Continuously differentiable spaces

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Dirac, in the early years of this century, invented the δ-function and it became immediately popular with the physcists who applied them widely. It was the problem of the mathematicians to find a good mathematical justification for the use of Dirac function. The whole difficulty was that no sound mathematician could accept the δ-function as a function. One could not reject it also, because it was so useful in getting valid results in physics and other applied sciences. It was L. SCHWARTZ who observed first that a good meaning could be given only if we expand our vision by defining a generalized concept of functions. After Schwartz's discovery of the theory of distribution [6], [7], many mathematicians gave their own meaning for the situation. To mention some of the important works (i) Mikusinski's theory of operators [4] (ii) Mikusinski—Sikorski theory of distributions [5] (iii) M. J. Lighthill's theory of generalized functions [2] (iv) Liverman's theory of distributions [3] (v) Gelfand and Shilov theory of distributions [1].

All these spaces develop spaces possessing essentially the following features (i) the structure of the vector space with translations (ii) the convergence structure (iii) differentiation — and these are compatible with each other. This leads to the concept of continuously differentiable spaces (CD spaces) and in § 1 we deal with some basic ideas and properties of CD spaces.

Once can see that the various spaces of generalized functions are instances of CD spaces. One natural question will be what the structure of a general CD-space is. In § 2 we exhibit a natural association between CD spaces on the one hand and topological vector spaces together with a prescribed endomorphism on the other. Since a canonical form for a general endomorphism of a topological vector space is not known even in such a simple case as the Hilbert space, we leave the investigation in this direction.

§ 1.

Definition 1. 1: A topological vector space X over a field*) K is (i) a topological vector space**) X over K and (ii) a family of linear transformations, $\tau_h(-\infty<h<\infty)$ such that $\tau_0=1$ (the identity transformation), $\tau_{(h_1+h_2)}=\tau_{h_1}\tau_{h_2}$ and $\tau_h(x)$ is a continuous function of both h and $x\in X$.

*) Unless explicity stated otherwise, K refers to the real number field.

**) Topology is given by a notion of sequential convergence.

Definition 1. 2: An element $a \in X$ where X is a topological translation vector space, is said to be differentiable if $\underset{h \to 0}{\text{Lt}} \frac{(\tau_h a - a)}{h}$ exists. This limit, when it exists, is called the derivative of a and denoted by a'. If the derivative exists for every $a \in X$, then X is called a differentiable space.

Remark 1. 1: The element 0 is always differentiable $(\tau_h(0)=0)$ and has the derivative 0.

Definition 1. 3: A topological vector space X is called an allowable space if whenever a_n converges to zero and if a_n' exists, then a_n' converges to zero.

Definition 1. 4: A topological translation vector space X is called a CD space if every element of the space has derivatives and if whenever a_n converges to zero, then a_n' also converges to zero. (i.e.) If it is both differentiable and an allowable space.

Remark 1. 2: The class of all topological vector spaces with linear continuous maps forms a category T. The class of all CD spaces with translation preserving linear continuous maps forms a subcategory G.

Definition 1. 5: We shall say that there is a minimal embedding of a topological translation vector space X into the category G if there exists a CD-space say $\overline{X}$ and a linear, differentiable (that is derivative preserving) continuous map t_1 from X to $\overline{X}$ such that whenever $t_2: X \to Y$ is a linear differentiable continuous map from X to the space Y of G, we can find a linear differentiable, continuous map $t_3: \overline{X} \to Y$ such that $t_3 t_1 = t_2$. We call $\overline{X}$, a minimal embedding of X.

Definition 1. 6: An element in a topological translation vector space X is called a polynomial P_k of degree less than or equal to a positive integer k if its k-th derivative exists and is zero and its $(k-1)$-th derivative is not zero. The k-th derivative of $a \in X$ is written as $a^{(k)}$. Thus $a^{(k)}=0$ if and only if $a=$some P_k.

Definition 1. 7: An element a of a topological translation vector space X is said to have a primitative b in X if there exists a $b \in X$ such that $b'=a$. This b will also be referred to as $\int a$ (this symbol can, of course, have more than one meaning). We shall speak of $\int\int a$ as $\int_2 a$ and so on.

Definition 1. 8: A topological translation vector space X is called a primitive space if every element of X has a primitive.

