On the regularity of the integrable solutions of the functional equations $\sum A_i f(\sum a_{ij} x_j) = \sum B_j f(x_j) + \sum C_{jk} g(x_j) g(x_k)$*)

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It is easy to see that the Cauchy functional equation

$$f(x+y) = f(x)+f(y),$$

Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2},$$

the equation

$$f(ax+y) = Af(x)+f(y) \qquad (aA \neq 0)$$

which was considered by Z. DARÓCZY [4] (cf. also J. ACZÉL [2], page 68), the equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

which general solution was found by J. ACZÉL and E. VINCZE ([3])**), the equations

$$f(x+y)+f(x-y) = 2f(x)+2f(y)+g(x)g(y)$$

and

$$f(x+y+z)+f(x+y-z)+f(x+y-z)+f(-x+y+z) =$$

$$= 4[f(x)+f(y)+f(z)+2[g(x)g(y)+g(x)g(z)+g(y)g(z)],$$

which are characteristic for the even polynomials of order 2 and 4 (cf. H. ŚWIATAK [10]), can be written generally as

$$\sum_{i=1}^{m} A_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) = \sum_{j=1}^{n} B_j f(x_j) + \sum_{\substack{j,k=1 \\ j \leq k}}^{n} C_{jk} g(x_j) g(x_k). \tag{1}$$

*) The results of this paper were presented without proofs during the II-nd International Colloquiumon Functional Equations in Miskolc, 30. IX. — 2. X. 1968.

**) This equation was considered already by Jensen [5], [6] but he found only continuous solutions.

The only regular solutions of the equations which suggested us to consider (1) are polynomials. However, it is not a property which could be characteristic for all the equations of the form (1) since e.g. the equation

$$f(x+y) = g(x)g(y)$$

is satisfied by the functions

$$f(x) = g(x) = e^{ax},$$

the equation

$$f(x-y)-f(x+y) = 2g(x)g(y)$$

by the functions

$$f(x) = \cos ax, \quad g(x) = \sin ax,$$

and the equation

$$f(x-y)-f(x+y) = -2f(y)+2g(x)g(y)$$

by the functions

$$f(x) = ax+\cos bx, \quad g(x) = \sin bx.$$

It is possible to formulate some assumptions on the coefficients A_i, B_j, C_{jk}, a_{ij} so that all the regular solutions of (1) were polynomials but this will be not done in this paper.

We are going to show that there exists another property which is characteristic for a larger class of equations (1) namely the regularity of all their locally integrable and locally bounded solutions f, g possessing primitive functions. Theorems which will be proved in this paper do not exhaust all the cases when equations (1) have the property mentioned above but they are rather general.

Lemma 1. *If the function*

$$\varphi(x, y) = x\left[af^{(\nu)}(y)+\frac{d^\nu}{dy^\nu}W\big(g(y)\big)\right]+bG(x)g^{(\nu)}(y),$$

where a, b are constants ($a\neq 0$, $b\neq 0$), $G'(x)=g(x)$, and $W(u)$ is a polynomial, has the first derivative with respect to y, then $f^{(\nu+1)}(y)$ and $g^{(\nu+1)}(y)$ exist.

Proof. If $G(x)=\alpha x$, then $g'(x)=\alpha$, $g^{(i)}(x)=0$ for $i=1, 2, \dots, \frac{d^\nu}{dy^\nu}W\big(g(y)\big)=$ $=\text{const}$ and the existence of $f^{(\nu+1)}(y)$ is obvious.

If $G(x)\neq\alpha x$, then there exist two constants $\bar{x}$, $\bar{\bar{x}}$ such that

$$\begin{vmatrix} \bar{x}, & G(\bar{x}) \\ \bar{\bar{x}}, & G(\bar{\bar{x}}) \end{vmatrix} \neq 0.$$

Therefore from the differentiability of the functions $\varphi(\bar{x}, y)$ and $\varphi(\bar{\bar{x}}, y)$ it follows that the functions $af^{(\nu)}(y)+\frac{d^\nu}{dy^\nu}W\big(g(y)\big)$ and $g^{(\nu)}(y)$ are differentiable.*) The dif-

*) A similar idea was used by R. Satô [9] (cf. also J. Aczél [1], [2] and J. H. B. Kemperman [7], [8]) to prove that if the functions f, g, h_i, k_i satisfying the equation

$$f(x+y)+g(x-y) = \sum_{i=1}^{n} h_i(x)k_i(y)$$

possess primitive functions, then they have derivatives of all orders.

ferentiability of the function $g^{(v)}(y)$ implies the differentiability of the function $\frac{d^v}{dy^v} W(g(y))$ and now we conclude that also the function $f^{(v)}(y)$ is differentiable.

