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Locally conformal Berwald spaces and Weyl structures

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Dedicated to Professor Dr. Masao Hashiguchi on his 65th birthday.

Abstract. A differentiable manifold M is called a *Finsler manifold* if, to each point of M, a normed structure on the tangent space is associated. In the present paper, we are concerned with special Finsler manifolds, called *Berwald spaces* and *locally conformal Berwald spaces*. These spaces have the property that each tangent space is isometric to a fixed normed space. We show that to an arbitrary locally conformal Berwald space, a natural conformal structure is associated, and state some properties of Weyl structure associated with it.

1. Introduction

Let $\pi : TM \to M$ be the tangent bundle of a connected differentiable manifold M. A *Finsler connection* on M is defined as a connection of the pull-back $E := \pi^*TM$. If a (convex) Finsler metric is given on M, then we can define a natural inner product on E and, from which, a Finsler connection $D : \Gamma(E) \to \Gamma(TTM^* \otimes E)$ satisfying some natural conditions (cf. Aikou [2]). For the general theory of Finsler connections, see also Abate–Patrizio [1] and Matsumoto [10].

A Finsler manifold is said to be *modeled on a Minkowski space* if D is the pull-back of a linear connection on M. As a special case, if the linear connection is symmetric, the space is called a *Berwald space*. In this case, it is proved that the linear connection is the Levi-Civita connection of a Riemannian metric on M(cf. Szabó [12]). As another special case, a Finsler manifold is said to be *Wagner* if D is skew-symmetric. Such a space has been studied in Hashiguchi [6]. In Finsler geometry, conformal changes of Finsler metrics are also interesting subjects. As shown in

Hashiguchi–Ichijyō [7], a Finsler space is conformal to a Berwald space if and only if it is a special Wagner space. In these special Finsler space, the essential property is that the norm of any vector field defined by the given Finsler metric is invariant by the parallel translation with respect to the linear connection.

In the present paper, we are concerned with conformal change of Finsler metrics, and discuss some related topics in Finsler geometry. Especially, we are interested in the problem whether a Finsler space is locally or globally conformal to a Berwald space. In this problem, a Weyl structure of a conformal class plays an important role.

2. Locally conformal Berwald spaces

Let M be a connected differentiable manifold of dimension n, and $\pi : TM \to M$ the tangent bundle of M. We denote by $\{\pi^{-1}(U), (x^i, y^i)\}$ the canonical covering on TM induced from a covering $\{U, (x^i)\}$ by a system of coordinate neighborhoods on M. Here and in the following, the Latin indices take the value $1, \cdots, n$.

Definition 2.1. A function L defined on the total space TM is called a *Finsler metric* if it satisfies the following conditions:

(1) $L(x,y) \geq 0$, and $L(x,y) = 0$ if and only if $y = \mathbf{0}$,

(2) $L(x,y)$ is smooth on $TM - \{\mathbf{0}\}$,

(3) $L(x, ky) = kL(x,y)$ for $\forall k > 0$,

(4) the fundamental tensor field $g_{ij}(x,y) := \frac{1}{2}\partial^2 L/\partial y^i \partial y^j$ is positive-definite.

A manifold M with a Finsler metric L is called a *Finsler manifold*, and denoted by (M, L).

Now we assume that there exists a linear connection D whose parallel displacement preserves the norm function $L(x,y)$ invariant. Denoting by $\omega^i_j = \sum \Gamma^i_{jk}(x) dx^k$ the connection form of D, the assumption is written as follows:

$$d_\omega L := \sum_k \left(\frac{\partial L}{\partial x^k} - \sum_{l,m} y^m \Gamma^l_{mk}(x) \frac{\partial L}{\partial y^l} \right) dx^k \equiv 0. \tag{2.1}$$

We suppose that there exists a linear connection D satisfying (2.1). If we put

$$G = \{A \in GL(n, \mathbb{R});\ f(A\xi) = f(\xi)\},$$

then G is a compact Lie group, and may be considered as a closed subgroup of the orthonormal group $O(n)$. If there exists a linear connection D satisfying (2.1), the structure group of TM is reducible to G, that is, M has a G-structure (Ichijyō [8]). By using the compactness of G and the existence of bi-invariant Haar measure on G, we can proof the following theorem by the same method as Theorem 1 in Szabó [12].