Theorem 1. 1. *Any primitive space X can be embedded minimally into a* CD *space $\overline{X}$.*

Proof. First we shall actually construct the CD space as follows: Consider ordered pairs (a, k) where $a \in X$ and k a non-negative integer. Define

$$(a, k)+(b, k) = (a+b, k)$$
$$(a, k)+(b, m) = \left(\int_m a + \int_k b, k+m\right)$$
$$\alpha(a, k) = (\alpha a, k),$$

where α is an element in the field. Define $(a, k)\sim(b, m)$ if and only if $\int_m a - \int_k b = P_{k+m}$. Now this relation $\sim$ is easily proved to be an equivalence relation compatible with the operations of addition and multiplication by reals and this divides the class of all pairs (a, k) into mutually disjoint classes like $\{(a, k)\}$-forming in a natural way a vector space. Every equivalence class is called a generalized function or a distribution. For sake of convenience we denote a distribution by (a, k) itself. We now define $\tau_h(a, k)$ to be $(\tau_h a, k)$ and we say (a_n, k_n) converges to (a, k) if $(a_n, k_n)\sim\sim(b_n, m)$; $(a, k)\sim(b, m)$ and b_n converges to b in X. It is easy to verify that these definitions depend only on the classes and not on the representative elements so chosen. The space of all distributions defined thus is the required CD space $\overline{X}$, for we note that (i) $[(\tau_h a, k)-(a, k)]/h$ converges to $(a, k+1)$ as h tends to zero, because $[(\tau_h a, k)-(a, k)]/h$ is $(\tau_h b-b/h, k+1)$ where b is a primitive of a. Also whenever (a_n, k_n) converges to zero, (a_n, k_n+1) also converges to zero. For, given $(a_n, k_n)\sim(b_n, m)$ and b_n converges to zero in X, we have $(a_n, k_n+1)\sim(b_n, m+1)$ and b_n converges to zero. Now, correspond $a\in X$ to $t_1(a)=(a, 0)$ in $\overline{X}$. If b corresponds to $(b, 0)$ and if $(a, 0)\sim(b, 0)$ then $a=b$ in X. Therefore, the map t_1 is one-one. It is also seen to be a differentiable and continuous map. Let now Y be any CD space and $t_2: X\to Y$ be a linear differentiable and continuous map. To each element (a, k) of $\overline{X}$, we can associate the k-th derivative of $t_2 a$ in Y. It is straightforward to verify this mapping is well-defined from $\overline{X}$ to Y and preserves translation. We need only to show that the map is continuous. For this, let (a_n, k_n) converges to zero in $\overline{X}$. This means $(a_n, k_n)\sim(b_n, m)$ and b_n converges to zero in X. Since t_2 is continuous, we find $t_2 b_n$ converges to zero in Y. Since Y is a C—D space the m-th derivative of $t_2 b_n$ converges to zero. Thus, by corresponding the m-th derivative of $t_2 b_n$ to (a_n, k_n) we see that the map is continuous.

Theorem 1. 2. *The direct product of topological translation vector spaces E_t is a* CD *space if and only if each E_t is a* CD *space.*

Proof. Let $\{x_t\}$ be a sequence of elements in $\prod_t E_t$. $\{x_t\}$ converges to 0 in $\prod_t E_t$ if and only if its projection in E_t converges to zero in E_t. Since E_t is a CD space $\{x_t^{(k)}\}$ exist and $\{x_t^{(k)}\}$ converges to zero in E_t. Consider $\{x_t^{(k)}\}$ in $\prod_t E_t$. $\{x_t^{(k)}\}$ converges to zero in $\prod_t E_t$. Hence $\prod_t E_t$ is a CD space. Conversely consider $(0, 0, \dots, 0, x_t 0, \dots \dots)$ in the direct sum $\sum_t E_t$. This converges to zero whenever $\{x_t\}$ converges to zero in E_t. Therefore, whenever $\{x_t^{(k)}\}$ converges to zero in E_t, $(0, 0, \dots \dots, 0, x_t^{(k)}, 0, \dots)$ also converges to zero in $\prod_t E_t$.

Theorem 1. 3. *Every translation topological closed subspace S of a* CD *space G is also a* CD *space.*

Proof. Let $f\in S$; to show $f'\in S$.

For, in G, $\operatorname{Lt}_{h\to 0} (\tau_h f-f)/h = f'\in G$ since G is a CD space. Since S is closed f' is in S. By induction $f^{(m)}$ is in S. If f_n converges to f in S, to show that $f_n^{(m)}$ converges in S. For, as elements of G, $f_n^{(m)}$ converges to $f^{(m)}$ in G since G is a CD space and whenever $f_n, f\in S$, $f_n^{(m)}$ and $f^{(m)}\in S$. Therefore $f_n^{(m)}$ converges to $f^{(m)}$ in S. Hence S is a CD space.

