The fully invariant subgroups of reduced algebraically compact groups

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1. Introduction. In this paper we give a description of the fully invariant subgroups of a reduced algebraically compact group. The approach is very much in the spirit of KAPLANSKY's ([5]) theory for primary modules.

The notation is the standard notation of [3].

2. Approach. We shall use following general and very easy theorem. For lattice theoretical notions and notation see [1].

2. 1 Theorem. *Let A be a module over a commutative ring R, $\mathbf{N}$ its lattice of fully invariant submodules, $\mathbf{H}$ some meet-semilattice (=poset with meet) and $U: A \to \mathbf{H}$ a function with the following properties.*

(1) *U is surjective.*
(2) *$U(af) \geqq U(a)$ for every $a \in A$ and every $f \in$ End A.*
(3) *$U(a+b) \geqq U(a) \wedge U(b)$ for all $a, b \in A$.*
(4) *Whenever $U(a) \geqq U(b)$ then there is $f \in$ End A such that $bf = a$.*
(5) *Whenever $N \in \mathbf{N}$ and $a, b \in N$ then there is $c \in N$ such that $U(c) = U(a) \wedge U(b)$.*

Then the set $\mathbf{H}^*$ of all dual ideals of $\mathbf{H}$, ordered by set inclusion, is a lattice, and the function

$$\alpha: \mathbf{H}^* \to \mathbf{N}: H\alpha = \{a: U(a) \geqq h \quad \text{for some} \quad h \in H\}$$

is a lattice isomorphism. Furthermore, $H\alpha = a$ End A for some $a \in A$ iff H is a principal dual ideal.

PROOF. It follows from (2) that, for all $a \in A$, $U(0) \geqq U(a)$ and $U(-a) \geqq U(a)$. Since U is surjective, $U(0)$ is the largest element of $\mathbf{H}$. Since $\mathbf{H}$ contains a largest element, the dual of the proof of [1], p. 25, Theorem 3, goes through, and $\mathbf{H}^*$ is a lattice under set inclusion. By the preceding remarks, and by (2) and (3) it follows easily that $H\alpha \in \mathbf{N}$ for every $H \in \mathbf{H}^*$. To show that α is surjective, let $N \in \mathbf{N}$, and put $H = \{U(a): a \in N\}$. Then $U(a)$, $U(b) \in H$ implies by (5) that $U(a) \wedge U(b) \in H$. If $U(a) \geqq U(b)$, $b \in N$, then (4) yields that $a \in N$, $U(a) \in H$. Thus $H \in \mathbf{H}^*$. Obviously $H\alpha \supset N$, and by (4) $H\alpha = N$. Finally we show that α is injective. Assume that $H\alpha = K\alpha$ for $H, K \in \mathbf{H}^*$. Let $h \in H$. By (1) there is $a \in A$ such that $h = U(a)$. Then $a \in H\alpha = K\alpha$, and therefore $h = U(\alpha) \geqq k$ for some $k \in K$. Thus $h \in K$, and $H \subset K$. By symmetry $H = K$. The last statement of the theorem is obvious.

Remarks. (a) Theorem 2. 1 is an abstraction of Kaplansky's ([5]) approach to describe the fully invariant submodules of fully transitive primary modules over a complete discrete valuation ring. In [5], $\boldsymbol{H}$ is the set of all Ulm sequences of elements of a primary module A. The set $\boldsymbol{H}$ is a partially ordered set under the natural pointwise ordering. By [5], Lemmas 26 and 24, $\boldsymbol{H}$ has greatest lower bounds which, in fact, are the pointwise infimums. Now, if $U(a)$ is the Ulm sequence of $a\in A$, then (1), (2), (3) of our theorem are automatically satisfied, (4) is Kaplansky's assumption of full transitivity, and (5) is essentially proved on p. 60 of [5]. Thus 2. 1 yields a description of the fully invariant submodules of A. In addition, Kaplansky's results show that in this case the lattice of dual ideals of Ulm sequences is dually isomorphic with the lattice of U-sequences (see [5]). This characterization of dual ideals cannot be duplicated for algebraically compact groups while everything else goes through as we shall show. Hence in Kaplansky's exposition U-sequences should only be introduced at the very end. (b) It would be interesting to know whether, or in which cases, (5) is a consequence of (1) through (4).