Theorem 2.1. *Let (M, L) be a Finsler manifold. Suppose that there exists a linear connection D satisfying* (2.1). *Then D is a metrical connection of a Riemannian metric on M.*

Let (M, L) be a Finsler manifold stated in Theorem 2.1. Then each tangent space is isometric to a fixed normed vector space $T_xM \cong \mathbb{R}^n$ with the norm function $f(\xi) = L(x, \xi)$. Such a special Finsler manifold is studied by Ichijyō [8].

A Finsler manifold (M, L) is said to be *modeled on a Minkowski space* $(\mathbb{R}^n,\ f)$, if there exists a linear connection D satisfying (2.1). Especially, if D is symmetric, (M, L) is called a *Berwald space*. A Berwald space (M, L) with *flat* D is called a *locally Minkowski space*.

As a special case of Theorem 2.1, we get

Theorem 2.2 (Szabó [12]). *Let (M, L) be a Berwald space. Then, there exists a Riemannian metric g on M whose Levi-Civita connection coincides with the symmetric connection D.*

Remark 2.1. The Riemannian metric g is not uniquely determined (cf. Szabó [12]). We call such a metric an *associated* Riemannian metric for L. For the notion of associated Riemannian metric, see also Laugwitz [9].

Example 2.1. Suppose that a Riemannian manifold (M, g) admits a parallel non-zero vector field Y. We may assume that Y is a unit vector field. Then M has a $1 \times O(n-1)$-structure, that is, the structure group of TM is reducible to $1 \times O(n-1) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}; A \in O(n-1) \right\}$. Let $s_U = \{s_1 = Y, s_2, \dots, s_n\}$ be an adapted frame of the $1 \times O(n-1)$-structure. For an arbitrary $\xi = \sum \xi^i s_i$, we define its new norm $L(x, \xi)$ by

$$L(x, \xi)^2 = \frac{1}{2} \left\{ \|\xi\|^2 + \sqrt{\|\xi\|^4 + 4(\xi^1)^4} \right\},$$

where $\| \cdot \|$ means the norm with respect to g. This norm L is invariant by the action of $1 \times O(n-1)$. Then, if we denote by ω the connection form of the Levi-Civita connection ∇ with respect to s_U, it satisfies $\omega_i^1 = \omega_1^j = 0$. Consequently $L(x,y)$ satisfies $d_\omega L = 0$.

In the following, we are concerned with conformal changes of Finsler metrics. A conformal change of a Finsler metric L is given by the change $L \to \tilde{L}(x,y) = e^{\sigma(x)} L(x,y)$ for a smooth function $\sigma(x)$ on M. For a linear connection D with the connection form ω, because of the homogeneity (3), we get

$$\begin{aligned} d_\omega \tilde{L} &= e^\sigma (d_\omega L + d\sigma \otimes L) \\ &= e^\sigma \sum_k \left\{ \frac{\partial L}{\partial x^k} - \sum_{l,m} y^m \left(\Gamma^l_{mk}(x) - \delta^l_m \frac{\partial \sigma}{\partial x^k} \right) \frac{\partial L}{\partial y^l} \right\} dx^k \\ &= e^\sigma d_{\omega - I \otimes d\sigma} L. \end{aligned}$$

The condition $d_\omega \tilde{L} = 0$ is equivalent to $d_{\omega - I\otimes d\sigma} L = 0$.

Definition 2.2. A Finsler space (M, L) is said to be *locally conformal Berwald* (*l.c. Berwald*) if there exists an open covering $\{U_\alpha\}$ with a family of local functions $\{\sigma_\alpha(x)\}$ satisfying

$$d_\omega(e^{\sigma_\alpha} L) = 0$$

for a symmetric linear connection D with the connection form ω. Especially, if $U_\alpha = M$, we say (M, L) is *globally conformal Berwald* (*g.c. Berwald*).