Theorem 1. 4. *If X is any* CD *space, and Y is a differentiable closed subspace of X, then the quotient space X/Y is a* CD *space.*

PROOF. First we shall show that X/Y is a differentiable space. For, consider an element $[x+Y]$ which is a coset, of X/Y. Define $[x+Y]^{(k)} = [x^{(k)}+Y]$ and this exists since X is a CD space. The above definition of derivative in X/Y is definite. In other words, if $[x+Y] = [z+Y]$ then $[x^{(k)}+Y] = [z^{(k)}+Y]$. For if $[x+Y] = = [z+Y]$ (i.e.) if $x-z$ is in Y, then $(x-z)^{(k)} = x^{(k)}-z^{(k)}$ is also in Y since Y is a differentiable space, (i.e.) $[x^{(k)}+Y] = [z^{(k)}+Y]$. Secondly, we shall show that X/Y, is an allowable space. Consider a sequence $[x_n+Y]$ of cosets in X/Y. Define $\{[x_n+Y]\}$ converges to zero in X/Y if and only if $\{z_n\}$ converges to z, where z_n is a suitable sequence of representative elements from each of the cosets $[x_n+Y]$.

To show that X/Y is an allowable space, we have to show that whenever, $[x_n+Y]$ converges to zero in X/Y then $[x_n+Y]^{(k)}$ if it exists, converges to zero in X/Y. $[x_n+Y]^{(k)}$ exists, since X/Y is a differentiable space. Given $[\{x_n+Y]\}$ converges to zero in X/Y, (i.e.) $\{x_n+Y\}$ converges to zero in X/Y, (i.e.) z_n converges to zero in X.

Since X is a CD space, $z_n^{(k)}$ exists and $(z_n^{(k)})$ converges to zero in X. Therefore, $\{x_n^{(k)}+Y\}$ converges to zero in X/Y, (i.e.) $\{[x_n^{(k)}+Y]\}$ converges to zero in X/Y. Hence X/Y is a CD space.

Theorem 1. 5. *Dual space of a* CD *space is, in a natural way a translation topological vector space, which by itself is a* CD *space.*

PROOF. Let X be a CD space and let X^* be its dual. If $T\in X^*$ define $\tau_h T$ as follows: for every $a\in X$, $\langle\tau_h T, a\rangle=\langle T, \tau_h a\rangle$, $\tau_h T$ is continuous from X to X. Thus X^* is a translation vector space. Now

$$\langle T', a\rangle = \operatorname*{Lt}_{h\to 0} \frac{\langle\tau_h T-T, a\rangle}{h} = \operatorname*{Lt}_{h\to 0} \frac{\langle\tau_h T, a\rangle-\langle T, a\rangle}{h} =$$

$$= \operatorname*{Lt}_{h\to 0} \frac{\langle T, \tau_h a\rangle-\langle T, a\rangle}{h} = \operatorname*{Lt}_{h\to 0} \frac{\langle T, \tau_h a-a\rangle}{h} = \langle T, a'\rangle.$$

Therefore T' exists. Similarly $\langle T^{(k)}, a\rangle=\langle T, a^{(k)}\rangle$ for all $a\in K$, and $k=1, 2, 3, \dots$. Thus X^* is a differentiable space. If T_n converges to 0 in the dual topology of X^*, then $\langle T_n^{(k)}, a\rangle=\langle T_n, a^{(k)}\rangle$ converges to zero. Therefore $T_n^{(k)}$ converges to zero in the dual topology of X^*. Thus X^* is an allowable space. So X^* is a CD space.

