Ω-Groups with Composition

By GÜNTER PILZ (Linz)

Introduction

In 1956 P. J. HIGGINS investigated groups with multiple operators, so-called Ω-groups, in his famous paper [5]. The present paper is devoted to the study of sets of functions, mapping an Ω-group into itself, which form again Ω-groups. A purely axiomatic discission is made possible by the definition of an "*Ω-group with composition*", more briefly called "*Ω-composition group*". In such a system the notions of "even" and "odd" functions can be generalized. Subsequently such Ω-composition groups are studied, in which each element is the sum of an even and an odd element. Finally, we deduce order-theoretic properties with the help of these notions. We restrict ourselves to functions of one variable.

1. Definitions and basic results

Let $\langle G, +, -, 0, \omega_1, \omega_2, \dots \rangle$ be an Ω-group with (in general non abelian) addition $+$, subtraction $-$ and zero element 0; $\omega_1, \omega_2, \dots$ denote the further operations (cf. [5], [6] and [7]).

An *Ω-composition group* is an Ω-group $\langle G, +, -, 0, \circ, \omega_2, \dots \rangle$ with an operation "$\circ$" of weight 2, called *composition*, fulfilling for all $g_i \in G$

(1) $$(g_1+g_2)\circ g_3 = g_1\circ g_3 + g_2\circ g_3$$

(2) $$(g_1\circ g_2)\circ g_3 = g_1\circ (g_2\circ g_3)$$

and

(3) $$\omega_i(g_1, \dots, g_{n_i})\circ g = \omega_i(g_1\circ g, \dots, g_{n_i}\circ g),$$

if the weight of ω_i is equal to $n_i > 0$, or

(4) $$\omega_j \circ g = \omega_j,$$

if ω_j is a 0-ary operation.

An *Ω-composition ring* is an Ω-composition group $\langle G, +, -, 0, \circ, \cdot, \omega_3, \dots \rangle$, where $\omega_2 = \cdot$ is a binary operation, such that $\langle G, +, -, 0, \cdot \rangle$ is a ring.

The element $c \in G$ is called *constant*. if

(5) $$c \circ 0 = c.$$

The set $C(G)$ of all constant elements of G is an Ω-subcomposition group. This can be easily verified. Each operation of weight 0 determines a constant element.

If there exists a right and at the same time left neutral element with respect to the composition, it will be denoted by j.

Remark. For the foundations of the theory of Ω-groups see [6]. Ω-composition groups of the form $\langle G, +, -, 0, \circ \rangle$ are called *near-rings*. A discussion can be found in [2]. Ω-composition rings of the type $\langle G, +, -, 0, \circ, \cdot \rangle$ are called *composition rings* (cf. [1] and [11]) or *TO-algebras* (see e.g. [8]).

Examples of Ω-composition groups: all sets of functions, mapping an Ω-group into itself, which are closed with respect to all operations (composition included), form Ω-composition groups, if the operations $+$, ω_i are transferred to functions in the usual manner and if composition means substitution of functions. Conversely, W. Nöbauer proved in [10] that each Ω-composition group is isomorphic to such an Ω-composition group of functions on a suitable Ω-group.

An important example is formed by the composition ring $R[x]$ of all polynomials over a commutative ring R with multiplicative unit 1. The operations are defined as in the general case above (cf. [9]). One can prove immediately:

$$C(R[x]) = R; \quad j = x.$$

An element $g_1 \in G$ is called *even*, if for all $h \in G$ the equation

(6) $$g_1 \circ (-h) = g_1 \circ h \quad \text{holds.}$$

The element $g_2 \in G$ is called *odd*, if

(7) $$g_2 \circ (-h) = -g_2 \circ h, \quad \text{for all} \quad h \in G.$$

In $R[x]$ all

$$\sum_{i=0}^{n} r_{2i} x^{2i}$$

are even, all

$$\sum_{i=0}^{n} r_{2i+1} x^{2i+1}$$

are odd.

To avoid trivial, but troublesome distinctions of several cases we postulate in addition to (1)—(3) that there is no $g \in G$, $g \neq 0$, with $g + g = 0$.