If (M, L) is l.c. Berwald, the metric L satisfies the following for a smooth local function σ_α on U_α:

$$d_{\omega - I\otimes d\sigma_\alpha} L = 0. \tag{2.4}$$

From the homogeneity of L, we get on $U_\alpha \cap U_\beta$

$$0 = d_{\omega - I\otimes d\sigma_\alpha} L - d_{\omega - I \otimes d\sigma_\beta} L = -d(\sigma_\beta - \sigma_\alpha) \cdot L.$$

Hence, the function $\sigma_{\alpha\beta} := \sigma_\beta - \sigma_\alpha$ is locally constant on $U_\alpha \cap U_\beta$, and so the forms $\{\omega - I \otimes d\sigma_\alpha\}$ define a global linear connection on M.

Moreover, because of $\sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta} = 0$, the cycle $\{\sigma_{\alpha\beta}\}$ defines a 1-Čeach cocycle with coefficients in $\mathbb{R}$, and a 1-Čeach cohomology class $[\sigma_{\alpha\beta}]$. Then, we have

Proposition 2.1. *A l.c. Berwald space is g.c. Berwald if and only if the cohomology class* $[\sigma_{\alpha\beta}]$ *is trivial. Hence, if the first Betti number* $b_1(M)$ *vanishes,* (M, L) *is g.c. Berwald.*

Proof. If the class $[\sigma_{\alpha\beta}]$ is trivial, there exists a 0-chain $\{c_\alpha\}$ with coefficients in $\mathbb{R}$ such that $\delta\{c_\alpha\} = \{\sigma_\alpha\beta\}$. Hence we have

$$e^{\sigma_\alpha + c_\alpha} L = e^{\sigma_\beta + c_\beta} L$$

on $U_\alpha \cap U_\beta$, that is, $\{e^{\sigma_\alpha + c_\alpha} L\}$ is globally defined. Moreover, because of $dc_\alpha = 0$, we get

$$d_\omega(e^{\sigma_\alpha + c_\alpha} L) = e^{\sigma_\alpha + c_\alpha} d_{\omega - I \otimes d(\sigma_\alpha + c_\alpha)} L = e^{\sigma_\alpha + c_\alpha} d_{\omega - I \otimes d\sigma_\alpha} L = 0,$$

which shows that (M, L) is g.c. Berwald. □

To close this section, we state on the conformal flatness of Finsler manifolds.

Definition 2.3. A Finsler space (M, L) is said to be *conformally flat* if it is locally conformal to a locally Minkowski space.

Then we have

Theorem 2.3. *Let* (M, L) *be a l.c. Berwald space. Then,* (M, L) *is conformally flat if and only if its symmetric connection* D *is flat.*

Proof. Suppose that (M, L) is a l.c. Berwald space whose symmetric connection D is flat. Then, there exists a covering $\{U_\alpha\}$ such that, on each U_α, there exists a function σ_α satisfying $d_\omega(e^{\sigma_\alpha} L) = 0$. Hence, if D is flat, $e^{\sigma_\alpha} L$ is locally Minkowski. Thus (M, L) is conformally flat. The converse is trivial. □

Because of $d_\omega(e^{\sigma_\alpha} L) = e^{\sigma_\alpha} d_{\omega - I \otimes d\sigma_\alpha} L$, if (M, L) is conformally flat, by Theorem 2.2, the forms $\{\omega - I \otimes d\sigma_\alpha\}$ define a metrical connection of an associated metric g. Hence we have $Dg = -2d\sigma_\alpha \otimes g$, which is equivalent to $D(e^{2\sigma_\alpha} g) = 0$. Consequently, D is the Levi-Civita connection of the local metric $e^{2\sigma_\alpha} g$. Since D is flat, g is conformally flat. Hence Theorem 2.3 means that a l.c. Berwald space is conformally flat if and only if its D is the Levi-Civita connection of a conformally flat Riemannian metric.

