Typically real polynomials

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1. Introduction. Let TR denote the class of normalized functions f analytic and typically real in the unit disk E. That is, f is of the form $f(z) = z+c_2z^2+ +c_3z^3+\dots$ in E and satisfies in E the condition $\operatorname{Im} f(z)\cdot\operatorname{Im}(z) \geqq 0$. This class of functions was introduced by W. ROGOSINSKI ([2]) and has been studied extensively. In this paper we initiate a study of polynomials $P_n(z) = z+a_2z^2+\dots+a_nz^n$ which belong to TR, that is, $P_n(z)$ is typically real in E. It is known that $|c_k|\leqq k$, $k=2, 3, \dots$. For $n\leqq 5$ we find the exact bounds on a_k, $k\leqq n$. We find also the coefficient regions for the cubic $z+a_2z^2+a_3z^3$ and the odd polynomial $z+a_3z^3+ +a_5z^5$. In everything which follows the a_k are real.

2. Main Theorem. Let $R(u)$ be a polynomial such that

$$R(\cos\theta) = \sum_{k=1}^{n} a_k \frac{\sin k\theta}{\sin\theta} = \frac{\operatorname{Im}\{P_n(e^{i\theta})\}}{\sin\theta}. \tag{1}$$

It follows that $P_n\in TR$ if and only if $R(\cos\theta)\geqq 0$ for all θ, $-\pi \leqq \theta < \pi$. Let $u=\cos\theta$. Then (1) can be written $R(u)=A\sum_{j=1}^{n-1} b_ju^j$, $-1\leqq u\leqq 1$. For fixed k, we determine the various forms $R(u)$ may assume in order that a_k be extremal.

Lemma 1. *Let b_j be real, $0\leqq j\leqq n-1$ and $b_{n-1}=1$ and suppose $\sum_{j=0}^{n-1} b_ju^j$ is either non-negative or non-positive for all u in $-1\leqq u\leqq 1$. Then there exist unique real a_j, $1\leqq j\leqq n$, $a_1=1$, such that*

$$\sum_{k=1}^{n} a_k \frac{\sin k\theta}{\sin\theta} = 2^{n-1} a_n \sum_{j=0}^{n-1} b_j u^j \tag{2}$$

and $P_n(z) = \sum_{k=1}^{n} a_k z^k$ belongs to the class TR.

PROOF. First, let us write

$$(\sin\theta)u^j = \sum_{k=1}^{j+1} c_{kj}\sin k\theta, \quad c_{kj} \text{ real}, \quad 1\leqq k\leqq j+1. \tag{3}$$

Then

$$(4)\qquad 2^{n-1}a_n\sum_{j=0}^{n-1}b_ju^j=2^{n-1}a_n\sum_{j=0}^{n-1}b_j\sum_{k=1}^{j+1}c_{kj}\frac{\sin k\theta}{\sin\theta}=2^{n-1}a_n\sum_{k=1}^{n}\sum_{j=k-1}^{n-1}b_jc_{kj}\frac{\sin k\theta}{\sin\theta}$$

which yields

$$(5)\qquad \frac{a_k}{a_n}=2^{n-1}\sum_{j=k-1}^{n-1}b_jc_{kj}.$$

In what follows denote the right-hand side of (5) by d_k.

It is clear, using induction and equation (3) that $c_{j+1,j}=2^{-j}$, so that (5) holds for $k=n$, that is, $d_n=1$. Let a_k/a_n be given by (5) for $2\leqq k\leqq n-1$. We note that $d_1\neq 0$, for if $d_1=0$ then $\sum_{k=1}^{n}d_k\frac{\sin k\theta}{\sin\theta}$ has constant sign and $\sum_{k=2}^{n}d_k\sin\theta\ \sin k\theta$ has constant sign which implies Re $\{z^{-1}(1-z^2)\sum_{k=2}^{n}d_kz^k\}$ does not change sign on $|z|=1$ and hence in $|z|<1$. But this polynomial has a zero at $z=0$ and hence must be identically zero, contradicting $d_n=1$. Therefore $d_1\neq 0$ and we define $a_n=1/d_1$. Thus the a_k, $2\leqq k\leqq n$, are uniquely determined by the equation $a_k=a_nd_k$ which proves (2). By a similar argument, using $a_1=1$, we can show Re $\{z^{-1}(1-z^2)\sum_{k=1}^{n}a_kz^k\}>$ >0 for $|z|<1$ which proves $P_n\in TR$.