A subset S of G is called a *base for equality*, if the implication (8) holds:

(8) $$g \circ s = h \circ s \quad \text{for all} \quad s \in S \quad \text{implies} \quad g = h \quad \text{(cf. [12]).}$$

Lemma 1. *Let $C(G)$ be a base for equality. Then the element $g \in G$ is already even, resp. odd, if*

(9a) $$g \circ (-c) = g \circ c \quad \textit{resp.} \quad \text{(9b)} \quad g \circ (-c) = -g \circ c$$

is valid for all $c \in C(G)$.

To prove this, regard $(g \circ (-h)) \circ c = g \circ (-h \circ c)$, g, $h \in G$. $h \circ c =: c_0 \in C(G)$. In case (a) one can calculate $g \circ (-h \circ c) = g \circ (-c_0) = g \circ c_0 = (g \circ h) \circ c$, therefore is $(g\circ(-h))\circ c = (g\circ h)\circ c$, which implies $g\circ(-h) = g\circ h$. In case (b) holds $(g\circ(-h))\circ c = g\circ(-c_0) = -g\circ c_0 = (-g\circ h)\circ c$, therefore $g\circ(-h) = g\circ h$.

Lemma 2. *Let $\langle G, +, -, 0, \circ, \omega_2, \ldots\rangle$ be an Ω-composition group with $j\in G$. If $g, h\in G$, g even, h odd, so is*

$$(10)\qquad g=h \quad \text{equivalent to} \quad g=h=0.$$

This implies that 0 is the only element which is even and odd at the same time.

(10) is valid because $g=h$ implies
$g = g\circ j = g\circ(-j) = h\circ(-j) = -h\circ j = -h$, and therefore $g=h=0$.

Theorem 1. *a) The set $E(G)$ of all even elements of $G = \langle G, +, -, 0, \circ, \omega_2, \ldots\rangle$ forms an Ω-sub-composition group of G containing $C(G)$.*
b) Let G contain j. Then $E(G)\neq G$ or $G=\{0\}$.

PROOF. a) Consider ω_r, having the weight $n_r>0$. Then $\omega_r(g_1, \ldots, g_{n_r})\circ(-g) = \omega_r(g_1\circ(-g), \ldots, g_{n_r}\circ(-g)) = \omega_r(g_1\circ g, \ldots, g_{n_r}\circ g) = \omega_r(g_1, \ldots, g_{n_r})\circ g$, if all g_i $(1\leqq i\leqq n_r)$ are even. This implies $\omega_r(g_1, \ldots, g_{n_r})\in E(G)$. If ω_r has weight 0, then $\omega_r\circ(-g) = \omega_r = \omega_r\circ g$, therefore all elements determined by 0-ary operations are in $E(G)$. If g lies in $E(G)$, so does $-g$, because of $(-g)\circ(-h) = -(g\circ(-h)) = -(g\circ h) = (-g)\circ h$. This implies $0\in E(G)$. Each $c\in C(G)$ fulfills $c\circ(-g) = c = c\circ g$, for all $g\in G$, which shows that $C(G)\subseteq E(G)$.

b) If $j\in G$, then j is odd: $j\circ(-g) = -g = -j\circ g$. $j=0$ implies $G=\{0\}$. If j is unequal to 0, then, by Lemma 2, $j\notin E(G)$.

Theorem 2. *Let $j\in G\neq\{0\}$, $\{\omega_i\}$ containing a binary operation ω_r, which is left and right distributive with respect to $+$ and with existing left and right neutral element e_r. Then*

a) *$e_r\in E(G)$. If ω_r is associative and if there exists a left and right inverse element $i_r(g_0)$ for $g_0\in E(G)$, then $i_r(g_0)$ is uniquely determined and is contained in $E(G)$.*
b) $C(G)\neq E(G)$.

PROOF. a) Let ω_r be an operation of the described kind. By definition we have $\omega_r(g, e_r)=\omega_r(e_r, g)=g$, for all $g\in G$. We define $h\in G$ by $h:= e_r-e_r\circ(-j)$. $h\circ(-g)= h\circ(-g) = e_r\circ(-g)-e_r\circ(-j)\circ(-g) = -(e_r\circ g-e_r\circ(-g)) = -h\circ g$. This shows that h is odd.

$$(11)\qquad \omega_r(h, h)\circ(-g) = \omega_r(h\circ(-g), h\circ(-g)) = \omega_r(-h, -h)\circ g.$$

