operatorname{div}_R(A) = \{A\}$. It follows that, considered as sets, $[A] = \operatorname{div}_R(A)$ so that g is 1—1, hence an isomorphism.

(5)$\Rightarrow$(4) Clear since $\mathscr{D}(R)$ is a group.

(4)$\Rightarrow$(3) Suppose $\mathscr{A}(R)$ is a group. Then if $[A]\in\mathscr{A}(R)$, there is $[B]\in\mathscr{A}(R)$ such that $[A]+[B]=0$; i.e., $[AB]=[R]$. Then $v(AB)=0$ for all $v\in F$ and so by theorem 3. 2, $AB=R$. Thus A is invertible and hence has a finite basis so that R is Noetherian and therefore Dedekind.

(3)$\Rightarrow$(2) This is found in [1], page 23.

4. Some further examples of AD-domains

Let R be an AD-domain with quotient field K and family F of essential valuations. Let L be a subfield of K and put $A=R\cap L$. We may suppose that L is the quotient field of A, for if T is the quotient field of A then $A\subseteqq T\subseteqq L$, and so $A\subseteqq R\cap T\subseteqq R\cap L=A$.

Proposition 4. 1. *If R is integral over A, then A is almost-Dedekind.*

PROOF. For $v\in F$, let v' denote the restriction of v to L. Let $F'=\{v'|v\in F$ and v is nontrivial on $L\}$. Thus if $v'\in F'$, then $v'(x)\neq 0$ for some non-zero element $x\in L$. It is clear that each $v'\in F'$ is a rank one discrete valuation since each $v\in F$ is a rank one discrete valuation. Since $R=\bigcap_{v\in F}R_v$ we have $A=R\cap L=(\bigcap_{v\in F}R_v)\cap L=\bigcap_{v\in F}(R_v\cap L)=$ $=\bigcap_{v'\in F}(R_v\cap L)$, for if $w\in F$ is trivial on L, then $L\subseteqq R_w$. For $v'\in F'$, let $A_{v'}=$ $=\{x\in L|v'(x)\geqq 0\}$, and let $Q(v')=\{y\in A|v'(y)>0\}$. Then $Q(v')=A\cap P(v)$, so $Q(v')$ is a non-zero minimal prime of A since $P(v)$ is a non-zero minimal prime of R and R is integral over A. We have $A_{Q(v')}=L\cap R_{P(v)}=L\cap R_v=A_{v'}$ for each $v'\in F'$. For if $\frac{x}{y}\in A_{Q(v)}$, then $v'\left(\frac{x}{y}\right)\geqq 0$; i.e., $\frac{x}{y}\in L\cap R_v=L\cap R_{P(v)}=A_{v'}$. Thus $A_{Q(v')}\subseteqq A_{v'}$. On the other hand, let $\frac{x}{y}\in A_{v'}$, $x,y\in A$. Then $v'\left(\frac{x}{y}\right)\geqq 0$. Since v' is a non-trivial, rank one discrete valuation on L, there is an irreducible element $\pi\in A_{v'}$, such that $\frac{x}{y}=\frac{\pi^n u}{w}$, where $n\geqq 0$ is an integer, $u,w\in A$ are such that $v(u)=0=v(w)$ and $\pi\in A$ without loss of generality. Then $w\in Q(v')$, and so $\frac{x}{y}\in A_{Q(v)}$; i.e., $A_{v'}=A_{Q(v')}$. Now let B be an ideal of A such that $v'(B)=0$ for all $v'\in F'$, where $v'(B)=\inf_{b\in B}v'(b)$, for each $v'\in F'$. Then $v(BR)=0$ for all $v\in F$. For if $v\in F$ is trivial on L, then $v(BR)=0$ clearly. If $v\in F$ is such that $v'\in F'$, then $v'(B)=v(BR)=0$. Thus BR is an ideal of R such that $v(BR)=0$ for all $v\in F$, and so $BR=R$. In this case, $B=A$. For if $B<A$, then $B\subseteqq M'$ for some maximal ideal M' of A. Since R is integral over A, there is a maximal ideal M' of R such that $M\cap A=M'$. Then $B\subseteqq M$, and so $BR\subseteqq M<R$, a contradiction. So we must have $B=A$. Thus F' is a family of valuations on L satisfying the conditions of theorem 3. 2.

Proposition 4. 2. *Let R be an AD-domain and let x be an indeterminate. Let S denote the multiplicatively closed set of $R[x]$ consisting of all monic polynomials. Then $(R[x])_S$ is AD.*

PROOF. Denote $(R[x])_S$ by R^1. *Let* $\mathscr{P}$ *be a prime ideal in* R^1, $(0) < \mathscr{P} < R^1$, *and consider* $R^1_{\mathscr{P}}$.

Case 1. $\mathscr{P} \cap R = (0)$. Then $K[x] \subseteqq R^1_{\mathscr{P}}$ and $K[x]$ is Dedekind. Hence $R^1_{\mathscr{P}}$ is Dedekind.

Case 2. $\mathscr{P} \cap R = P \neq (0)$. Then P is a prime ideal in R, and $\mathscr{P} \cap R[x] = PR[x]$. For if not, then $\mathscr{P} \cap S \neq 0$, and $\mathscr{P} = R^1$, a contradiction.

Let $M = R^1 - \mathscr{P}$. Then $M_1 = R[x] - PR[x] \subseteqq M$. Now $R[x]_P = R_P[x] \subseteqq R^1_{\mathscr{P}}$. Let S' denote the set of monic polynomials in $R_P[x]$. Then $(R_P[x])_{S'}$ is Dedekind since R_P is Dedekind by Lemma 2. 1 of [2]. We have $(R_P[x])_{S'} \subseteqq R^1_{\mathscr{P}}$. For, let $z \in (R_P[x])_{S'} = (R[x]_P)_{S'}$. Then

$$z = \frac{f(x)}{m} \Big/ x^n + \frac{a_{n-1}}{m_{n-1}} x^{n-1} + \dots + \frac{a_0}{m_0}, \text{ where } f(x) \in R[x], m, m_0, \dots, m_{n-1} \in R - P,$$

$$a_0, \dots, a_{n-1} \in R.$$

Then $z = \dfrac{f(x)}{m} \Big/ [(m'x^n + a'_{n-1}x^{n-1} + \dots + a'_0)/m']$, where $m' = m_0 m_1 \dots m_{n-1}$. Then $z = m'f(x)/(mm'x^n + \dots + ma'_0)$ and $g(x) = mm'x^n + \dots + ma'_0$ does not belong to $\mathscr{P}$ since $mm' \in R - P$, and so $g(x) \in R[x] - PR[x]$. Thus, $z \in R^1_{\mathscr{P}}$. Since $(R_P[x])_{S'} \subseteqq R^1_{\mathscr{P}}$ and $(R_P[x])_{S'}$ is Dedekind, $R^1_{\mathscr{P}}$ is also Dedekind.

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