Theorem 1. 6. *The inductive limit of* CD *spaces is a* CD *space in a natural way.*

PROOF. We know that the inductive limit X of topological linear spaces X_α is a topological linear space. The inductive limit as made into a translation topological linear space in the following way. If $a\in X$, then $a\in X_\alpha$ for some α. Therefore $\tau_h\, a$ is in X_α since X_α is a translation space, and so $\tau_h a$ is in X, which we define to be the translation of $a\in X$ by h. If all X_α are differentiable spaces, the inductive limit is also a differentiable space. For, if $a\in X$, then $a\in X_\alpha$ for some α. So $a^{(k)}$ is in X_α since X_α is a differentiable space and so $a^{(k)}$ is in X. If all the X_α' s are allowable spaces, the inductive limit is also an allowable space. For, if a_n converge to zero in X, all a_n are in X_α for some α, and a_n converges to zero in X_α. Since X_α is an allowable space $a_n^{(k)}$ converges to zero in X_α and hence in X. Hence the theorem. Similarly we have,

Theorem 1. 7. *The projective limit of* CD *spaces is also a* CD *space.*

Theorem 1. 8. *Any* CD *space X can be densely embedded in a complete* CD *space.*

PROOF. We know that any linear topological space X can be embedded in a complete linear topological space $\bar{X}$. It is enough if we recognize that, this completion is a CD space if the initial space is a CD space. If $a\in\bar{X}$, define $\tau_h a=\operatorname{Lt}_n \tau_h a_n$ where $a=\operatorname{Lt} a_n$ and a_n is a Cauchy sequence in X converging to a. Define

$$a' = \operatorname*{Lt}_{h\to 0} \frac{\tau_h a - a}{h} = \operatorname*{Lt}_{h\to 0} \operatorname*{Lt}_{n\to\infty} \frac{\tau_h a_n - a_n}{h} = \operatorname*{Lt}_{h\to 0} \operatorname*{Lt}_{n\to\infty} \frac{\tau_h a_n - a_n}{h} = \operatorname*{Lt}_{n\to\infty} a_n'$$

Now a_n is a Cauchy sequence. Therefore, a_n' is also a Cauchy sequence in X. $\operatorname{Lt}_{n\to\infty} a_n'$ exists in $\bar{X}$, (i.e.) a' exists in $\bar{X}$. Hence $\bar{X}$ is a differentiable space. To show that $\bar{X}$ is a CD space we have to show that whenever x_n converges to zero in $\bar{X}$ then $x_n^{(k)}$ also converges to zero in $\bar{X}$. Now $x_m=\operatorname{Lt}_m x_{nm}$, where x_{nm}, is a Cauchy seqeunce in X. Now $\operatorname{Lt}_n x_n=0$. Therefore $\operatorname{Lt}_n\operatorname{Lt}_m x_{nm}=0$ so $\operatorname{Lt}_n\operatorname{Lt}_m x_{nm}^{(k)}=0$ since X is a CD space. But $\operatorname{Lt}_m x_{nm}^{(k)}=x_n^{(k)}$ and $\operatorname{Lt}_n x_n^{(k)}=0$ hence the theorem.

Theorem 1. 9. *The set of all endomorphisms on a* CD *space G which commutes with translations is a* CD *space — where we declare that the endomorphisme* $\Theta_n\to\Theta$ *if and only if for each x in G,* $\Theta_n x\to\Theta x$ *in G.*

PROOF. Let E be the set of all endomorphism on G.

Let F, K, H be elements of E. $F{:}g\to Fg$ where $g\in G$; $K{:}g\to Kg$.

(i) $F+K{:}g\to(F+K)g$ where we define $(F+K)g = Fg+Kg$. Now we have to show that this commutes with translation

$$(F+K)\tau_h g = F(\tau_h g)+K(\tau_h g) = \tau_h Fg+\tau_h Kg = \tau_h(Fg+Kg) = \tau_h(F+K)g$$

(ii) $\alpha F{:}g\to\alpha Fg=\alpha(Fg)$.

(iii) $F{:}g\to Fg$ then $F^{(m)}{:}g\to F^{(m)}\,g$ where we define $F^{(m)}g=Fg^{(m)}$. Here again we have to show that this commutes with translation. For

$$F^{(m)}\tau_h g=F(\tau_h g)^{(m)}=F(\tau_h g^{(m)})=\tau_h(Fg^{(m)}).$$

(iv) F_n converges to zero, means $F_n g$ converges to zero for every g in G where $F_n{:}g\to F_n g$. If F_n converges to zero, then $F_n^{(m)}{:}g\to F_n^{(m)}=F_n g^{(m)}$ converges to zero as n tends to ∞, for every $g\in G$. Therefore $F_n^{(m)}$ converges to zero in E for $n=1, 2, \dots$. Therefore E is a CD space.

Remark: In the definition 1. 1, instead of 1. 1 (i), if we take a vector space with an L-convergence compatible with the vector space structure, we can define a CD space in a similar way and the above theorems are valid. In this case we see that the space of Mikusinski operators becomes a CD space.