It is well known that every reduced algebraically compact group A is of the form $A=\oplus^* A_p$, the sum being extended over all primes p, where each A_p is a complete module over the ring of p-adic integers. See [4], where a comprehensive account of algebraically compact groups is given, pp. 77—78. This fact enables us to reduce the problem to the case of p-adic algebraically compact groups A_p as follows.

2. 2. Theorem. *Let $A=\oplus^* A_p$ where the sum is extended over all primes p, and where each A_p is a reduced module* (complete or not) *over the ring of p-adic integers. Assume that for every p there is a meet-semilattice $\boldsymbol{H}_p$ and a function $U_p\colon A_p\to\boldsymbol{H}_p$ satisfying properties* (1) *through* (5). Let $\boldsymbol{H}=\boldsymbol{H}_2\times\ldots\times\boldsymbol{H}_p\times\ldots$. *Define $(\ldots h_p\ldots)\geqq(\ldots k_p\ldots)$ iff $h_p\geqq k_p$ for every p. Then $\boldsymbol{H}$ is a meet-semilattice. Define $U\colon A\to\boldsymbol{H}\colon U((\ldots a_p\ldots))=(\ldots U_p(a_p)\ldots)$. Then U satisfies* (1) *through* (5) of 2. 1.

Proof. (a) It is clear that $\boldsymbol{H}$ is partially ordered and has (pointwise) meets in the given partial order.

(b) We claim that End A is the ring direct product of the rings End A_p. It is clear that each A_p is fully invariant in A since A_p consists exactly of those elements of A which are q-divisible for all $q\neq p$. Observe further that $A/\oplus_p A_p$ is divisible. Define $\alpha\colon$ End $A\to$ End $A_2\times\ldots\times$ End $A_p\times\ldots$ by $f\alpha=(\ldots f_p\ldots)$ where $f_p=f|A_p$. Clearly α is surjective and homomorphic. If $f\alpha=g\alpha$ then $\oplus A_p\subset\ker(f-g)$ and $A(f-g)\cong A/\ker(f-g)$ and is divisible since $A/\oplus A_p$ is divisible. Since A is reduced $A(f-g)=0$, and it follows that α is injective. In the remainder we shall identify f and $(\ldots f_p\ldots)$.

(c) It is quite obvious that U satisfies (1): If $(\ldots h_p\ldots)\in\boldsymbol{H}$, then there is $a_p\in A_p$ with $U_p(a_p)=h_p$. Now $(\ldots a_p\ldots)\in A$ and $U((\ldots a_p\ldots))=(\ldots h_p\ldots)$. Condition (2) is clearly satisfied since for every $f\in$ End A we have $U((\ldots a_p\ldots)f)=U((\ldots a_pf\ldots))=$ $=(\ldots U_p(a_pf)\ldots)\geqq(\ldots U_p(a_p)\ldots)=U((\ldots a_p\ldots))$. Condition (3) is satisfied since $U((\ldots a_p\ldots)+(\ldots b_p\ldots))=U((\ldots a_p+b_p\ldots))=(\ldots U_p(a_p+b_p)\ldots)\geqq(\ldots U_p(a_p)\wedge U_p(b_p)\ldots)=$ $=(\ldots U_p(a_p)\ldots)\wedge(\ldots U_p(b_p)\ldots)=U((\ldots a_p\ldots))\wedge U((\ldots b_p\ldots))$. Condition (4) follows from the same property of the U_p. To prove (5), let $(\ldots a_p\ldots)$, $(\ldots b_p\ldots)\in N$ where N is some fully invariant subgroup of A. For each p, let N_p denote the smallest fully invariant subgroup of A_p containing a_p and b_p. By hypothesis there is $c_p\in N_p$

such that $U_p(c_p)=U_p(a_p)\wedge U_p(b_p)$. By definition of N_p, c_p must be of the form $c_p = a_pf_p+b_pg_p$ with f_p, $g_p\in \mathrm{End}\, A_p$. Put $f=(\dots f_p\dots)$ and $g=(\dots g_p\dots)$. Then $(\dots a_p\dots)f+(\dots b_p\dots)g = (\dots c_p\dots)\in N$ and $U((\dots c_p\dots)) = (\dots U_p(c_p)\dots) = (\dots U_p(a_p)\wedge \wedge U_p(b_p)\dots) = U((\dots a_p\dots))\wedge U((\dots b_p\dots))$.