Lemma 2. *If the function*

$$\psi(y, z) = af^{(v)}(z) + \frac{d^v}{dy^v} V(g(z)) + bg(y)g^{(v)}(z),$$

where a, b are constants ($a \neq 0$, $b \neq 0$), and $V(u)$ is a polynomial, is differentiable with respect to z, then $f^{(v+1)}(z)$ and $g^{(v+1)}(z)$ exist.

PROOF. If $g(y) \equiv \alpha$, then the existence of $g^{(v+1)}(z)$ is obvious. We have then $\frac{d^v}{dz^v} V(g(z)) = \text{const}$ and therefore the differentiability of the function $\psi(y, z)$ with respect to z implies the existence of $f^{(v+1)}(z)$.

If $g(y) \not\equiv \alpha$, then there exist $\bar{y}$ and $\bar{\bar{y}}$ such that $g(\bar{y}) \neq g(\bar{\bar{y}})$ and the differentiability of the function $g^{(v)}(z)$ follows from the equality

$$\psi(\bar{y}, z) - \psi(\bar{\bar{y}}, z) = b[g(\bar{y}) - g(\bar{\bar{y}})]g^{(v)}(z)$$

since the function $\psi(\bar{y}, z) - \psi(\bar{\bar{y}}, z)$ is differentiable.

The existence of $g^{(v+1)}(z)$ implies the differentiability of the function $\frac{d^v}{dz^v} V(g(z))$ and, in view of the assumption $a \neq 0$, the function $f^{(v)}(z)$ is also differentiable.

Theorem I. *Suppose that locally integrable and locally bounded functions $f(x)$, $g(x)$ satisfy equation (1), where $n > 1$, $a_{i1} \neq 0$ for $i = 1, \dots, m$, $B_2 \neq 0$, and $C_{12} \neq 0$. If the functions $f(x)$, $g(x)$ and $g(x)^2$ possess primitive functions, they are functions of class C^∞.*

PROOF. Let us denote the primitive functions by $F(x)$, $G(x)$ and $\tilde{G}(x)$. We may assume without loss of generality that $F(0) = G(0) = \tilde{G}(0) = 0$.

Setting $x_1 = t$, $x_2 = y$, $x_3 = \dots = x_n = 0$ into (1) yields

$$(2) \qquad \sum_{i=1}^{m} A_i f(a_{i1}t + a_{i2}y) = B_1 f(t) + B_2 f(y) + K + C_{11} g(t)^2 + C_{12} g(t) g(y) + \\ + C_{22} g(y)^2 + Lg(t) + Mg(y),$$

where

$$K = \sum_{j=3}^{n} B_j f(0) + \sum_{\substack{j,k=3 \\ j \leq k}}^{n} C_{jk} g(0)^2, \quad L = \sum_{k=3}^{n} C_{1k} g(0), \quad M = \sum_{k=3}^{n} C_{2k} g(0).$$

Integrating (2) with respect to t from 0 to x we obtain after simple computations

$$(3) \qquad \sum_{i=1}^{m} \frac{A_i}{a_{i1}} [F(a_{i1}x + a_{i2}y) - F(a_{i2}y)] - B_1 F(x) - Kx - C_{11} \tilde{G}(x) - LG(x) = \\ = x[B_2 f(y) + W(g(y))] + C_{12} G(x) g(y),$$

where

$$W(g(y)) = C_{22} g(y)^2 + Mg(y).$$

The left-hand side of (3) is differentiable with respect to y and therefore also the right-hand side of this equation is differentiable with respect to y. But the right-hand side of (3) has the same form as the function $\varphi(x, y)$ with $\nu=0$ in Lemma 1 and we conclude hence that $f'(y)$ and $g'(y)$ exist.