Now we shall show to construct an example of conformally flat Finsler metric on M which admits a conformally flat Riemannian metric.

Example 2.2. Let M be the unit sphere S^n in the Euclidean space $\mathbb{R}^{n+1}$. The induced metric on S^n is given by

$$ds^2 = \frac{4}{(1+|x|^2)^2} \sum_{i=1}^{n} dx^i \otimes dx^i.$$

Its metric tensor is given by $g_{ij} = \frac{4}{(1+|x|^2)^2}\delta_{ij}$. This metric is conformal to the flat metric $ds_0^2 = \sum_{i=1}^{n} dx^i \otimes dx^i$, that is, (S^n, ds) is conformally flat. We denote by D the flat connection of ds_0^2 with the connection form ω.

We define a new norm function $L(x,\xi)$ on TM by

$$L(x,\xi)^2 := \frac{2}{(1+|x|^2)^2} \left\{ \|\xi\|^2 + \sqrt{\|\xi\|^4 + 4(\xi^1)^4} \right\},$$

where we put $\|\xi\|^2 = \sum_{i=1}^{n} (\xi^i)^2$. Then it is easily seen that D satisfies $d_\omega \left\{ (1+|x|^2)L \right\} = 0$. Hence (S^n, L) is a conformally flat Finsler manifold.

3. Weyl structures

The discussions in the previous section suggest us to consider a conformal structure and its Weyl structure if a given Finsler space is l.c. Berwald. In this section, we state some basic facts on Weyl structures on a conformal manifold (cf. Gauduchon [5], Tod [13]).

Let C be a conformal structure on M, that is, C is the set of conformally equivalent classes of Riemannian metrics on M. A *Weyl structure* on a conformal manifold (M, C) is a linear connection D satisfying the following conditions:

(1) D preserves the conformal class C,

(2) D is symmetric.

Here and in the following, we denote a linear connection by its covariant derivation $D : \Gamma(TM) \to \Gamma(TM \otimes TM^*)$.

The $CO(n)$-principal bundle defined by C is denoted by P, where $CO(n)$ is the conformal group of degree n. The condition (1) means that a Weyl structure D is to be induced by a connection on P. In the following, we shall identify any connection on P with a linear connection on M. Then, (1) is equivalent to the existence of a 1-form θ satisfying

$$Dg = -2\theta \otimes g \tag{3.1}$$

for any representative g of C. If we take another representative $g' = e^{2f(x)}g$, the corresponding 1-form θ' is given by

$$\theta' = \theta - df. \tag{3.2}$$

From the condition (2), we have a representation of D as follows:

$$D = \nabla + \tilde{\theta}, \tag{3.3}$$

where ∇ is the Levi-Civita connection of g, and the $\mathrm{End}(TM)$-valued 1-form $\tilde{\theta}$ is defined by

$$\tilde{\theta}_X Y = \theta(X)Y + \theta(Y)X - g(X,Y)\Theta, \tag{3.4}$$

for the dual Θ of θ, that is, $\theta(X) = g(X, \Theta)$ for $\forall X \in \Gamma(TM)$. We note that the 1-form θ is given by

$$\theta = \frac{1}{n}(\mathrm{Tr.}\,\omega - \mathrm{Tr.}\,\omega^g) = \frac{1}{n}(\mathrm{Tr.}\,\omega - d\log(\det g)), \tag{3.5}$$

where ω (resp. ω^g) is the connection form of D (resp. ∇).

Remark 3.1. A Weyl structure D is said to be *closed* if $d\theta = 0$. Since the form $\mathrm{Tr.}\,\omega$ defines a connection D^E of the line bundle $E = \wedge^n TM$ and $\mathrm{Tr.}\,\Omega$ is the curvature of D^E, a closed Weyl structure defines a flat connection of E.