Lemma 2. *Let $P_n(z)$ be a polynomial of degree n and let k be fixed, $2\leqq k\leqq n$. Suppose that among all polynomials in TR of degree n the k^{th} coefficient a_k assumes its extreme value for $P_n(z)$. Then it suffices to assume that all the zeros of $R(u)$ are real.*

Proof. Let b be real, $c\geqq 0$ and suppose $u=b+i\sqrt{c}$ is a zero of $R(u)$. Then $R(u)=2^{n-1}a_n(u^2-2bu+b^2+c)Q(u)$. Let Q and b be fixed. Then a_k, $1\leqq k\leqq n$, depends upon c. By Lemma 1, each $c\geqq 0$ determines a polynomial $P_n(z)$ which belongs to the class TR. Since each coefficient in $R(u)/a_n$ is linear in a_k/a_n, $1\leqq k\leqq n$, and each coefficient in $2^{n-1}(u^2-2bu+b^2+c)Q(u)$ is linear in c we have $1/a_n=$ $=A_1c+B_1$ and $a_k/a_n=A_kc+B_k$, $2\leqq k\leqq n$, A_k and B_k constants for $1\leqq k\leqq n$. Hence $a_k=(A_kc+B_k)(A_1c+B_1)^{-1}$ and the extreme values for a_k must occur when $c=0$ (assuming $a_n\neq 0$).

Lemma 3. *Under the hypothesis of Lemma 2, it suffices to assume that all the zeros of $R(u)$ are situated in the closed interval $[-1,1]$.*

Proof. Suppose $R(u)=2^{n-1}a_n(u-b)Q(u)$ where $b>1$ (or $b<-1$). By Lemma 1, $P_n\in TR$ for each b in the open interval $(1,\infty)$ (or $(-\infty,-1)$). By an argument similar to the one given in the proof of Lemma 2 we see that no extreme value of a_k can occur in $(1,\infty)$ unless a_k, $2\leqq k\leqq n$, is independent of b, in which case we may take $b=1$.

Since all zeros of $R(u)$ lying in the open interval $(-1,1)$ must be zeros of even multiplicity we have the following result.

Theorem 1. *Let $P_n(z)$ be a polynomial of degree n $(a_n\neq 0)$ and let k, $1<k\leqq n$, be fixed. If among all polynomials of degree n belonging to the class TR the k^{th} co-*

efficient a_k assumes its extreme value for $P_n(z)$, then $R(u)$ has the form

(6)
$$R(u) = \pm 2^{n-1} a_n(1 \pm u) \prod_{j=1}^{\frac{n-2}{2}} (u-\gamma_j)^2$$

for n even, where $-1 \leqq \gamma_j \leqq 1,\ 1 \leqq j \leqq (n-2)/2$ *and*

(7)
$$R(u) = -2^{n-1} a_n(1-u^2) \prod_{j=1}^{\frac{n-3}{2}} (u-\gamma_j)^2$$

or

(8)
$$R(u) = 2^{n-1} a_n \prod_{j=1}^{\frac{n-1}{2}} (u-\gamma_j)^2$$

for n odd, where $-1 \leqq \gamma_j \leqq 1,\ 1 \leqq j \leqq (n-1)/2$.

When n is even we find $\max a_k$ and $\min a_k$, for all $P_n \in TR$, by taking $R(u)$ in the form given by (6), where the positive signs are chosen. Indeed, $-P_n(-z) = \sum_{k=1}^{n} (-1)^{k-1} a_k z^k$ belongs to the class TR and leaves the coefficients with odd subscript unchanged while changing the sign of the coefficients with even subscript. Further, if R is given by (6) with positive signs chosen, then

(9)
$$\operatorname{Im}\left\{\frac{-P_n(-e^{i\theta})}{\sin\theta}\right\} = -R(-u) = -2^{n-1} a_n(1-u) \prod_{j=1}^{\frac{n-2}{2}} (u-\beta_j)^2 \quad \text{where } \beta_j = -\gamma_j$$

which implies the extreme values for the coefficients with odd subscript will be the same for either choice of sign in (6) while $|a_k|$ will be the same for either choice of sign. Thus for even k, $\min a_k = -\max |a_k|$ and $\max a_k = \max |a_k|$ where the extrema are taken over all $P_n \in TR$.