§ 2.

Now we show that the mapping which associate with a CD space the system consisting of a given topological vector space with the derivating mapping is one-one. Thus the knowledge of the topological vector space together with this continuous mapping itself is tantamount to knowing the CD spaces with all its translations.

The essence of this idea is that the infinitesmal operator corresponding to the 'velocity' completely unfolds the translations.

We shall denote the CD space consisting of the topological vector space V, the translations E_h, and the 'derivating mapping D by (V, E_h, D). Here we restrict ourselves to locally convex spaces in which the translation operators take each convex neighbourhood into a multiple of itself. In short, the translation operators are continuous in the seminorm topology specified by each of the closed convex circled neighbourhoods. We also assume that the polynomials (i.e.) the elements whose derivatives of sufficiently large orders vanish — form a dense subset.

Lemma 2. 1. *If the* CD *space* (V, E_h, D) *allows sufficiently many continuous linear functions to distinguish points, in particular if the space is locally convex and if* $D=0$ *then* $E_h=I$.

Proof. Consider the real-valued function $r(h)=(E_h f, g)$; f, g being arbitrary elements of V and V^*. Then

$$\big(r(h+\varepsilon)-r(h)\big)/\varepsilon = \big((E_{h+\varepsilon}f, g)-(E_h f, g)\big)/\varepsilon = \big(E_\varepsilon(f_1-f_1), g\big)/\varepsilon.$$

Where $f_1=E_h f$ and this tends to $(Df_1, g)=0$ as ε tends to zero. Thus $(E_h f, g)=$ $=\text{constant}=(E_0 f, g)=(f, g)$ for all f, g, and so $E_h f=f$.

Theorem 2. 1. *If* (V, E_h, E) *is a* CD *space and also* (V, E_h, F) *and if each* E_h *commutes with each* F_h, *then* $(V, E_h F_h, E+F)$ *is also a* CD *space.*

Proof. We need only verify that $(G_h f - f)/h - Ef - Ff$ tends to zero as h tends to zero. $(G_h=E_h F_h)$. We note that

$$\frac{G_h f-f}{h}-Ef-Ff=\left(\frac{E_h F_h-f}{h}\right)-Ef-Ff=$$

$$=E_h\left(\frac{F_h f-f}{h}-Ff\right)+(E_h-I)Ff+\left(\frac{E_h f-f}{h}-Ef\right)$$

Since the last two terms clearly tend to zero, we need only verify that $E_h\left(\frac{F_h f-f}{h}\right)-Ff$ tends to zero as h tends to zero. For this, for an arbitrary neighbourhood U of the origin, we choose h_1 so small that for $h<h_1$, $(F_h f-f)/h-Ff$ belong to U. Next we consider the norm specified by U. Since each F_h is bounded in this, all the F_h are uniformly bounded and we can get an r such that $F_h U\subset rU$ for every h. If now we choose $h_2<h_1/r$ and h_1, we get $E_h\left(\frac{(F_h f-f)}{h}-Ef\right)$ belongs to U for $h<h_2$.

Lemma 2. 2. *If* (V, E_h, E) *is a* CD *space, so is* $\big(V, F_h(=E_h)-E\big)$.

PROOF: $$\operatorname*{Lt}_{h\to 0} \frac{F_h f-f}{h} = \operatorname*{Lt}_{h\to 0} \frac{F_{-h} f-f}{-h} = -\operatorname*{Lt}_{h\to 0} \frac{E_h-f}{-h} = -Ef.$$

Theorem 2. 2. *If* (V, E_h, D) *and* (V, F_h, D) *are* CD *spaces and if each E-translate commutes with each F-translates and if in both systems, there are sufficiently many continuous linear functionals to distinguish points, then* $E_h=F_h$.

PROOF. By Theorem 2. 1 and lemma 2. 2, $(V, E_h\, F_{-h}, 0)$ is a CD space and by Lemma 2. 1, $E_h F_{-h}=I$ and $E_h=F_h$.

Lemma 2. 3. *If in a* CD *space* (V, E_h, D) *an element* f *is a polynomial of degree less than* n, *(i.e.) if* $D^n f=0$ *then* $E_h f$ *is also a polynomial of degree less than* n.

PROOF. Since each E_h commutes with each E_k and since D is $\operatorname*{Lt}_{h\to 0} \frac{E_h-E_0}{h}$ we have each E_h commutes with D. Thus $D^n E_h f=E_h D^n f=0$, showing that $E_h f$ is a polynomial of degree $<n$ if (and only if) f is one such.