By virtue of (3.5), the closeness of a Weyl structure D does not depend on the choice of the representative $g \in C$. If D_1, D_2 are Weyl structures of (M, C), there exists a 1-form θ satisfying $D_1 = D_2 + \tilde{\theta}$, where $\tilde{\theta}$ is defined by (3.4). Hence the set of all Weyl structures of (M, C) is an affine space modeled on the vector space of all 1-form on M. Let D be a closed Weyl structure of a conformal class C. Since the closeness of the 1-form θ in (3.1) is independent on the choice of a representative g, we put $\alpha(D) := [\theta] \in H^1(M, \mathbb{R})$. Then, the mapping $\alpha : D \to \alpha(D)$ is an homomorphism from the affine space of closed Weyl structures to the de Rham vector space $H^1(M, \mathbb{R})$, and $\ker\alpha$ is the set of Levi-Civita connections of global metrics in C.

The pair (g, θ) of a metric g in the conformal class C and the corresponding 1-form θ is said to be *distinguished* if θ is co-closed with respect to g, that is,

$$\delta\theta := -\sum g^{jk}\nabla_j\theta_k = 0.$$

Let θ be the g-harmonic representative of an element of $H^1(M, \mathbb{R})$. For the Levi-Civita connection ∇ of g, the Weyl structure defined by (3.3) is closed, and the pair (g, θ) is distinguished. Assume that M be compact and orientable, and choose an arbitrary metric g in the conformal class C. If D is a closed Weyl structure, the corresponding 1-form θ is closed, and by Hodge's decomposition, θ may be written as $\theta = H\theta + df$, where $H\theta$ means the harmonic part and f is a smooth function on M. Then, by (3.2), the 1-form θ corresponding to $\tilde{g} = e^{2f(x)}g$ is harmonic. Especially, if the 1-st Betti number $b_1(M)$ vanishes, D is the Levi-Civita connection of a global metric in C.

The following theorem is also fundamental.

Theorem 3.1 (Gauduchon [5]). *Let (M, C) be a compact, orientable conformal manifold of $\dim M \geq 3$. For any Weyl structure D on (M, C), there exists a metric g in the conformal class C, unique up to a constant factor, such that the pair (g, θ) is distinguished.*

4. Ricci curvatures

We shall consider the curvature of the Weyl structure D on a conformal manifold (M, C). Using the relation (3.3), we get the following relation between the curvatures R^D of D and R^∇ of ∇ (cf. Besse [3]).

$$R^D(X,Y)Z = R^\nabla(X,Y)Z - (d^\nabla\tilde{\theta})_{X,Y}Z - \tilde{\theta}_X\tilde{\theta}_Y Z + \tilde{\theta}_Y\tilde{\theta}_X Z,$$

where $d^\nabla : \Gamma(M, \wedge^k TM^* \otimes \mathrm{End}(TM)) \to \Gamma(M, \wedge^{k+1} TM^* \otimes \mathrm{End}(TM))$ is the covariant derivation defined by ∇. The Ricci curvature Ric^D of D is given by

$$\begin{aligned}\mathrm{Ric}^D(X,Y) &= \mathrm{Ric}^\nabla(X,Y) - (n-1)(\nabla_X\theta)(Y) + (\nabla_Y\theta)(X) \\ &\quad + (n-2)\theta(X)\theta(Y) + \left\{\delta\theta - (n-2)|\theta|^2\right\} g(X,Y),\end{aligned} \tag{4.1}$$

for the one $\mathrm{Ric}^\nabla(X,Y)$ of ∇. The scalar curvature S^D is given by

$$S^D = S^\nabla + 2(n-1)\delta\theta - (n-1)(n-2)|\theta|^2 \tag{4.2}$$

for the one S^∇ of ∇. Then we have

Proposition 4.1. *Let D be a Weyl structure on a conformal manifold (M, C). The following conditions are equivalent.*