From $\omega_r(h, h) = \omega_r(h+0, h) = \omega_r(h, h)+\omega_r(0, h)$ one gets

$$(12)\qquad \omega_r(0, h) = \omega_r(h, 0) = 0$$

and therefore

$$0 = \omega_r(h, 0) = \omega_r(h, h-h) = \omega_r(h, h)+\omega_r(h, -h).$$

Summarizing these results one gets

$$(13)\qquad \omega_r(-h, -h) = -(-\omega_r(h, h)) = \omega_r(h, h).$$

Combining (11) with (13) proves that $\omega_r(h,h)\in E(G)$.

$$\omega_r(h,h)=\omega_r\big(e_r-e_r\circ(-j),\,e_r-e_r\circ(-j)\big)=$$
$$=\omega_r(e_r,e_r)-\omega_r\big(e_r,e_r\circ(-j)\big)-\omega_r\big(e_r\circ(-j),e_r\big)+\omega_r(-e_r,-e_r)\circ(-j)=$$
$$=e_r-e_r\circ(-j)-e_r\circ(-j)+\big(-(-e_r)\big)\circ(-j)=e_r-e_r\circ(-j)=h.$$

This shows that h is at the same time even and odd, by lemma 2 we get $h=0$. This means $e_r\circ j=e_r=e_r\circ(-j)$ and this implies $e_r\circ g=e_r\circ(-g)$ for all $g\in G$. Therefore $e_r\in E(G)$.

Given $g_0\in E(G)$ with existing inverse element $i_r(g_0)$ with respect to ω_r. Let ω_r be associative. If $i'_r(g_0)$ is also an inverse element of g_0, then from

$$i_r(g_0)=\omega_r\big(i_r(g_0),e_r\big)=\omega_r\big(i_r(g_0),\omega_r(g_0,i_r(g_0))\big)=$$
$$=\omega_r\big(\omega_r(i_r(g_0),g_0),i'_r(g_0)\big)=\omega_r\big(e_r,i'_r(g_0)\big)=i'_r(g)$$

the uniqueness of the inverse element follows.

By definition, (14) holds:

$$\omega_r\big(g_0,i_r(g_0)\big)=e_r \tag{14}$$

This implies

$$\omega_r\big(g_0,i_r(g_0)\big)\circ(-g)=\omega_r\big(g_0\circ(-g),i_r(g_0)\circ(-g)\big)=\omega_r\big(g_0\circ g,i_r(g_0)\circ(-g)\big)=$$
$$=e_r\circ(-g)=e_r\circ g=\omega_r\big(g_0,i_r(g_0)\big)\circ g=\omega_r\big(g_0\circ g,i_r(g_0)\circ g\big).$$

Because of the proved uniqueness of the inverse element one gets $i_r(g_0)\circ g=$ $=i_r(g_0)\circ(-g)$ and therefore $i_r(g_0)\in E(G)$.

b) Just like in (13) one gets
$\omega_r(j,j)\circ(-g)=\omega_r(-g,-g)=\omega_r(-j,-j)\circ g=\omega_r(j,j)\circ g$ and therefore $\omega_r(j,j)\in E(G)$. If $\omega_r(j,j)$ is contained in $C(G)$, then $\omega_r(j,j)\circ 0=\omega_r(0,0)=0$. It follows $0=\omega_r(j,j)\circ e_r=\omega_r(e_r,e_r)=e_r$, and finally $g=\omega_r(g,e_r)=\omega_r(g,0)=0$ for all $g\in G$, which implies the excluded case $G=\{0\}$. Therefore $E(G)\neq C(G)$, and the theorem is completely proved.

Corollary. If G denotes an Ω-composition ring with multiplicative unit 1, then $1\in E(G)$.

Furthermore, the following conclusions hold. Each sum of even (odd) elements is even (odd). If g is any element of G, then the composition of g with an even element is again even. In the case of abelian addition $E(G)$ is therefore a left ideal in the near-ring $\langle G,+,-,0,\circ\rangle$.

Theorem 3. *Let G be an Ω-composition ring, $j\in G$: j be no left nullifier with respect to multiplication. Then $E=E(G)$ has the same cardinal number like the set $U=U(G)$ of all odd elements of G:$E\sim U$.*

Proof. Consider the mapping $\varphi: g\to j\cdot g$ for all $g\in E$. One verifies immediately: $j\cdot g\in U$. Therefore $\varphi(E)=j\cdot E\subseteqq U$. φ is injective, because $j\cdot g_1=j\cdot g_2$ implies $g_1=g_2$. One gets $E\sim j\cdot E\subseteqq U$, and, by a similar argument, $U\sim j\cdot U\subseteqq E$. This implies $U\sim j\cdot U\subseteqq E\sim j\cdot E\subseteqq U$, therefore $\operatorname{card} U=\operatorname{card} j\cdot U\leqq\operatorname{card} E=$ $=\operatorname{card} j\cdot E\leqq\operatorname{card} U$, and from this one gets $\operatorname{card} E=\operatorname{card} U$.