Now, suppose that we have already proved the existence of $f^{(\nu)}(y)$ and $g^{(\nu)}(y)$. Differentiating (2) ν times by y and integrating the resulting equation with respect to t we obtain

$$\sum_{i=1}^{m} \frac{A_i a_{i2}^{\nu}}{a_{i1}} [f^{(\nu-1)}(a_{i1}x+a_{i2}y)-f^{(\nu-1)}(a_{i2}y)] =$$

$$= x\left[B_2 f^{(\nu)}(y)+\frac{d^{\nu}}{dy^{\nu}} W(g(y))\right]+C_{12}G(x)g^{(\nu)}(y).$$

The differentiability of the left-hand side of the last equality implies the differentiability of its right-hand side and $f^{(\nu+1)}(y)$ and $g^{(\nu+1)}(y)$ exist on the basis of Lemma 1.

By the induction principle the functions $f(y)$ and $g(y)$ are functions of class C^{∞}.

Theorem 2. *Suppose that the locally integrable and locally bounded functions $f(x)$ and $g(x)$ satisfy equation* (1) *wherein* $n \geqq 3$, $a_{i1} \neq 0$ for $i=1, \dots, m$, $B_3 \neq 0$, $C_{12} \neq 0$. *If the functions $f(x)$, $g(x)$, $g(x)^2$ possess primitive functions, they are functions of class C^{∞}.*

Proof. If $B_2 \neq 0$, we may apply Theorem 1. If $B_2=0$, then repeating the same considerations as in the proof of Theorem 1 we get equation (3) and we conclude that the first derivative of the function $g(y)$ exists. However, equation (3) cannot be in this case used to prove the existence of $f'(y)$ and we have to look for another method of proof.

Setting $x_1=t$, $x_3=y$, $x_2=x_4=\dots=x_n=0$ into (1) yields

$$\text{(4)} \quad \sum_{i=1}^{m} A_i f(a_{i1}t+a_{i3}y)-B_1 f(t)-K^*-C_{11}g(t)^2-L^* g(t)-C_{13}g(t)g(y)-$$

$$-M^* g(y)-C_{33}g(y)^2 = B_3 f(y),$$

where

$$K^* = \sum_{j=4}^{n} B_j f(0)+\left[C_{22}+\sum_{k=4}^{n} C_{2k}+\sum_{\substack{j,k=1\\ j\leqq k}}^{n} C_{jk}\right] g(0)^2,$$

$$L^* = \left[C_{12}+\sum_{k=4}^{n} C_{1k}\right] g(0), \quad M^* = \left[C_{23}+\sum_{k=4}^{n} C_{3k}\right] g(0).$$

Integrating (5) with respect to t from 0 to x yields

$$\text{(5)} \quad \sum_{i=1}^{m} \frac{A_i}{a_{i1}} [F(a_{i1}x+a_{i3}y)-F(a_{i3}y)]-B_1 F(x)-K^* x-C_{11}\tilde{G}(x)-L^* G(x)-$$

$$-C_{13}G(x)g(y)-M^* xg(y)-C_{33}xg(y)^2 = B_3 xf(y).$$

The left-hand side of equation (5) is differentiable with respect to y since the existence of $g'(y)$ is proved. Therefore the right-hand side of this equation is differentiable. Since $B_3 \neq 0$, this proves the existence of $f'(y)$.

Suppose that we have already proved the existence of $g^{(v)}(y)$ and $f^{(v)}(y)$. Then the existence of $g^{(v+1)}(y)$ and $f^{(v+1)}(y)$ can be proved as follows. We differentiate (5) v times with respect to y and we obtain

$$\sum_{i=1}^{m} \frac{A_i a_{i3}^{v}}{a_{i1}} [f^{(v-1)}(a_{i1}x + a_{i3}y) - f^{(v-1)}(a_{i3}y)] =$$

$$= x\left[B_3 f^{(v)}(y) + \frac{d^v}{dy^v} W\big(g(y)\big)\right] + C_{13} G(x) g^{(v)}(y),$$

where

$$W\big(g(y)\big) = M^* g(y) + C_{33} g(y)^2.$$

The left-hand side of the last equality is differentiable qirth respect to y and, in view of Lemma 1, $g^{(v+1)}(y)$ and $f^{(v+1)}(y)$ exist. Thus, by the induction principle, the functions f and g are functions of class C^∞.