(1) *$d\theta = 0$, that is, D is closed,*

(2) *Ric^D is symmetric,*

(3) *D is locally the Levi-Civita connection of a local metric in C.*

PROOF. The equivalence of (1) and (3) is obvious from the Weyl change (3.2). From (4.1), we get

$$\begin{aligned}\mathrm{Ric}^D(X,Y) - \mathrm{Ric}^D(Y,X) &= n\{(\nabla_X\theta)(Y) - (\nabla_Y\theta)(X)\}\\ &= nd\theta(X,Y).\end{aligned}$$

This shows the equivalence of (1) and (2). □

From (4.2), we get the following integral formula:

$$\int_M |\theta|^2 dv = \frac{1}{(n-1)(n-2)}\int_M (S^\nabla - S^D)dv,$$

which shows

Corollary 4.1. *If $S^D \geq S^\nabla$ holds on a compact and orientable M, then the 1-form θ vanishes identically, and $D = \nabla$.*

In a compact and orientable (M,g), the following formulae are known. For any vector field X on M,

$$\int_M \left\{\mathrm{Ric}^\nabla(X,X) + |\nabla X|^2\right\} dv = \int_M \left\{\frac{1}{2}|d\xi|^2 + |\delta\xi|^2\right\} v,$$

$$\int_M \left\{\mathrm{Ric}^\nabla(X,X) - |\nabla X|^2 - |\delta X|^2\right\} dv = -\frac{1}{2}\int_M |L_X g|^2 dv,$$

where ξ is the dual of X and $\delta X = -\sum_j \nabla_j X^j$. From these identities, we have

Proposition 4.2 (WATANABE [14], YANO [15]). *Let (M,g) be a compact and orientable Riemannian manifold. For any vector field X on (M,g), the following inequalities hold:*

$$\int_M \{\mathrm{Ric}^\nabla(X,X) + |\nabla X|^2\} dv \geq 0,$$

$$\int_M \{\mathrm{Ric}^\nabla(X,X) - |\nabla X|^2 - (\delta X)^2\} dv \leq 0.$$

The equalities hold if and only if X is a harmonic vector field and a Killing vector field on (M,g) respectively.

In the following, we assume that M is compact, orientable and $\dim M \geq 3$. By Theorem 3.1, for any Weyl structure D on (M,C), there exists a unique (up to a constant factor) Riemannian metric g in C such that the pair (g,θ) is distinguished. Then, we apply Proposition 4.2 to the dual vector field Θ.

First we consider the case where the Weyl structure D is closed.

Theorem 4.1. *Let (M, C) be a compact and orientable conformal manifold of $\dim M \geq 3$, and D a Weyl structure on (M, C) with distinguished pair (g, θ). Suppose that D is closed.*

(1) *If the Ricci curvature Ric^{∇} of g is non-negative, the dual Θ of θ is a Killing vector field on (M, g).*

(2) *If Ric^{∇} is positive-definite, D coincides with ∇.*

Proof. Because of $\delta\theta = \delta\Theta = 0$ for the distinguished pair (g, θ), we have

$$\int_M \{\mathrm{Ric}^{\nabla}(\Theta, \Theta) + |\nabla\Theta|^2\} dv \geq 0, \tag{4.3}$$

$$\int_M \{\mathrm{Ric}^{\nabla}(\Theta, \Theta) - |\nabla\Theta|^2\} dv \leq 0. \tag{4.4}$$

Since D is closed and (g, θ) is distinguished, θ is harmonic, that is, the dual Θ is a harmonic vector field on (M, g). Hence, Proposition 4.2 implies

$$\int_M \{\mathrm{Ric}^{\nabla}(\Theta, \Theta) + |\nabla\Theta|^2\} dv = 0.$$

If Ric^{∇} is non-negative, we have $\mathrm{Ric}^{\nabla}(\Theta, \Theta) = 0$ and $\nabla\Theta = 0$. Then, the equality of (4.4) also holds, that is, Θ is a Killing vector field on (M, g). Furthermore, if Ric^{∇} is positive-definite, as is well-known (cf. Bochner [4]), the first Betti number $b_1(M)$ vanishes: $H^1(M, \mathbb{R}) = \{0\}$. Hence there exists no harmonic 1-form other than zero, and so $\theta = 0$, which means that D coincides with ∇. □