2. Cleavable Ω-composition groups

An element g of an Ω-composition group G let be called *cleavable*, if it can be written in the form

(15) $$g = \bar{g} + \underline{g} \quad \text{with} \quad \bar{g} \in E(G), \quad \underline{g} \in U(G).$$

If all elements of G are cleavable, then we say that G is *cleavable*.

Let $S(G)$ be the set of all cleavable elements of G. If the conditions of theorem 2 are fulfilled, then

(16) $$C(G) \subset E(G) \subset S(G) \subset G \quad \text{holds.}$$

$E(G) \neq S(G)$ is valid, because, for example, $\omega_r(j, j) + j$ is not even, but cleavable.

The composition ring (and therefore the near-ring, too) $R[x]$ is cleavable:

$$\sum_{i=0}^{n} r_i x^i = \sum_{i=0}^{\left[\frac{n}{2}\right]} r_{2i} x^{2i} + \sum_{i=0}^{\left[\frac{n-1}{2}\right]} r_{2i+1} x^{2i+1}$$

We will give another example of a cleavable Ω-composition group (resp. Ω-composition ring). Let T be the near-ring (composition ring) generated by the real numbers R and the functions $x \to x$, $x \to \sin x$ and $x \to \cos x$ in the near-ring (composition ring) $C(R)$ of all continuous functions from R into R. $C(R)$ is cleavable, for each function $f(x) \in C(R)$ can be divided by

(17) $$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

into an even and an odd part.

We show the cleavability of T assuming that T is a composition ring. In the case of near-rings the proof is similar if one desits from forming products.

Let M be the subset of T containing all elements of T which can be formed from $R, x, \sin x, \cos x$ by a finite number of multiplications and compositions. The total number of these operations which lead to $m \in M$ we call the *step* of m.

Lemma 3. *T is the set of all elements of the form*

(18) $$a = \sum_{\alpha \in A} a_\alpha$$

with $a_\alpha \in M$, A being a finite set of indices.

Proof.* Let W be the set of all elements of the kind (18). $W \subseteqq T$ is trivial. All the elements generating T are contained in M and therefore in W. Thus we have only to show that W is a composition ring.

If $a, b \in W$, so is $a - b$. By the distributive laws we get $a \cdot b \in W$. Take

$$a = \sum_{\alpha \in A} a_\alpha, \quad b = \sum_{b \in B} b_\beta \qquad (a_\alpha, b_\beta \in M).$$

*) The present kind of the proof is due to Prof. W. Nöbauer, whom I wish to thank very much for his suggestions.

It follows

$$a \circ b = \big(\sum_{\alpha \in A} a_\alpha\big) \circ \big(\sum_{\beta \in B} b_\beta\big) = \sum_{\alpha \in A} a_\alpha \circ \big(\sum_{\beta \in B} b_\beta\big).$$

It is sufficient to show that for all $a_\alpha \in M$

$$(19) \qquad a_\alpha \circ \big(\sum_{\beta \in B} b_\beta\big) \in W$$

We verify this by induction on the step of a. Is the step $=0$, then $a_\alpha = r \in R$, x, $\sin x$ or $\cos x$. For $a_\alpha = r, x$ (19) is trivial, for $a_\alpha = \sin x$ or $\cos x$ (19) holds because of the trigonometric addition theorems. Assume now the lemma to be proved for all $a_\alpha \in M$ with step less than n. If $a_\alpha \in M$, then $a_\alpha = k \cdot 1$ or $a_\alpha = k \circ 1$ with $k, 1 \in M$. Let a_α have the step n. Then k, 1 have lower steps. If $a_\alpha = k \cdot 1$, this implies

$a_\alpha \circ \big(\sum_{\beta \in B} b_\beta\big) = a_\alpha \circ b = (k \cdot 1) \circ b = (k \circ b) \cdot (1 \circ b) \in W$ and, if $a_\alpha = k \circ 1$:
$a_\alpha \circ \big(\sum_{\beta \in B} b_\beta\big) = a_\alpha \circ b = (k_\alpha \circ 1) \circ b = k \circ (1 \circ b) \in W$, because of $1 \circ b \in W$. This shows that $T = W$ and the lemma is proved.