Theorems 2. 1 and 2. 2 suffice to describe the lattice of fully invariant subgroups in some special cases. The following is a particularly simple example.

2. 3. Example. Let $A = \oplus^*\{A_p:p\in\Pi\}$ where Π is a set of primes and each A_p is a (non-zero) direct sum of cyclic groups of order p. For each p, let $\boldsymbol{H}_p=\{0, \infty\}$ with $\infty>0$, and define $U_p:A_p\to H_p$ by $U_p(0)=\infty$ and $U_p(a_p)=0$ for $a_p\neq 0$. Then obviously (1) through (5) of 2. 1 are satisfied. Applying 2. 2 and then 2. 1 we obtain that the lattice $\boldsymbol{N}$ of fully invariant subgroups of A is isomorphic with the lattice of dual ideals of the partially ordered set $\boldsymbol{H}$ whose elements are at most countable vectors with entries 0 or ∞. The fully invariant subgroups of A generated by a single element are exactly the subgroups of the type $\oplus^*\{A_p:p\in\Pi'\}$ where $\Pi'\subset\Pi$. The dual ideal of $\boldsymbol{H}$ consisting of all vectors with a finite number of 0 components is — for infinite Π — an example of a dual ideal which is not principal. The corresponding fully invariant subgroup is the torsion subgroup $\oplus\{A_p:p\in\Pi\}$ of A. The lattice of fully invariant subgroups is the same for all A with infinite Π.

3. The „Torsion" case. A reduced p-adic algebraically compact group A is of the form $A = B^*\oplus C$ where the "*torsion part*" B^* is the p-adic completion of a direct sum of cyclic p-groups, and C is a torsion free module over the ring of p-adic integers which is complete in the p-adic topology; see [4], p. 79. In this section we handle the "torsion" case of a completion B^* of a direct sum B of cyclic p-groups. Write $B=\oplus B_i$ where B_i is a direct sum of cyclic groups of order p^i. Without loss of generality we assume that B^* is the group of all elements $(x_i)\in\oplus^* B_i$ such that the heights $H(x_i)\to\infty$ as $i\to\infty$. Let $\boldsymbol{H}$ denote the poset of all Ulm-sequences of elements of B^* with the natural componentwise order. The following lemma shows both that $\boldsymbol{H}$ has pointwise meets and that 2. 1 (5) holds.

3. 1. Lemma. *Let $x=(x_i)$ and $y=(y_i)$ be given elements of B^*. Let $z_i=x_i$ if $H(x_i)\leqq H(y_i)$ and let $z_i=y_i$ if $H(y_i)<H(x_i)$. Then $z=(z_i)\in B^*$, $H(p^kz) = \min\{H(p^kx),\ H(p^ky)\}$ and there exist f, $g\in \mathrm{End}\, B^*$ such that $z = xf+yg$.*

Proof. Since $H(z_i)=\min\{H(x_i),\ H(y_i)\}$ it is clear that $z=(z_i)\in B^*$. Notice that, for a, $b\in B_i$, $H(a)\leqq H(b)$ implies $H(p^ka)\leqq H(p^kb)$. Hence $H(p^kz) = \min H(p^kz_i)=\min\{H(p^kx_i),\ H(p^ky_i)\}=\min\{H(p^kx),\ H(p^ky)\}$. Now let $f:\oplus^* B_i\to \oplus^*\{B_i:z_i = x_i\}$ and $g:\oplus^* B_i\to\{B_i:z_i = y_i\}$ be the projections. Then clearly f and g restricted to B^* are endomorphisms of B^* and $xf+yg = z$.

It remains to show that B^* is fully transitive (property (4)). We need some observations which are interesting in themselves. The subgroups $(p^hB^*)[p^e]$ are a basic type of fully invariant subgroup. We abbreviate $[h, e]=(p^hB^*)[p^e]$. We use the $[h, e]$ to construct new fully invariant subgroups as follows. Let $\{h_i\}$ be a sequence of non-negative integers such that $h_i\to\infty$ as $i\to\infty$, and let $\{e_i\}$ be any sequence of non-negative integers. Every infinite series Σz_i, $z_i\in[h_i, e_i]$, converges (in the p-adic topology) since $h_i\to\infty$ as $i\to\infty$, and $\{\Sigma z_i:z_i\in[h_i, e_i]\}$ is clearly a subgroup and fully invariant since, for all $\alpha\in \mathrm{End}\, B^*$, α is continuous in the p-adic topology of

B^*, and thus $(\Sigma z_i)\alpha = \Sigma(z_i\alpha)$. We shall show next that all fully invariant subgroups generated by a single element are of this type.

