Relative projectivity and a property of Jacobson radical

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§ 1. Introduction

Let RG denote the group ring of a group G over a commutative unitary ring R. If H is a subgroup of G, and $G = \cup x_i H$ is a fixed coset decomposition, then each element of RG can be written as $\sum x_i p_i$ where $p_i \in RH$. Then we say that (R, G, H) has property ϱ with respect to the coset representation $\{x_i\}$, if whenever $\sum x_i p_i \in \in \text{rad}\, RG$ then each $p_i \in \text{rad}\, RH$ refers to the Jacobson Radical: {For several characterizations of this radical and its relations to the structure of rings we refer to [4], [7], [10] and [11]}.

For normal subgroups H of G, we characterize property ϱ by the fact that every RG module induced by an irreducible RH module is completely reducible: {Th. (3. 5)}. Some conditions for a subgroup to have this property, are obtained in Th. (3. 6) and Cor. (3. 8).

Further, we say that (RG, RH) is a projective pairing if every G module is H projective in the sense of [3] {see also [2], [6]}. For normal subgroups H of G, we show that projective pairing implies Property ϱ: {Th. (2. 4), (3. 3) and Cor. (3. 4)}.

§ 2. Generalities on modules

Let R be a ring with unity element 1 and P be a subring (all subrings of unitary rings will be assumed to contain the unity element of the ring). Suppose that R is a free right module over P with a basis $\{x_i | i \in I\}$ where I is some index set. Every element of R has the form $\sum x_i p_i$ with each $p_i \in P$.

Given a left R module M, we can obtain the restriction M_p as a P module merely by restricting the operatros to P. On the other hand, given any left P module N, we can form the induced module $N^R = \oplus \sum x_i \otimes N$ as an R module, where the symbol $x_i \otimes N$ stands for the tensor product $x_i P \otimes_P N$, and the direct sum is not necessarily a module sum even over P. However, if any x_i centralises P, i.e. $x_i p = p x_i$ for all $p \in P$, then $x_i \otimes N$ can be looked upon as a P module; and if this is the case with each i, then the above direct sum becomes a direct sum of P modules. We then make the following:

Definitions 1. We shall say that $\{R, P\}$ has property ϱ with respect to the basis $\{x_i\}$ if $\sum x_i p_i \in \text{rad}\, R$ implies that each $p_i \in \text{rad}\, P$.

2. We shall say that $\{R, P\}$ is a projective pairing, if every exact sequence of R-modules $0 \to N \to L \to M \to 0$, for which the associated sequence of restrictions $0 \to N_P \to L_P \to M_P \to 0$ splits over P, is itself split over R.
Before analysing the above two properties further, we recall the following known form of a standard result:

Lemma 2. 1. *Let R be a ring with minimum condition of left ideals. Then an R module M is completely reducible if and only if* rad $R \subseteqq$ *annihilator of M in R.* ([9], [5].)

Next we recall that for finite cardinality of the index set I, it has been shown in [8] that property ϱ with respect to one basis implies the same with respect to any other basis, and that it is a transitive property.

Here we first of all show a relation between the two properties defined above. Our result below contains Th. 3 of [8].

Theorem 2. 4. *Let R be a unitary ring which is a free right module over a subring P having minimum condition of left ideals, and let $\{x_i\}$ with $x_1 = 1$, $i \in I$, be a finite P basis for R. If for each $p \in P$ and each $i \in I$, $px_i = x_{p(i)}\sigma_i(p)$, where $i \to p(i)$ induces a permutation in the index set I, and σ_i are automorphisms of P, then projective pairing for $\{R, P\}$ implies property ϱ for $\{R, P\}$.*

Proof. Let M be an arbitrary irreducible P module. Then we consider the induced R module $M^R = \oplus \sum_{i \in I} x_i \otimes M$. Looking upon P as a set of permutations on I, let $C(i)$ be the P cycle to which i belongs. Put $W_i = \oplus \sum_{j \in C(i)} x_j \otimes M$. Then each W_i is a left P module and $M^R = \oplus \sum W_i$ as a direct sum of P modules.