By the help of this lemma we get now

Theorem 4. *T is cleavable.*

PROOF. If $a \in T$, we have by the preceding lemma:

$$a = \sum_{\alpha \in A} a_\alpha \qquad (a_\alpha \in M).$$

The theorem is proved if we show that each $m \in M$ is even or odd. We do this again by induction on the step of m. If m has step 0, then again $m = r$, x, $\sin x$ or $\cos x$. The assertion holds, because r and $\cos x$ are even, x and $\sin x$ are odd. Now let m have step n. Again we can write m as $m = k \cdot 1$ or $m = k \circ 1$, with $k, 1 \in M$ having a lower step than n. By induction hypothesis and theorem 1 m is either even or odd. This proves the theorem.

Finally, we remark two useful statements in cleavable Ω-composition groups:

(20) $g \circ (-h) = (\bar{g} - \underline{g}) \circ h$ *for all* $g, h \in G$

(21) $g \in U(G)$ *implies* $g \circ 0 = 0$.

3. Applications to the theory of ordered Ω-composition groups

An Ω-composition group $\langle G, +, -, 0, \circ, \omega_2, \dots \rangle$ is called *fully* (resp. *partially*) *ordered*, if it is a fully (partially) ordered universal algebra (see [3]) and if

(22) $\langle G, +, -, 0 \rangle$ is a fully (partially) ordered group

(23) $g, h \geqq 0$ implies $g \circ h \geqq 0$.

The Ω-composition group is called *fully (partially) ordered in the wider sense*, if (23) needs only to be valid if $h \notin C(G)$. (cf. [4], [11], [12]).

For Ω-composition rings one postulates furthermore that $\langle G, +, -, 0, \circ, \cdot \rangle$ is a fully (partially) ordered ring.

The symbols $|g|$ and sign g are defined as usual. As an example of a composition ring which is ordered in the wider sense but not ordered take again $R[x]$ with the "lexicographic" order (cf. [11]): call $\sum_{i=0}^{n} r_i x^i > 0$ if and only if $r_n > 0$ in a given order in R. It can easily be shown by a counterexample that $R[x]$ cannot be ordered. The composition ring $\{\sum_{i=1}^{n} r_i x^i,\ r_i \in R\}$ is, in opposite to $R[x]$, ordered in the lexicographic order. A detailed discussion of partially and fully ordered composition rings can be found in [11] and [12].

Theorem 5. *Let G be a fully ordered, cleavable Ω-composition group with abelian addition. Let be $g \in G$, $h \in G$. (If G is only fully ordered in the wider sense one has to postulate $h \notin C(G)$.) Then*

$$(24) \qquad |g \circ h| = |g| \circ |h|, \quad \textit{if} \quad h \geqq 0 \quad \textit{or} \quad g \in E \cup U.$$

$$(25) \qquad |g \circ h| \leqq |g| \circ |h|, \quad \textit{if} \quad h < 0 \quad \textit{and} \quad \operatorname{sign} \bar{g} = \operatorname{sign} \underline{g}.$$

$$(26) \qquad |g \circ h| \geqq |g| \circ |h|, \quad \textit{if} \quad h < 0 \quad \textit{and} \quad \operatorname{sign} \bar{g} \neq \operatorname{sign} \underline{g}.$$

Remark. In (25) *and* (26) *the relations* $<$ *and* $>$ *can hold actually.*

Proof. To (24):

(i) $g \geqq 0$, $h \geqq 0$. Then $g \circ h \geqq 0$ and (24) is verified.

(ii) $g < 0$, $h \geqq 0$. Then $|g| \circ |h| = (-g) \circ h = -(g \circ h) \geqq 0$, which implies (24).

(iii) $g \in E = E(G)$, $h < 0$ (for $h \geqq 0$ see (i) and (ii)). $g \circ (h) = g \circ h$. If $g \geqq 0$, then $g \circ (-h) = g \circ h \geqq 0$; if $g < 0$, then $g \circ h \geqq 0$ and therefore $|g \circ h| = -(g \circ h) = (-g) \circ (-h) = |g| \circ |h|$.