3. 2. Lemma. *Given* $x=(x_i)\in B^*$. *Let* $h_i = H(x_i)$ *and* $e_i = E(x_i)$ (*=exponent*). *Then* $x \operatorname{End} B^* = \{\Sigma z_i : z_i \in [h_i, e_i]\}$.

PROOF. Clearly $x \in \{\Sigma z_i : z_i \in [h_i, e_i]\}$, and, since $\{\Sigma z_i : z_i \in [h_i, e_i]\}$ is a fully invariant subgroup, $x \operatorname{End} B^* \subset \{\Sigma z_i : z_i \in [h_i, e_i]\}$. Conversely, let Σz_i, $z_i \in [h_i, e_i]$, be given. Let $b_i \in B_i$ be such that $x_i = p^{h_i} b_i$. Then $\langle b_i \rangle$ is a direct summand of B_i. Write $z_i = p^{h_i} y_i$. There is $\alpha \in \operatorname{End} B^*$ such that $b_i\alpha = y_i$. Then $x\alpha = (\Sigma x_i)\alpha = \Sigma(x_i\alpha) = \Sigma p^{h_i}(b_i\alpha) = \Sigma p^{h_i} y_i = \Sigma z_i$. Hence $\Sigma z_i \in x \operatorname{End} B^*$, and this concludes the proof.

Remark. It can be shown that every fully invariant subgroup of B^* generated by a single element is uniquely of the form $\{\Sigma z_k : z_k \in [h_{i_k}, e_{i_k}]\}$ where $0 < i_1 < i_2 \ldots$; each i_k is a *relevant integer* for B^*, i.e. $B_{i_k} \neq 0$; $0 \leqq h_{i_1} < h_{i_2} \ldots, h_{i_k} < i_k$; and $0 < e_{i_1} < e_{i_2} < \ldots$. This approach is the one chosen by Shiffmann [7] for p-groups without elements of infinite height.

We are now ready to prove

3. 3. Lemma. B^* *is fully transitive.*

PROOF. Let $x, y \in B^*$ such that $U(x) \geqq U(y)$, i.e. $H(p^k x) \geqq H(p^k y)$ for all k. We have to show that $x \in y \operatorname{End} B^*$. Write $x=(x_i)$, $y=(y_i)$, and put $h_i = H(y_i)$, $e_i = E(y_i)$. By 3. 2 we need to show that $x \in \{\Sigma z_i : z_i \in [h_i, e_i]\}$. Since $H(p^k x) = \min\{H(p^k x_i)\}$ and similarly $H(p^k y) = \min\{H(p^k y_i)\}$, the hypothesis says that for every given j and k there is i such that $H(p^k x_j) \geqq H(p^k y_i)$. Consider $x_j \neq 0$, and put $h = H(x_j)$, $e = E(x_j)$. Note that $e + h = j$. By hypothesis there is i such that $h + e - 1 = H(p^{e-1} x_j) \geqq H(p^{e-1} y_i) = h_i + e - 1$. Hence $h \geqq h_i$. Further $1 \leqq E(p^{e-1} y_i) = e_i - e + 1$, or $e \leqq e_i$. Thus $x_j \in [h_i, e_i]$. Now let $z_i = \Sigma\{x_j : x_j \in [h_i, e_i]$ but $x_j \notin [h_k, e_k]$ with $k < i\}$. Then $z_i \in [h_i, e_i]$ and $x = \Sigma z_i$. This proves 3. 3.

We summarize the result.