From (2) of Theorem 4.1, we have

Proposition 4.3. *Let (M, C) and D be the same as in Theorem 4.1. Suppose that the 1-form θ is parallel: $\nabla\theta = 0$. If the Ricci curvature Ric^D is positive definite, D coincides with ∇.*

Proof. Since the pair (g, θ) is distinguished, Ric^D is given by

$$\begin{aligned} \mathrm{Ric}^D(X, Y) = \mathrm{Ric}^{\nabla}(X, Y) &- (n-1)(\nabla_X\theta)(Y) + (\nabla_Y\theta)(X) \\ &+ (n-2)\theta(X)\theta(Y) - (n-2)|\theta|^2 g(X, Y), \end{aligned}$$

from which and $\nabla\theta = 0$ we have

$$\begin{aligned} \mathrm{Ric}^D(X, X) &= \mathrm{Ric}^{\nabla}(X, X) + (n-2)\{\theta(X)^2 - |\theta|^2 g(X, X)\} \\ &= \mathrm{Ric}^{\nabla}(X, X) + (n-2)\{g(X, \Theta)^2 - g(\Theta, \Theta) g(X, X)\} \\ &\leq \mathrm{Ric}^{\nabla}(X, X). \end{aligned}$$

The last inequality means that, if the Ricci curvature Ric^D of D is positive-definite, Ric^∇ is also positive-definite. Hence the proposition has been proved. □

Secondly, we consider the case where the dual vector field Θ of the corresponding 1-form θ in (3.1) is a Killing vector field on (M, g).

Theorem 4.2. *Let (M, C) be a compact and orientable conformal manifold of $\dim M \geq 3$, and D a Weyl structure on (M, C) with distinguished pair (g, θ). Suppose that the dual Θ of the 1-form θ is a Killing vector field on (M, g).*

(1) *If the Ricci curvature Ric^∇ of g is non-positive, then D is closed and θ is harmonic.*

(2) *If Ric^∇ is negative-definite, then D coincides with ∇.*

Proof. Since Θ is a Killing vector field on (M, g), (4.4) implies

$$\int_M \{\mathrm{Ric}^\nabla \Theta, \Theta) - |\nabla\Theta|^2\} dv = 0.$$

If Ric^∇ is non-positive, we have $\mathrm{Ric}^\nabla(\Theta, \Theta) = 0$ and $\nabla\Theta = 0$, from which θ is closed, that is, the Weyl structure D is closed. Consequently, θ is harmonic:

$$\triangle\theta := (d\delta + \delta d)\theta = 0.$$

If Ric^∇ is negative-definite, it is also well-known that there exists no Killing vector field other than zero. Hence we have $\Theta = 0$, and $\theta = 0$. This means that $D = \nabla$. □

The Ricci curvature Ric^D of a Weyl structure D is expressed as in (4.1) by the Ricci curvature Ric^∇ and the corresponding 1-form θ. A Weyl structure D is said to be an *Einstein-Weyl structure* if the symmetric part of Ric^D is proportional to g, and a Riemannian manifold (M, g) is said to be *Einstein-Weyl* if it admits a Einstein-Weyl structure on (M, C), where C is the conformal class constructed from g (see e.g., Pedersen-Swann [11]). If a Riemannian manifold (M, g) admits a closed Einstein-Weyl structure D, the space is locally conformal to Einstein space, and if $\theta \equiv 0$, the space is Einstein.