Theorem 3. *Suppose that locally integrable and locally bounded functions $f(x)$ and $g(x)$ satisfy equation* (1), *wherein $n \geqq 3$, $a_{i1} \neq 0$ for $i = 1, \ldots, m$, $B_3 \neq 0$, and $C_{23} \neq 0$. If the functions $f(x)$, $g(x)$ and $g(x)^2$ possess primitive functions, they are functions of class C^∞.*

Proof. If $C_{12} \neq 0$, we may apply Theorem 2. If $C_{13} \neq 0$, then it suffices to replace the index 2 by 3 to apply Theorem 1. Therefore suppose that $C_{12} = C_{13} = 0$ and denote the primitive functions of $f(x)$, $g(x)$ and $g(x)^2$ by $F(x)$, $G(x)$ and $\tilde{G}(x)$. Without loss of generality we may assume that $F(0) = G(0) = \tilde{G}(0) = 0$.

Setting $x_1 = t$, $x_2 = y$, $x_3 = z$, $x_4 = \cdots = x_n = 0$ into (1) yields

$$(6) \quad \sum_{i=1}^{m} A_i f(a_{i1}t + a_{i2}y + a_{i3}z) = B_1 f(t) + B_2 f(y) + B_3 f(z) + \tilde{K} + C_{11} g(t)^2 +$$

$$+ C_{22} g(y)^2 + C_{33} g(z)^2 + C_{23} g(y) g(z) + \tilde{L} g(t) + \tilde{M} g(y) + \tilde{N} g(z),$$

where

$$\tilde{K} = \sum_{j=4}^{n} B_j f(0) + \sum_{\substack{j,k=4 \\ j \leqq k}}^{n} C_{jk} g(0)^2, \quad \tilde{L} = \sum_{k=4}^{n} C_{1k} g(0),$$

$$\tilde{M} = \sum_{k=4}^{n} C_{2k} g(0), \quad \tilde{N} = \sum_{k=4}^{n} C_{3k} g(0).$$

Integrating (6) with respect to t from 0 to 1 we obtain

$$(7) \quad \sum_{i=1}^{m} \frac{A_i}{a_{i1}} [F(a_{i1} + a_{i2}y + a_{i3}z) - F(a_{i2}y + a_{i3}z)] - h(y) =$$

$$= B_3 f(z) + \tilde{N} g(z) + C_{33} g(z)^2 + C_{23} g(y) g(z),$$

where

$$h(y) = B_1 F(1) + C_{11} \tilde{G}(1) + \tilde{L} G(1) + B_2 f(y) + \tilde{K} + C_{22} g(y)^2 + \tilde{M} g(y).$$

Now, notice that the left-hand side of (7) is differentiable with respect to z and that the right-hand side of this equation satisfies all the assumptions of Lemma 2 with $v=0$ because $B_3 \neq 0$, $C_{23} \neq 0$, and because we may take $V(u) = \tilde{N}u + C_{33}u^2$. Thus the functions $f(z)$ and $g(z)$ are differentiable on the basis of Lemma 2.

Assuming that there exist $f^{(v)}(z)$ and $g^{(v)}(z)$ and differentiating (7) v times by z we obtain

$$\sum_{i=1}^{m} \frac{A_i a_{i3}^{v}}{a_{i1}} [f^{(v-1)}(a_{i1}+a_{i2}y+a_{i3}z)-f^{(v-1)}(a_{i2}y+a_{i3}z)] =$$

$$= B_3 f^{(v)}(z) + \frac{d^v}{dz^v} V\big(g(z)\big) + C_{23} g(y) g^{(v)}(z).$$

The right-hand side of this equality satisfies all the assumptions of Lemma 2 and therefore there exist $f^{(v+1)}(z)$ and $g^{(v+1)}(z)$.

By the induction principle, the functions f and g are functions of class C^∞.

Theorem 4. *Suppose that locally integrable and locally bounded functions $f(x)$ and $g(x)$ satisfy equation* (1), *wherein $n \geqq 2$, $a_{i1} \neq 0$ for $i=1, \dots, m$, $B_1 \neq 0$, $C_{12} \neq 0$, and*

(8) *for every index i there exists an index $j=j(i)$ $(j \neq 1)$ such that $a_{ij} \neq 0$. If the functions $f(x)$, $g(x)$ and $g(x)^2$ possess primitive functions, they are functions of class C^∞.*

Proof. The existence of $g'(x)$ can de proved analogously as in the proof of Theorem 1.

If $B_j \neq 0$ for some $j \neq 1$, we can apply either Theorem 1 or Theorem 2.