3. 4. Theorem. *Let B^* be the p-adic completion of a direct sum of cyclic p-groups, $\mathbf{N}$ its lattice of fully invariant subgroups and $\mathbf{H}$ the poset of all Ulm sequences of elements of B^*. Then $\mathbf{H}$ is a meet-semilattice with meets taken pointwise. If $\mathbf{H}^*$ is the lattice of dual ideals of $\mathbf{H}$, then*

$$\alpha : \mathbf{H}^* \to \mathbf{N} : H\alpha = \{x : U(x) \geqq h \quad \text{for some} \quad h \in H\}$$

is a lattice isomorphism. Furthermore $H\alpha = x \operatorname{End} B^*$ if and only if H is a principal dual ideal.

For applications it is important to know precisely which sequences $\{h_0, h_1, \ldots\}$ belong to $\mathbf{H}$. The answer is contained in the following proposition.

3. 5 Proposition. *Let B^* and $\mathbf{H}$ be as before. A sequence $\{h_0, h_1, \ldots\}$ of non-negative integers and the symbol ∞ belongs to $\mathbf{H}$ if and only if*

(a) $0 \leqq h_0 \leqq h_1 \leqq \ldots$ *and* $h_i < h_{i+1}$ *if* $h_{i+1} \neq \infty$.

(b) *A gap* $h_j + 1 < h_{j+1}$ *occurs only if* $B_{h_j+1} \neq 0$.

(c) *If all h_i are finite then the sequence has infinitely many gaps.*

PROOF. We show first the necessity of the conditions (a), (b), and (c). Let $x=(x_i)\in B^*$ such that $h_i=H(p^ix)$.
(a) This condition is well-known and trivial.
(b) We have $h_j=\min H(p^jx_i)$. If $h_j=H(p^jx_i)$ for some $i\neq h_j+1$, then $H(p^{j+1}x_i)=$ $=h_j+1\geqq H(p^{j+1}x)$ and there cannot be a gap between h_j and h_{j+1}. Thus, if there is a gap between h_j and h_{j+1}, then $\infty\neq h_j=H(p^jx_{h_j+1})$ and $0\neq x_{h_j+1}\in B_{h_j+1}$.
(c) Deny the claim. Then there is n such that $H(p^{n+i}x)=h_n+i$ for all i. This means that for each i there is k_i such that $H(p^{n+i}x_{k_i})\leqq h_n+i$. Since $H(p^{n+i}x_{k_i})=$ $=n+i+H(x_{k_i})$ it follows that $H(x_{k_i})\leqq h_n-n$ for all i. It is easy to see that the set $\{k_i\}$ is infinite, and so $H(x_{k_i})\leqq h_n-n$ contradicts the fact that $H(x_i)\to\infty$ as $i\to\infty$.

To establish the converse let $h=\{h_0,h_1,\dots\}$ be given satisfying (a), (b) and (c). Note first that, for each i, $h_i-i=(h_i-h_{i-1}-1)+(h_{i-1}-h_{i-2}-1)+\dots+$ $+(h_1-h_0-1)+h_0$. The quantity $h_j-h_{j-1}-1\geqq 1$ iff there is a gap between h_{j-1} and h_j. Hence $h_i-i=h_j-j$ if there is no gap between h_i and h_j. If there is a gap between h_i and h_j and $j>i$, then $h_i-i<h_j-j$. We obtain $x=(x_i)$ as follows. If a gap occurs between h_j and h_{j+1}, then $B_{h_j+1}\neq 0$ and we choose $x_{h_j+1}\in B_{h_j+1}$ such that $H(x_{h_j+1})=h_j-j$. Note that $E(x_{h_j+1})=j+1$. Otherwise let $x_i=0$. That $x\in B^*$ follows from (c). We wish to calculate $H(p^ix)=\min H(p^ix_j)$ for given i. We only need to consider components x_{h_j+1} with a gap between h_j and h_{j+1} since all other components are 0. If $j<i$, then $p^ix_{h_j+1}=0$. Hence $H(p^ix)=H(0)=\infty$ if there are no gaps h_j, h_{j+1} with $j\geqq i$. In this case $h_i=\infty$ and we have $H(p^ix)=h_i$. For $j\geqq i$, $H(p^ix_{h_j+1})=i+(h_j-j)$. Hence, if h_j, h_{j+1} is the first gap with $j\geqq i$, then $H(p^ix)=H(p^ix_{h_j+1})=i+(h_j-j)=i+(h_i-i)=h_i$. This concludes the proof.