Now $p \in \text{rad } P$ implies that $pW_i = \sum_{j \in C(i)} x_{p(j)} \otimes \sigma_j(p)M = 0$ since $\sigma_j(p) \in \text{rad } P$ and M is P irreducible. Thus rad $P \subseteqq$ annihilator of W_i in P for each i. Then by (2. 1), each W_i is completely reducible over P and hence, so is M^R over P.

Now let $0 \to N \to M^R \to L \to 0$ be any R exact sequence. Then this splits as a P exact sequence since M^R is completely reducible over P. But as $\{R, P\}$ is a projective pairing, this sequence splits as an R exact sequence also. Then M^R is completely reducible as an R module.

Finally let $\sum x_i p_i \in \text{rad } R$ where each $p_i \in P$. Then from complete reducibility of M^R, we have $(\sum x p_i) M^R = 0$. In particular $(\sum x_i p_i)(1 \otimes m) = 0$ or $\sum x_i \otimes p_i m = 0$ for each $m \in M$. This implies that $p_i M = 0$ for each i.

Since M was an arbitrary P irreducible module, so we conclude that each $p_i \in \text{rad } P$. This gives property ϱ for the pair $\{R, P\}$. Q.E.D.

In this general setting a complete characterization of property ϱ is not obtained here. In case of group rings we shall give a more satisfactory result in the next section. Here we complement (2. 4) by:

Theorem 2. 5. *Let a ring R be a free right module over a subring P with a finite P basis $\{x_i\}$ and let R have minimum condition of left ideals. Let $\{R, P\}$ have property ϱ. Then for an R module M, if M_P is completely reducible over P, then so is M over R.*

PROOF. Let $0 \to N \to M \to L \to 0$ be any R exact sequence and $\sum x_i p_i \in \operatorname{rad} R$. Then by hypothesis each $p_i \in \operatorname{rad} P$ and M_P is completely reducible over P. Hence $(\sum x_i p_i)M = \sum x_i(p_i M) = 0$. Thus rad $R \subseteq$ annihilator of M in R, so that by (2. 1), M is completely reducible over R. Q.E.D.

§ 3. Applications to group rings

In order to apply the above concepts to group rings, we begin with:

Definitions 3. Let $G = \cup x_i H$ be a fixed coset decomposition of a group G with respect to its subgroup H, and let each element of RG be expressed as $\sum x_i p_i$ where each $p_i \in RH$. Then we shall say that $\{R, G, H\}$ has property ϱ with respect to the coset representatives $\{x_i\}$, if $\{RG, RH\}$ has property ϱ with respect to the basis $\{x_i\}$.

Definition 4. The class $C(H)$ of subgroups $H_t = \langle H; x_{i_1}, x_{i_2}, \dots, x_{i_t}\rangle$, generated by H and a finite number of coset representatives $\{x_{i_j}\}$, will be called the covering class of H in G.

If $\{T_i\}$ is any other coset representation in G over H, then each $T_i = X_{i_j} \cdot h_{i_j}$ for some X_{i_j} in $\{X_i\}$ and $h_{i_j} \in H$. Hence we have

Lemma 3. 1. *$C(H)$ is independent of the choice of coset representation in G over H.*

For group ring, we can prove a stronger version of Th. 1 in [8]:

Theorem 3. 2. *If $\{R, G, H\}$ has property ϱ with respect to one coset representation, then it has so with respect to any other coset representation.*

PROOF. Observe that each element of RG is a finite sum $\sum_{g \in G} r_g \cdot g$ with each $r_g \in R$, and the elements of R commute with those of G. Now let $\{x_i\}$ and $\{y_i\}$ be two coset representations in G over H. Then $y_i = x_i h_i$ for some x and some $h_i \in H$. Hence given $\sum y_i p_i \in RG$, we can write it as $\sum x_i h_i p_i \in RG$.