Due to Tod [13], if D is an Einstein-Weyl structure and (g, θ) is the distinguished pair, the vector field Θ dual to the 1-form θ is a Killing vector field on (M, g). Hence, if (M, g) is Einstein-Weyl, we have

$$\mathrm{Ric}^\nabla(X, Y) = \frac{1}{n} S^\nabla g(X, Y) + \frac{n-2}{n} |\theta|^2 g(X, Y) - (n-2)\theta(X)\theta(Y).$$

This equation is called the *Einstein-Weyl equation*. Then, (4.2) leads to

$$\mathrm{Ric}^{\nabla}(X,X) = \frac{1}{n}S^{D}g(X,X) + (n-2)\{|\theta|^{2}g(X,X) - \theta(X)^{2}\}. \tag{4.5}$$

Proposition 4.4. *Let (M,C) and D be the same as in Theorem 4.1. Suppose that D is a closed Einstein-Weyl structure on (M,C) and its scalar curvature S^{D} is positive. Then (M,g) is an Einstein manifold.*

Proof. Let (g,θ) be the distinguished pair of D. Since D is a closed Einstein-Weyl structure, θ is a harmonic form on M. If S^{D} is positive, (4.5) shows that Ric^{∇} of (M,g) is positive-definite, and so $b_1(M)$ vanishes. Hence D coincides with the Levi-Civita connection of g, and so (M,g) is Einstein. □

5. Some remarks on Finsler spaces

A Finsler space (M,L) is l.c. Berwald if and only if there exists a open covering $\{U_\alpha\}$ with the family of local smooth functions $\{\sigma_\alpha\}$ satisfying (2.4) for a symmetric linear connection D. Because of $d\sigma_\alpha = d\sigma_\beta$ on $U_\alpha \cap U_\beta$, it defines a closed 1-form θ satisfying

$$d_{\omega - I\otimes\theta}L = 0.$$

Conversely, if there exists a closed 1-form θ and a symmetric connection D satisfying this equation, we get (2.4) and so, (M,L) is l.c. Berwald. Then we have our main theorem.

Theorem 5.1. *Let (M,L) be a Finsler manifold. Then (M,L) is l.c. Berwald if and only if there exists a closed Weyl structure D of a conformal class C satisfying*

$$d_{\omega}L = -\theta \otimes L, \tag{5.1}$$

where θ is the corresponding closed 1-form.

Proof. It is sufficient to show that the connection D satisfying (5.1) is a closed Weyl structure. By Theorem 2.1, if (M,L) is l.c. Berwald, the connection defined by the form $\omega - I\otimes\theta$ is metrical with respect to an associated Riemannian metric g. Hence the connection D satisfies

$$Dg = -2\theta \otimes g.$$

This means that D is a closed Weyl structure of C, where C is the conformal class determined by g. □

Remark 5.1. By Proposition 4.1, the closeness of D in the theorem above may be replaced by the symmetry of Ric^D.

Applying the results in the previous section, we shall show the effects of Weyl structure of C to (M,L), and consider the case where a l.c. Berwald space becomes a g.c. Berwald space.

By definitions, a l.c. Berwald space is g.c. Berwald if and only if there exists a global metric g in the class C such that the connection D coincides with the Levi-Civita connection ∇ of g. Hence, from Corollary 4.1 and Theorem 5.1, we have

Proposition 5.1. *Let (M,L) be a compact and orientable l.c. Berwald space. If there exists a metric g in C whose scalar curvature S^∇ satisfies $S^D \geq S^\nabla$, then (M,L) is a g.c. Berwald space.*

Theorem 5.2. *Let (M,L) be a compact and orientable Finsler manifold of $\dim M \geq 3$ which is l.c. Berwald. For the distinguished pair (g,θ) of the Weyl structure D, if one of the following conditions is satisfied, then (M,L) is g.c. Berwald.*

(1) *$\nabla\theta = 0$ and the Ricci curvature Ric^D of D is positive-definite.*

(2) *The vector field Θ dual to θ is a Killing vector field, and the Ricci curvature Ric^∇ of g is negative-definite.*

Proof. The first statement is derived from Proposition 4.3, and the second from (2) in Theorem 4.2. □

From Proposition 4.4, we have

Proposition 5.2. *Let (M,L) be the same as in Theorem 5.2. Suppose that D is a Einstein-Weyl structure. If the scalar curvature S^D of D is positive, then (M,L) is g.c. Berwald and an Einstein metric is associated with L.*

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