As an application we discuss p-pure fully invariant subgroups of B^*. These groups made their appearance in a paper by *G.* Roch [6]. Roch establishes invariants for a certain class of groups ("konvergenzfreie Gruppen"). In his approach, the p-pure fully invariant subgroups of B^* (called "Torsionstypen") play a fundamental role.

3. 6 Lemma. *If N is a non-zero p-pure fully invariant subgroup of B^* then N contains the maximal torsion subgroup $\bar{B}$ of B^*.*

PROOF. By our previous results, $N=\{x\in B^*: U(x)\geqq h$ for some $h\in H\}$ where $H\in\boldsymbol{H}^*$. If the first component of each $h\in H$ is >0 then $N\subset pB^*$ and $N=N\cap pB^*=$ $=pN$. This means that N is p-divisible contradicting $N\neq 0$. Hence there is $h=$ $=\{0,\dots\}\in H$. If x has order p then $U(x)=\{H(x),\infty,\infty,\dots\}\geqq h$, i.e. $x\in N$. By induction on the order of $x\in B$ we shall show that $\bar{B}\subset N$. Given $x\in\bar{B}$, $E(x)>1$. By induction hypothesis $0\neq px\in N$. Since N is p-pure there is $y\in N$ such that $px=py$. Since $x-y$ has order p or 1, $x-y\in N$, hence $x\in N$.

It is now easy to see when two p-pure fully invariant subgroups are isomorphic.

3. 7 Proposition. *Two p-pure fully invariant subgroups of B^* are isomorphic if and only if they are equal.*

PROOF. If N, M are non-zero p-pure fully invariant subgroups of B^* and $N\cong M$ then this isomorphism and its inverse extend uniquely to endomorphisms of B^* since N, M are dense topological subgroups of B^* and B^* is complete. Since N, M are fully invariant $N\subset M$ and $M\subset N$.

A very useful description of the p-pure fully invariant subgroups of B^* is obtained as follows. For $h=\{h_i\}\in \boldsymbol{H}$ and n a natural number let $h+n = \{h_n, h_{n+1}, \dots, h_{n+i}, \dots\}$. If $h=U(x)$ then $h+n = U(p^n x)$. In particular $h+n \in \boldsymbol{H}$.

3. 8 Proposition. *Let $H\in \boldsymbol{H}^*$. The fully invariant subgroup N corresponding to H is p-pure in B^* if and only if $h\in H$ whenever $h+n \in H$ for some n.*

PROOF. Assume first that H has the stated property. Let $x\in B^*$ be such that $p^n x\in N$. Then $U(x)+n = U(p^n x)\in H$ and therefore $U(x)\in H$, $x\in N$. Thus N is p-pure. Conversely, let N be p-pure and $N\neq 0$. Then $N\supset \bar{B}$ and $p^n x\in N$ implies $x\in N$. Given $h=U(x)$ with $h+n = U(p^n x)\in H$. Then $p^n x\in N$, therefore $x\in N$ and $h=U(x)\in H$.

To obtain sharp existence theorems we concentrate on the simplest case of p-pure fully invariant subgroups *generated by single elements*, i.e. the unique smallest p-pure fully invariant subgroups containing given single elements. By counting Ulm sequences, constructing suitable Ulm sequences, and by checking when they generate the same dual ideal of the type in 3. 8, we can prove

3. 9 Theorem. *If B^* is unbounded, then*
(a) *there are exactly $2^{\aleph_0}$ different and hence non-isomorphic p-pure fully invariant subgroups of B^* generated by single elements;*
(b) *B^* is not countably generated as a p-pure fully invariant subgroup of itself.*

Remark. By a theorem of CASTAGNA [2] every endomorphism of B^* is the sum of two automorphisms of B^* if $p>2$. Hence, for such p, the characteristic subgroups of B^* containing $\bar{B}$ are just the fully invariant subgroups between $\bar{B}$ and B^*.

4. The mixed case. In this section we consider groups $A=B^*\oplus C$ where B^* is the p-adic completion of a direct sum of cyclic p-groups, and where C is a reduced torsion-free module over the ring of p-adic integers (complete or not). Let $\boldsymbol{H}$ be the set of all Ulm sequences of elements of A with the natural pointwise order. We shall show that 2. 1 applies. For B^* we use again the representation used in the previous section. A close look at the elements of A and their Ulm sequences is necessary.