Now if $\{R, G, H\}$ has property ϱ with respect to the coset representatives $\{x_i\}$, and $\sum y_i p_i \in \operatorname{rad} RG$ then $\sum x_i h_i p_i \in \operatorname{rad} RG$ whence each $h_i p_i \in \operatorname{rad} RH$. Since each h_i is a unit in RH, so this implies that each $p_i \in \operatorname{rad} RH$. Hence $\{R, G, H\}$ has property ϱ with respect to the coset representatives $\{y_i\}$ also. Q.E.D.

By virtue of this theorem, throughout this section, we shall drop mentioning particular coset representation chosen, with respect to which $\{R, G, H\}$ has property ϱ.

Now recall that a subgroup H of a group G is called subnormal if there is a chain,

$$H = S_0 \Delta S_1 \Delta \dots \Delta S_n = G$$

of subgroups S_i such that $S_i \Delta S_{i+1}$; i.e. S_i a normal subgroup of S_{i+1}.

Then an immediate application of Th. (2. 4) and an obvious induction, gives us:

Theorem 3. 3. Let H be a subnormal subgroup of finite index in a group G and $\{S_i\}$ be as defined above. If $\{RS_i, RS_{i-1}\}$ has projective pairing for each i, then $\{R, G, H\}$ has property ϱ.

Cor. (3. 4). If $H \Delta G$ and $[G:H] < \infty$, then projective pairing of $\{RG, RH\}$ implies property ϱ for $\{R, G, H\}$.

Next we give a characterization of property ϱ for certain types of group rings.

Theorem 3. 5. *Let $H \Delta G$ and R be a unitary ring such that RG is artinian. Then $\{R, G, H\}$ has property ϱ if and only if for every irreducible RH module M, the induced RG module M^G is completely reducible.*

PROOF. Suppose firsly that for every irreducible RH module M, the induced RG module M^G is completely reducible over RG. Let $G = \cup x_i H$ be a coset decomposition of G over H, and $\sum x_i p_i \in \text{rad } RG$, where each $p_i \in RH$. Then from the complete reducibility of M^G, we have $(\sum x_i p_i) M^G = 0$.

In particular, if M is an arbitrary RH irreducible module, then for every $m \in M$, $(\sum x_i p_i)(1 \otimes m) = \sum x_i \otimes p_i m = 0$. Then from the independence of the $\{x_i\}$ over RH, we conclude that for each i and each $m \in M$, $p_i m = 0$; i.e. $p_i M = 0$, whence each $p_i \in \text{rad } RH$. This implies that $\{R, G, H\}$ has property ϱ. [Note that for this part of the proof we have neither made use of the normality of H nor of the minimum condition in RG.]

For the converse part, let $\{R, G, H\}$ have property ϱ. Then $\sum x_i p_i \in \text{rad } RG$ implies that each $p_i \in \text{rad } RH$. Now let M be an arbitrary irreducible module over RH. Then the induced RG module M^G has the form $M^G = \oplus \sum x_i \otimes M$. Since H is normal in G, so each $x_i \otimes M$ is an irreducible RH module, [1].

Also for each i, $h \in H$ implies $hx_i = x_i \cdot \varphi_i(h)$ where $\varphi(h) = x_i^{-1} h x_i$ induces an automorphism of H, which can be extended by linearity to RH. Then $\sum x_i p_i \in \text{rad } RG$ implies that

$$(\sum x_i p_i)(x_j \otimes M) = \sum x_i x_j \varphi_j(p_i) \otimes M = \sum x_{ij} h_{ij} \otimes \varphi_j(p_i) M$$

where $x_i x_j = x_{ij} h_{ij}$ for some x_{ij} in $\{x_i\}$ and some $h_{ij} \in H$.

Since each $\varphi_j(p_i) \in \text{rad } RH$ and M is an irreducible RH module, so $\varphi_j(p_i) M = 0$ for each i and j.