3. Coefficient bounds. Using the preceding results we calculate the extreme values for a_k, $2 \leqq k \leqq n$, $2 \leqq n \leqq 5$.

$n=2$. It is easy to verify that $P_2(z) = z + a_2 z^2$ is typically real if and only if $|a_2| \leqq 1/2$.

$n=3$. The polynomial $P_3(z) = z + a_2 z^2 + a_3 z^3$ belongs to TR if and only if $R(u) = 4a_3 u^2 + 2a_2 u + 1 - a_3 \geqq 0$. According to Theorem 1, $R(u) = -4a_3(1-u^2)$ which yields $a_2 = 0$, $a_3 = -1/3$ or $R(u) = 4a_3(\gamma^2 - 2\gamma u + u^2)$, $|\gamma| \leqq 1$, which yields $|a_2| \leqq 1$, $a_3 \leqq 1$. Hence $|a_2| \leqq 1$, $-\frac{1}{3} \leqq a_3 \leqq 1$ with equality for the polynomials $z + z^2 + \frac{1}{2} z^3$, $z - \frac{1}{3} z^3$ and $z + z^3$.

In the case $n=3$ we can find the coefficient region V in the a_2, a_3 plane. The equations of the boundary (∂V) of V are determined in part by finding the envelope of the family of lines bounding the half-planes $R(u) = 2ua_2 + (4u^2-1)a_3 + 1 \geqq 0$. The envelope is the ellipse $a_2^2 + 4(a_3 - \frac{1}{2})^2 = 1$. A short calculation shows that ∂V is that portion of the line $2a_2 - 3a_3 = 1$ between the points $\left(0, -\frac{1}{3}\right)$ and $\left(\frac{4}{5}, \frac{1}{5}\right)$, the upper arc of the ellipse between the points $\left(\frac{4}{5}, \frac{1}{5}\right)$ and $\left(-\frac{4}{5}, \frac{1}{5}\right)$ and the portion of the line $-2a_2 - 3a_3 = 1$ between the points $\left(-\frac{4}{5}, \frac{1}{5}\right)$ and $\left(0, -\frac{1}{3}\right)$.

If in addition to being typically real $P_3(z)$ is univalent in the unit disk, it was shown in [1] that the coefficient region for the univalent cubic is the intersection of V and the half-plane $a_3 \leqq \frac{1}{3}$.

$n=4$. For $n=4$, $R(u)=1-a_3+(2a_2-4a_4)u+4a_3u^2+8a_4u^3 \geqq 0$ and by Theorem 1 we have $R(u) = 8a_4(1+u)(\gamma-u)^2, |\gamma| \leqq 1$ which yields $a_4 = [2(4\gamma^2-2\gamma+1]^{-1}$, $a_3 = (1-2\gamma)(4\gamma^2-2\gamma+1)^{-1}$, $a_2 = (1-4\gamma+2\gamma^2)(1-2\gamma+4\gamma^2)^{-1}$. First, let us note $a_4 = 2^{-1}[(2\gamma-\frac{1}{2})^2+\frac{3}{4}]^{-1} \leqq \frac{2}{3}$ with equality for $\gamma=\frac{1}{4}$, that is, $P_4(z) = z+\frac{1}{6}z^2+\frac{2}{3}z^3+\frac{2}{3}z^4$. A simple argument involving only elementary calculus shows that $-1/3 \leqq a_3 \leqq 1$, $|a_2| \leqq (1+\sqrt{7})/3$ with equality for the polynomials $z-\frac{1}{3}z^2-\frac{1}{3}z^3+\frac{1}{6}z^4, z+z+z^3+\frac{1}{2}z^4$ and $z+\frac{1+\sqrt{7}}{3}z^2+\frac{6+4\sqrt{7}}{21}z^3+\frac{14-\sqrt{7}}{42}z^4$.