4. 1 Let $x_B=(x_i)\in B^*$, and let n be any natural number. Let $x'_B=\Sigma\{x_i : H(x_i)<n\}$ and let $x''_B = x-x'_B$. Let $k_x=E(x'_B)$. Then $x_B = x'_B+x''_B$; $H(x''_B)\geqq n$; $H(p^i x_B)\geqq n+i$ for all $i\geqq k_x$, and $H(p^i x_B) = H(p^i x'_B) < n+i$ for $i<k_x$. Thus k_x is the smallest i with $H(p^i x_B)\geqq n+i$.

4. 2 Let $x\in A$, $x = x_B+x_C$, $x_B\in B^*$, $x_C\in C$. Suppose $x_C\neq 0$, and let $n=H(x_C)$. If $x_B = x'_B+x''_B$ as in 4. 1, then $H(p^i x) = H(p^i x_B) = H(p^i x'_B)$ for $0\leqq i<k_x$, and $H(p^i x) = H(p^i x_C) = n+i$ for $i\geqq k_x$.

The next lemma states both that $\boldsymbol{H}$ is a meet-semilattice with pointwise meets, and that 2. 1 (5) holds.

4. 3 Lemma. *Let $x, y\in A$. Then there exist $z\in A$, and $f, g \in \operatorname{End} A$ such that $z = xf+yg$ and $U(z)$ is the pointwise infimum of $U(x)$ and $U(y)$.*

PROOF. Write $x = x_B+x_C$, $y = y_B+y_C$. By previous results there are f_B, $g_B\in \operatorname{End} B^*$ and $z_B\in B^*$ such that $z_B = x_B f_B+y_B g_B$, and $H(p^i z_B)=\min\{H(p^i x_B),$

$H(p^i y_B)\}$. This proves the lemma if $x_C = y_C = 0$. Otherwise we may assume that x_C has finite height $n = H(x_C)$, and that $n \leqq H(y_C)$. Let $z_C = x_C$, and $z = z_B + z_C$. There is $f \in \operatorname{End} A$ such that $f = f_B$ on B^* and $f = 1$ on C, and there is $g \in \operatorname{End} A$ such that $g = g_B$ on B^* and $g = 0$ on C. Then $xf + yg = x_B f + x_C f + y_B g + y_C g = z_B + x_C = z$.

We shall show that $U(z)$ is the pointwise infimum of $U(x)$ and $U(y)$. From 4. 1 we obtain with $n = H(x_C)$ numbers k_x, k_y, k_z for x_B, y_B, z_B respectively. Since, by definition of z_B, $H(p^i z_B) = \min\{H(p^i x_B), H(p^i y_B)\}$ it follows easily that $k_z = \max\{k_x, k_y\}$. Using 4. 2 we obtain for $i < k_z$ that $H(p^i z) = H(p^i z_B) = \min\{H(p^i x_B), H(p^i y_B)\} = \min\{H(p^i y), H(p^i y)\}$, while for $i \geqq k_z$ we have $H(p^i z) = n + i = H(p^i x) \leqq H(p^i y)$. This proves the lemma.

The next and last lemma shows that A is fully transitive.

4. 4 Lemma. *Given $x, y \in A$ such that $U(x) \geqq U(y)$. Then there is $f \in \operatorname{End} A$ such that $yf = x$.*

PROOF. Let $x = x_B + x_C$, $x_B \in B^*$, $x_C \in C$, and similarly $y = y_B + y_C$. If $y_C = 0$, and $x_C \neq 0$, then by 4. 1 there is k such that $H(p^i y) \geqq H(x_C) + 1 + i$ for $i \geqq k$. But for sufficiently large i we have by 4. 2 that $H(p^i x) = H(x_C) + i$, leading to $H(p^i y) \geqq H(p^i x) + 1$ contrary to the hypothesis. Thus $y_C = 0$ implies $x_C = 0$, and in this case the lemma holds since B^* is fully transitive.