Lemma 2. 4. If f is a polynomial of degree less than n, then $E_h(f)$ has an expansion in the form
$$E_h(f) = f+hDg+h^2 D^2 f/2!+\cdots .$$

PROOF. Set $r(h)=(\tau_h f, g)$ where f, g are fixed elements of V and V^*. Then

$$(d/dh)\,r(h) = \operatorname*{Lt}_{\varepsilon\to 0}\big(r(h+\varepsilon)-r(h)\big)/\varepsilon = \operatorname{Lt}\big((\tau_{h+\varepsilon}f, g)-(\tau_h f, g)\big)/\varepsilon =$$

$$= \operatorname{Lt}\left(\frac{\tau_h(\tau_\varepsilon f-f)}{\varepsilon}, g\right) = \left(\left(\tau_h \operatorname{Lt} \frac{\tau_\varepsilon f-f}{\varepsilon}\right), g\right) = (\tau_h Df, g)$$

Similarly $(d^n/dh^n)\, r(h)=(\tau_h D^n f, g)=0$.
Thus $r(h)$ is a polynomial and we have

$$f(h) = r(0)+hr'(0)+h^2 r''(0)/2!+\cdots$$

(i.e.)

$$\tau_h(f, g) = (f, g)+h(Df, g)+h^2(D^2 f, g)/2! \;\ldots\; \text{(finite series)}.$$

Since this is true for every g, we get

$$\tau_h f = f+hDf = h^2 D^2 f/2!+\cdots .$$

Lemma 2. 5. *Let* (V, E_h, D) *be a* CD *space and let* T *be a continuous linear map of* V *into* V *which commutes with* D. *Then* $TE_h=E_h T$ *for each* h.

PROOF. Since the polynomials are dense, it is enough if we prove that $TE_h q= =E_h Tq$ for each polynomial q. This follows from the polynomial expansion (lemma 2. 4) and the hypothesis that T commutes with D.

Theorem 2. 3. *If* (V, E_h, D) *and* (V, F_h, D) *are* CD *spaces, then* $E_h=F_h$.

PROOF. Since each E_h and F_k commutes with D, each E_h and F_k commute. Thus by theorem 2. 2, $E_h = F_h$.

Remark: We note that this result can be interpreted to mean that if (V, E_h, E) and (W, E_h, F) are CD spaces and if there is an isomorphism between (V, E) and (W, F) it is also an isomorphism between (V, E_h, E) and (W, F_h, F). We have only to identify (V, E) with (W, F) and apply the preceeding result.

This also means that the association of system (V, D) consisting of a topological vector space and a given continuous linear map of V into V with the CD space (V, E_h, F) is a one-one map. Thus the study of CD spaces — which are locally convex, which have a dense set of polynomials and in which the derivative operator is continuous in each seminorm has been reduced to the study of a locally convex space together with a single continuous linear operator D.

It is well known that the structure of the general continuous linear operator even in Hilbert spaces is not fully known. One could ask whether an inherent characterisation could be given of the derivative operators in such cases as Schwartz distributions. We reserve these for subsequent investigations. One can now ask whether every locally convex vector space together with an endomorphism can be associated as above with a CD space. We do not have an answer either. But a first guess that E_h can be reconstructed in terms of D by an exponential expansion $1 + hD + h^2 D^2/2! + \dots$ is not true as seen by the following counter-example on the space of Schwartz distributions. This is of course does not negate the possibility of reconstructing (V, E_h, D) from (V, D)-as is shown by the case of the distributions themselves and the earlier propositions.

Counter-example: The space of infinitely differentiable functions with compact support is a CD space convergence being uniform convergence over each compact set and also the uniform convergence of the derivatives of the r-th order. $(r = 1, 2, \dots.)$ The derivative D is the usual derivative. If our expectation that $\tau_h = \text{Exp. } hD$ should be correct, we would have that for every smooth compact function f, the series $1 + Df + D^2 f/2! \dots$ corresponding to Exp. D converges compactly — alongwith the r-th order derivatives. When we set $f = \big(1/(1+x^2)\big)$, we get $\frac{1}{(2n)!}\frac{d^{2n}f}{dx^{2n}}$ has the value $(-1)^n$ at the origin — and is not a cauchy sequence and the question of compact convergence of the series Exp. D is ruled out.

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