$n=5$. When $n=5$, $R(u)$ takes the form $R(u) = 1-a_3+a_5+(2a_2-4a_4)u+(4a_3-12a_5)u^2+8a_4u^3+16a_5u^4$ and according to Theorem 1, $R(u)$ must be of the form

$$R(u) = -16a_5(1-u^2)(u-\gamma)^2 \quad |\gamma| \leqq 1 \tag{10}$$

or

$$R(u) = 16a_5(u-\gamma_1)^2(u-\gamma_2)^2, \quad |\gamma_1| \leqq 1, \quad |\gamma_2| \leqq 1. \tag{11}$$

If $R(u)$ is given by (10) then $a_5 = -\frac{1}{2}(1+6\gamma^2)^{-1}$, $a_4 = \gamma(1+6\gamma^2)^{-1}$, $a_3 = \frac{1}{2}(1-4\gamma^2)(1+6\gamma^2)^{-1}$ and $a_2 = -4\gamma(1+6\gamma^2)^{-1}$. Again, simple arguments lead to the following inequalities, $-1/2 \leqq a_5 < 0$, equality for $\gamma=0$; $|a_4| \leqq \sqrt{6}/6$, equality for $\gamma=\sqrt{6}/6$; $-3/14 \leqq a_3 \leqq 1/2$, equality for $\gamma=1$ and 0; $|a_2| \leqq \sqrt{6}/3$, equality for $\gamma=-\sqrt{6}/6$.

If $R(u)$ is given by (11) we have, setting $\gamma_1=b$ and $\gamma_2=c$, $a_5 = \frac{1}{2}(1+2b^2+2c^2+8bc+8b^2c^2)^{-1/2} = \frac{1}{2}((b+c)^2+8\left(bc+\frac{1}{4}\right)^2+1/2)^{-1}$ which gives $0 < a_5 \leqq 1$ with equality for $b=-c$ and $bc=-1/4$, that is $b=1/2$, $c=-1/2$. Also, we get $a_4 = -4a_5(b+c)$, $a_3 = a_5(3+4b^2+4c^2+16bc)$ and $a_2 = -8a_5(b+c)(1+2bc)$. Again, long but elementary calculations yield $|a_4| \leqq 1$, $-(\sqrt{5}-1)/2 \leqq a_3 \leqq \frac{1+\sqrt{5}}{2}$ and $|a_2| \leqq \sqrt{2}$. The sharp bounds for a_5, that is, $a_5 = -\frac{1}{2}$ and $a_5=1$ are given by $P_5(z) = z+\frac{1}{2}z^3-\frac{1}{2}z^5$ and $P_5(z) = z+z^3+z^5$, respectively. The sharp bounds for a_4 occur for $P_5(z) = z - z^2 + z^3 - z^4 + \frac{1}{2} z^5$, for a_3, $P_5(z) = z + \frac{1 \pm \sqrt{5}}{2} z^3 + \frac{5 \pm \sqrt{5}}{10} z^5$ and for a_2, $P_5(z) = z - \sqrt{2} z^3 + \frac{5}{4} z^3 - \frac{\sqrt{2}}{2} z^4 + \frac{1}{4} z^5$.

Employing the methods of Theorem 1 will yield bounds on the coefficients for $n > 5$, however, the end result does not seem to justify the laborious calculations involved.

4. Coefficient regions. It is interesting to note that in the case of the odd polynomial $P_5(z) = z + a_3 z^3 + a_5 z^5$ we can find the coefficient region V in the a_3, a_5 plane. The boundary of V is determined in a manner similar to the case $n = 3$, that is, by determining the envelopes of half-planes $R(u) = (4u^2 - 1) a_3 + (16u^4 - 12u^2 + 1) a_5 + 1 \geqq 0$ which yields the ellipse $a_3^2 - 2a_3 a_5 + 5a_5^2 - 4a_5 = 0$.

Since $R(u)$ is an even function of u and $R(u) \geqq 0$ for all u, $0 \leqq u \leqq 1$, $R(0) = -a_3 + a_5 + 1 \geqq 0$ and $R(1) = 3a_3 + 5a_5 + 1 \geqq 0$ are two boundary half-planes. The line $R(0) = 0$ is tangent to the ellipse at $(3/2, 1/2)$ and intersects the line $R(1) = 0$ at $(1/2, -1/2)$. The line $R(1) = 0$ intersects the ellipse at $(-1/2, 1/10)$. Hence the coefficient region is bounded by the line $-a_3 + a_5 + 1 = 0$ from $(1/2, -1/2)$ to $(3/2, 1/2)$, the upper arc of the ellipse from $(3/2, 1/2)$ to $(-1/2, 1/10)$ and the line $3a_3 + 5a_5 + 1 = 0$ from $(-1/2, 1/10)$ to $(1/2, -1/2)$. The extreme values of a_5 are $a_5 = 1$ and $a_5 = -1/2$ and the extreme values for a_3 are $a_3 = \frac{1}{2}(1 \pm \sqrt{5})$.