Now assume that $y_C \neq 0$ and $x_C \neq 0$. Then, by 4. 2, for sufficiently large i, $H(p^i y) = H(p^i y_C) = H(y_C) + i$ and $H(p^i x) = H(p^i x_C) = H(x_C) + i$. Since $U(x) \geqq U(y)$ we find $H(x_C) \geqq H(y_C)$. This is the fact which is used below. But $H(x_C) \geqq H(y_C)$ also for $x_C = 0$ so that from now on x_C may or may not be zero. Put $n = H(y_C)$. According to 4. 1 write $y = y'_B + y''_B + y_C$ and $x = x'_B + x''_B + x_C$. Let k_y and k_x be as in 4. 1 with respect to $n = H(y_C)$.

We shall show that $k_x \leqq k_y$. In fact, if $k_y \leqq i < k_x$, then $n + i = H(p^i y) \leqq H(p^i x) \leqq H(p^i x_B) < n + i$, a contradiction. It follows that $U(y'_B) \leqq U(x'_B)$ since for $i < k_x$ we have $H(p^i y'_B) = H(p^i y) \leqq H(p^i x) = H(p^i x'_B)$ and, for $i \geqq k_x$, $H(p^i x'_B) = \infty$. Since B^* is fully transitive there is $f_B \in \operatorname{End} B^*$ such that $y'_B f = x'_B$. Since $H(y''_B) \geqq n$, $H(x''_B) \geqq n$, and $H(x_C) \geqq n$ (from above), there is $f \in \operatorname{End} A$ such that $f = f_B$ on B^* and $y_C f = x''_B + x_C - y''_B f_B$. Then $yf = y'_B f_B + y''_B f_B + x''_B + x_C - y''_B f_B = x$.

We summarize the result.

4. 5 Theorem. *Let $A = B^* \oplus C$ where B^* is the p-adic completion of a direct sum of cyclic p-groups, and where $\mathbf{C}$ is a reduced torsionfree module over the ring of p-adic integers (complete or not). Let $\mathbf{N}$ be the lattice of fully invariant subgroups of A and $\mathbf{H}$ the poset of all Ulm sepuences of elements of B^*. Then $\mathbf{H}$ is a meet-semilattice with meets taken pointwise. If $\mathbf{H}^*$ is the lattice of dual ideals of $\mathbf{H}$, then*

$$\alpha : \mathbf{H}^* \to \mathbf{N} : H\alpha = \{x : U(x) \geqq h \quad for\ some \quad h \in H\}$$

is a lattice isomorphism. Furthermore, $H\alpha = x \operatorname{End} B^*$ if and only if H is a principal dual ideal.

Remark. If $x \in A = B^* \oplus C$, $x = x_B + x_C$, then $U(x) = U(x_B) \wedge U(x_C)$. The sequences $U(x_B)$ are characterized in 3. 5, while $U(x_C) = \{n, n+1, n+2, \dots\}$ where $n = H(x_C)$. This characterizes the elements of H.

The following easy proposition — which we state without proof — gives a simple description of the fully invariant subgroups of $A = B^*\oplus C$ in terms of the fully invariant subgroups of B^*.

4. 6 Proposition. *Let $A = B^*\oplus C$ where B^* is the p-adic completion of a direct sum of cyclic p-groups, and where C is a reduced torsion-free module over the ring of p-adic integers (complete or not). Then the set of fully invariant subgroups of A is the set of all groups of the form $N+p^nA$ where N is a fully invariant subgroup of B^*, and $0\leqq n\leqq\omega$.*

5. The general case. Theorems 4. 5, 2. 2, and 2. 1 immediately yield the following result.

Theorem. *Let A be a reduced algebraically compact group. Write $A=\oplus^* A_p$, where each A_p is a reduced algebraically compact module over the ring of p-adic integers, and the summation is extended over all primes p. For every $a=(\dots a_p\dots)\in A$ let $U(a)=(\dots U(a_p)\dots)$, and let **H** be the set all $U(a)$ ordered componentwise. Then **H** is a meet-semilattice with pointwise meets. Let $\mathbf{H}^*$ be the lattice of dual ideals of **H**. Then the function*

$$\alpha:\mathbf{H}^*\to\mathbf{N}:H\alpha = \{a\in A: U(a)\geqq h \quad \text{for some} \quad h\in H\}$$

*is a lattice isomorphism of $\mathbf{H}^*$ with the lattice **N** of all fully invariant subgroups of A. Furthermore, $H\alpha=a$ End A for some $a\in A$ if and only if H is principal.*

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