This shows that rad RG is contained in the annihilator of M^G in RG. Hence M^G is completely reducible by (2. 1). Q.E.D.

We recall here the Theorem of Clifford, ([1]) which states that if $H \Delta G$ and M is an irreducible RG module, then the restriction M_H is completely reducible over RH. In this context the above Th. (3. 5) gives a criterion in the reverse direction, i.e. a criterion as to when an irreducible RH module can be lifted to a completely reducible RG module.

Finally, we use the notion of covering class defined above, in order to determine some subgroups H in a group G such that $\{R, G, H\}$ has property ϱ.

Theorem 3. 6. *Let R be a unitary ring and H be a subgroup of a group G. (i) If $\{R, S, H\}$ has property ϱ for each S in $C(H)$, then $\{R, G, H\}$ has property ϱ. (ii) If R is a field and $\{R, G, H\}$ has property ϱ, then for each normal subgroup S in $C(H)$, $\{R, S, H\}$ has property ϱ.*

PROOF. (i) Suppose $\{R, S, H\}$ has property ϱ for each S in $C(H)$. Let $\{x_i\}$ be a complete system of coset representatives in G over H and $r = \sum x_i p_i \in \text{rad } RG$, where each $p_i \in RH$. Only a finite number of x_i occur with non-zero coefficients in r. Let these be $\{x_{i_1}, \ldots, x_{i_t}\}$, and put $S = \langle H, x_{i_1}, x_{i_2}, \ldots, x_{i_t} \rangle$ in $C(H)$.

Let $\{y_j\}$ be a complete system of coset representatives in G over S, where $y_1=1$. Since $r\in \text{rad } RG$, so it has a quasi-inverse r^* in RG such that $r^*+r-r^*r=0$, [4]. Let $r^*=\sum y_i q_j$ where each $q_j\in RS$. Then we have

$$y_1\cdot(q_1+r-q_1r)+\sum_{j\neq 1} y_j(q_j-q_jr)=0$$

where r obviously belongs to RS. Then from the independence of $\{y_i\}$ over RS, we obtain $q_1+r-q_1r=0$ and $q_j(1-r)=0$ for each j. Since $1-r$ is a unit in RG as $r\in \text{rad } RG$, so each $q_j=0$ for $j\neq 1$, while $q_1+r-q_1\cdot r=0$. Hence $r^*=q_1\in RS$, so that $r\in \text{rad } RS$. The property ϱ for $\{R, S, H\}$ implies that each p_i is in rad RH, since $x_{i_1}, \dots, x_{i_t}$ can be taken as a part of coset representative system in S over H. From this we conclude property ϱ for $\{R, G, H\}$.

(ii) Next let R be a field, $\{R, G, H\}$ have property ϱ and S be a normal subgroup in $C(H)$. If M is any irreducible RG module, then by Clifford's Theorem mentioned above, M_S is a completely reducible RS module.

Now let $S=\bigcup x_i H$ be a coset decomposition of S over H, and extend this to a coset decomposition $G=[\bigcup y_j H]\cup[\bigcup x_i H]$ in G over H.

Suppose $\sum x_i p_i\in \text{rad } RS$ where each $p_i\in RH$. Then from the complete reducibility of M_S, we conclude that $(\sum x_i p_i)M=0$. Since this is true for an arbitrary irreducible RG module M, so $\sum x_i p_i\in \text{rad } RG$ as well. Then property ϱ for $\{R, G, H\}$ implies that each $p_i\in \text{rad } RH$, whence property ϱ for $\{R, S, H\}$ follows. Q.E.D.

From the latter part of the proof of the above theorem, we easily extract the following results:

Cor. (3. 7). If R is a field and S is a normal subgroup of a group G, then rad $RS\subseteqq$ rad RG.

Cor. (3. 8). If R is a field and $\{R, G, H\}$ has property ϱ for some subgroup H of a group G, then $\{R, S, H\}$ has property ϱ for all normal subgroups S containing H.

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