If $B_2 = \dots = B_n = 0$, then setting $x_1 = x$, $x_j = u_j$ for $j = 2, \dots, n$ into (1) yields

$$\sum_{i=1}^{m} A_i f\left(a_{i1}x + \sum_{j=1}^{n} a_{ij}u_j\right) = \tag{9}$$

$$= B_1 f(x) + C_{11} g(x)^2 + \sum_{k=2}^{n} C_{1k} g(u_k) g(x) + \sum_{\substack{j,k=2 \\ j \leqq k}}^{n} C_{jk} g(u_j) g(u_k).$$

After $n-1$ integrations with respect to $u_2, \dots, u_n$ we obtain

$$H(x) - C_{11} g(x)^2 - \hat{K} g(x) - \hat{L} = B_1 f(x), \tag{10}$$

where

$$H(x) = \int_0^1 \dots \int_0^1 \sum_{i=1}^{m} A_i f\left(a_{i1}x + \sum_{j=2}^{n} a_{ij}u_j\right) du_2 \dots du_n,$$

$$\hat{K} = \sum_{k=2}^{n} C_{1k} \int_0^1 \dots \int_0^1 g(u_k)\, du_2 \dots du_n, \quad \hat{L} = \sum_{\substack{j,k=2 \\ j \leqq k}}^{n} C_{jk} \int_0^1 \dots \int_0^1 g(u_j) g(u_k)\, du_2 \dots du_n.$$

In view of the condition (8), the function $H(x)$ is differentiable. Since, moreover, $g'(x)$ exists, the left-hand side of (10) is differentiable. Therefore the right-hand side of (10) is also differentiable and since $B_1 \neq 0$, $f'(x)$ exists.

Assuming the existence of $f^{(\nu)}(x)$ and $g^{(\nu)}(x)$ we can prove the existence of $g^{(\nu+1)}(x)$ analogously as in the proof of Theorem 1. Now, differentiating (9) ν times with respect to x yields

$$\sum_{i=1}^{m} A_i a_{i1}^{\nu} f^{(\nu)}\left(a_{i1}x+\sum_{j=2}^{n} a_{ij}u_j\right) =$$

$$= B_1 f^{(\nu)}(x)+c_{11}\sum_{i=0}^{\nu}\binom{\nu}{i} g^{(i)}(x)g^{(\nu-i)}(x)+\sum_{k=2}^{n} C_{1k}g(u_k)g^{(\nu)}(x).$$

Integrating the last equality $n-1$ times by $u_2, \ldots, u_n$ from 0 to 1 yields

$$H^{(\nu)}(x)-c_{11}\sum_{i=0}^{\nu}\binom{\nu}{i} g^{(i)}(x)g^{(\nu-i)}(x)-\hat{K}g^{(\nu)}(x) = B_1 f^{(\nu)}(x), \tag{11}$$

where $H(x)$ and $\hat{K}$ are the same as in equation (10).

Assumption (8) and the existence of $f^{(\nu)}(x)$ guarantee the differentiability of the function

$$H^{(\nu)}(x) = \int_0^1 \ldots \int_0^1 \sum_{i=2}^{m} A_i a_{i1}^{\nu} f^{(\nu)}\left(a_{i1}x+\sum_{j=2}^{n} a_{ij}u_j\right) du_2 \ldots du_n.$$

Since $B_1 \neq 0$ and $g^{(\nu+1)}(x)$ exists, equality (11) implies the existence of $f^{(\nu+1)}(x)$.

By the induction principle the functions $f(x)$ and $g(x)$ are functions of class C^{∞}.

Remark. Assumption (8) is an essential one as shows the following example. The equation

$$f(x+y)-f(-x-y)+f(-x) = f(x)+C_{11}g(x)^2+C_{12}g(x)g(y)+C_{22}g(y)^2,$$

where $C_{11}+C_{12}+C_{22} = 0$, satisfies all the assumptions of Theorem 4 excepting (8) and is satisfied by the functions

$$f(x) = |x|^{2n-1}, \quad g(x) \equiv \alpha.$$

The function $f(x)$ is locally integrable and locally bounded and it possesses the primitive function

$$F(x) = \begin{cases} -\dfrac{x^{2n}}{2n} & \text{for} \quad x \leqq 0 \\ \dfrac{x^{2n}}{2n} & \text{for} \quad x > 0 \end{cases}$$

but it is not a function of class C^{∞}.

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