(iv) $g \in U = U(G)$. If $g \geqq 0$, then $g \circ (-h) = -g \circ h \geqq 0$ $(h < 0)$ and therefore $g \circ h \leqq 0$. $|g| = g$, $|h| = -h$. This implies $|g| \circ |h| = g \circ (-h) = g \circ h = |g \circ h|$. If $g < 0$ then $|g \circ h| = g \circ h$ and $|g| \circ |h| = (-g) \circ (-h) = g \circ h = |g \circ h|$.

To (25):

(i) Let g be > 0. $\operatorname{sign} \bar{g} = \operatorname{sign} \underline{g}$ implies $\bar{g} > 0$, $\underline{g} > 0$. $h' := -h > 0$. $|g| \circ |h| = g \circ (-h) = g \circ h'$. Therefore $|g \circ h| = |(\bar{g} + \underline{g}) \circ h| \leqq |\bar{g} \circ h| + |\underline{g} \circ h| = |\bar{g} \circ (-h')| + |\underline{g} \circ (-h')| = |\bar{g} \circ h'| + |-\underline{g} \circ h'| = \bar{g} \circ h' + \underline{g} \circ h' = (\bar{g} + \underline{g}) \circ h' = g \circ h' = |g| \circ |h|$, and (25) is proved in this case.

(ii) $g = 0$ trivially implies (25).

(iii) $g < 0$ implies $\bar{g} < 0$, $g < 0$. $g' := -g > 0$. (25) (i) tells us that $|g \circ h| = |(-g') \circ h| = |-(g' \circ h)| = |g' \circ h| \leqq |g'| \circ |h| = |g| \circ |h|$.

To (26):

(i) If $\bar{g} = 0$ or $\underline{g} = 0$, then (26) is valid because of (24).

(ii) Let $\bar{g}$ be > 0, consequently $\underline{g} < 0$. $\underline{g} := -\underline{g} > 0$. Then $|g \circ h| = |g \circ (-h')| = |(\bar{g} + \underline{g}) \circ (-h')| = |g \circ h' + g' \circ h| - |\bar{g} \circ h' - \underline{g} \circ h'| = \bar{g} \circ h' + \underline{g}' \circ h'$. On the other hand, $|g| \circ |h| = |\bar{g} - \underline{g}'| \circ |h| \leqq |\bar{g}| \circ |h'| + |\underline{g}'| \circ |-h'| = \bar{g} \circ h' + \underline{g} \circ h' = |g \circ h|$. This verifies (26).

(iii) Let $\bar{g}$ be < 0, then $\underline{g} > 0$ and $\bar{g}' := -\bar{g} > 0$. $g' := -g = -\bar{g} - \underline{g} = \bar{g}' + \underline{g}'$, with $\bar{g}' > 0$, $\underline{g}' < 0$. Applying (26) (ii) one arrives at

$$|g \circ h| = |-g \circ h| = |g' \circ h| \geqq |g'| \circ |h| = |g| \circ |h|.$$

To show the remark, consider again the polynomial ring $R[x]$ with the lexicographic order. At first we take

$g := x^2+x-1 = (x^2-1)+x,$

$h := -x-1 < 0, h \notin C(R[x]) = R.$ We have $|g \circ h| = |(x^2+x-1)\circ(-x-1)| = |x^2+x-1| = x^2+x-1$, but

$$|g| \circ |h| = (x^2+x-1)\circ(x+1) = x^2+3x+1,$$

therefore $|g \circ h| < |g| \circ |h|$. Then we take

$g := x^2-x+1 = (x^2+1)-x > 0, \quad h := -x+1 < 0; \ h \notin R.$

$|g \circ h| = |(x^2-x+1)\circ(-x-1)| = |x^2-x-1| = x^2-x+1,$

but

$|g| \circ |h| = (x^2-x+1)\circ(x-1) = x^2-3x+3,$ therefore

$|g \circ h| > |g| \circ |h|$, and theorem 5 is completely proved.

Specializing $G = R[x]$, theorem 5 can be formulated in this way:

$|g \circ h| = |g| \circ |h|$, if $h \geqq 0$ or g contains only even or odd degrees.

$|g \circ h| \leqq |g| \circ |h|$, if $h < 0$ and the coefficients of the greatest even and odd degrees have the same sign.

$|g \circ h| \geqq |g| \circ |h|$, if $h < 0$ and the coefficients of the greatest even and odd degrees have opposite signs.

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