It was proved in [1] that $P_n(z) = z + a_2 z^2 + \ldots + a_n z^n$ is univalent in $|z| \leqq 1$ if and only if Lim Sup $(|b_n(\alpha)|)^{1/n} \leqq 1$ for all α, $|\alpha| = 1$, $\alpha \neq 1$, where

$$b_n(\alpha) = \frac{(-1)^n}{1-\alpha} \det(c_{ij}) \tag{12}$$

where $c_{ij} = a_2 \lambda_2(\alpha)$, $i = j$, $c_{ij} = 1$, $i = j+1$, $c_{ij} = 0$, $j > i+1$, $c_{ij} = a_{k+2} \lambda_{k+2}(\alpha)$, $i = j+k$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $k = 0, 1, \ldots, n-1$ and $\lambda_1(\alpha) = 1$, $\lambda_k(\alpha) = 1 + \alpha + \ldots + \alpha^{k-1}$. If we apply this condition to the polynomial $P_5(z) = z + a_3 z^3 + a_5 z^5$ and denote $\det(c_{ij})$ by R_n we get the recursive relation $R_n + a_3 \lambda_3(\alpha) R_{n-2} + a_5 \lambda_5(\alpha) R_{n-4} = 0$. The roots of the auxillary equation are of the form

$$[(-a_3 \lambda_3(\alpha) \pm (a_3^2 \lambda_3^2(\alpha) - 4a_5 \lambda_5(\alpha))^{1/2})/2]^{1/2}.$$

Hence a necessary and sufficient condition on the complex numbers a_3 and a_5 for the polynomial $z + a_3 z^3 + a_5 z^5$ to be univalent in $|z| < 1$ is that

$$|a_3 \lambda_3(\alpha) \pm (a_3^2 \lambda_3^2(\alpha) - 4a_5 \lambda_5(\alpha))^{1/2}| \leqq 2 \tag{13}$$

for all α satisfying $|\alpha| = 1$, $\alpha = 1$.

Set $\gamma = 4\cos^2\left(\frac{\theta}{2}\right)$. Then $\lambda_5(\alpha)/\lambda_3^2(\alpha) = 1 - \gamma/(\gamma-1)^2 = A/4$. The condition (13) now becomes $|(\gamma-1)a_3||-1 \pm (1 - Aa_5/a_3^2)^{1/2}| \leqq 2$. An analysis of this inequality leads to families of half-planes whose boundaries are given by $(\gamma-1)a_3 - ((\gamma-1)^2 - \gamma)a_5 = 1$ and $-(\gamma-1)a_3 - ((\gamma-1)^2 - \gamma)a_j = 1$. The intersection of these half-planes determines a convex region in the $a_3 a_5$ plane which is the intersection of the two ellipses $a_3^2 \pm 2a_3 a_5 + 5a_5^2 - 4a_5 = 0$ and the three half-planes,

$a_5 \leqq 1/5$, $3a_3 + 5a_5 \geqq -1$ and $3a_3 - 5a_5 \geqq 1$. This is the coefficient region for $P(z)$. The region is symmetric with respect to the a_5-axis. In the right half-plane $a_3 \geqq 0$, the boundary consists of the segment of the line $3a_3 - 5a_5 = 1$ between the point $(0, -1/5)$ and the point $(1/2, 1/10)$, the boundary of the ellipse $a_3^2 + 2a_3a_5 + 5a_5^2 - 4a_5 = 0$ from the point $(1/2, 1/10)$ to the point $(3/5, 1/5)$ and the line segment from the point $(3/5, 1,5)$ to the point $(0, 1/5)$. We note that the point $(3/5, 1/5)$ on the boundary yields the greatest value of a_3 and a_5. Thus the extremal polynomial is $z + \frac{3}{5} z^3 + \frac{1}{5} a_5$.

In the univalent case one can consider the coefficient regions for the trinomial $z + a_k z^k + a_{2k-1} z^{2k-1}$ employing the above method. One obtains a difference equation whose roots r_k can be found, then employ an analysis of the inequality $|r_k| \leqq 1$.

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