Composite sets of polynomials of several complex variables

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1. Introduction. Let $\{q_j^{(1)}(z)\}, \{q_j^{(2)}(z)\}, \dots, \{q_j^{(k)}(z)\}$, or in short $\{q_j^{(i)}(z)\}$ be a given[1]) finite number of basic sets[2]) of polynomials of single variable and consider the product element $q_{j_1}^{(1)}(z_1) \cdot q_{j_2}^{(2)}(z_2) \dots q_{j_k}^{(k)}(z_k)$. If, for any mode of arrangement, we put

$$q_{j_1}^{(1)}(z_1) \cdot q_{j_2}^{(2)}(z_2) \cdot \dots q_{j_k}^{(k)}(z_k) = p_j(z_1, z_2, \dots, z_k), \tag{1.1}$$

the sequence $\{p_j(z_1, z_2, \dots, z_k)\}$ is a set of polynomials of the complex variables $z_1, z_2, \dots, z_k$. This set is here defined as the *composite set of polynomials* whose *constituents* are the sets $\{q_j^{(i)}(z)\}$.

We propose to establish, in the present paper, certain convergence properties of the composite sets in terms of those of their constituent sets. To achieve this aim a study of basic sets of polynomials of several complex variables has first to be carried out. Such study has been initiated by Mursi and Makar [1, 2], whereby the representation in polycylindrical regions has been considered. This study will be here modified on the assumption that the regions of representations shall be spherical. However, an abbreviated study[3]) of functions of several complex variables has to be introduced in view to reveal those properties relevant to our present work.

2. Functions of several complex variables. To avoid lengthy scripts, the follow ing notations are adopted throughout the work.

$$\begin{cases} m_1, m_2, \dots, m_k = \mathbf{m}; \quad z_1, z_2, \dots, z_k = \mathbf{z}; \quad 0, 0, \dots, 0 = \mathbf{0}; \\ m_1 + m_2 + \dots + m_k = (\mathbf{m}); \quad |z_1|^2 + |z_2|^2 + \dots + |z_k|^2 = |\mathbf{z}|^2; \\ z_1^{m_1} \cdot z_2^{m_2} \dots z_k^{m_k} = \mathbf{z}^{\mathbf{m}}; \quad t_1^{m_1} \cdot t_2^{m_2} \cdot \dots t_k^{m_k} = \mathbf{t}^{\mathbf{m}}. \end{cases} \tag{2.1}$$

In these notations $m_1, m_2, \dots, m_k$ are non-negative integers whole $t_1, t_2, \dots, t_k$ are non-negative numbers (presumably less than 1). Also, square brackets are used here in functional notation to express the fact that the function is either a function of several complex variables or one related to such function.

[1]) Throughout this work the integer i assumes the values 1, 2, ..., k, while the integer j, whether suffixed or not, assumes the non-negative values.

[2]) The reader is supposed to be acquainted with the theory of basic sets of polynomials of single complex variable, as given by Whittaker [3, 4].

[3]) In this study, I am guided by the ideas of Academician M. M. DJERBASHIAN of the Academy of Science, Armenia, to whom I am much obliged.

In the space of the complex variables $z_1, z_2, \ldots, z_k$ an open shperical region of radius r; $r>0$, is here denoted by S_r and its closure by $\bar{S}_r$. In terms of the introduced notations, these regions satisfy the inequalities

$$(2.2) \qquad S_r : |\mathbf{z}| < r, \quad \bar{S}_r : |\mathbf{z}| \leqq r.$$

Suppose now that the function $f[\mathbf{z}]$, given by

$$(2.3) \qquad f[\mathbf{z}] = \sum_{\mathbf{m}=0} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}},$$

is regular in $\bar{S}_r$, where $|f[\mathbf{z}]| \leqq M$. From (2. 2) we easily see that

$$\bar{S}_r \supset [|z_i| \leqq r t_i : |\mathbf{t}| = 1],$$

where $\mathbf{t}$ is the vector $(t_1, t_2, \ldots, t_k)$. Hence

$$|a_{\mathbf{m}}| \leqq \frac{M}{r^{(\mathbf{m})} \mathbf{t}^{\mathbf{m}}}; \quad (|\mathbf{t}| = 1) \leqq \inf_{|\mathbf{t}|=1} \frac{M}{r^{(\mathbf{m})} \mathbf{t}^{\mathbf{m}}}$$

That is to say

$$|a_{\mathbf{m}}| \leqq \frac{M \sigma_{\mathbf{m}}}{r^{(\mathbf{m})}},$$

where

$$(2.4) \qquad \sigma_{\mathbf{m}} = \inf_{|\mathbf{t}|=1} \frac{1}{\mathbf{t}^{\mathbf{m}}} = \frac{\{(\mathbf{m})\}^{\frac{1}{2}(\mathbf{m})}}{m_1^{\frac{1}{2}m_1} m_2^{\frac{1}{2}m_2} \ldots m_k^{\frac{1}{2}m_k}},$$

on the assumption that $m_i^{\frac{1}{2}m_i} = 1$, whenever $m_i = 0$.

On the other hand, suppose that, for the function $f[\mathbf{z}]$, given by (2. 3),

$$(2.5) \qquad \limsup_{(\mathbf{m}) \to \infty} \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}} \right\}^{\frac{1}{(\mathbf{m})}} = 1/\varrho; \qquad (\varrho > 0).$$

then it can be easily proved that the function $f[\mathbf{z}]$ is a regular in the open sphere S_ϱ. The number ϱ, given by (2. 5), is thus conveniently called the *radius of regularity* of the function $f[\mathbf{z}]$.

If the function $f[\mathbf{z}] = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ is an integral function, it can be proved, in exactly the same way as in single variable case, that the *order* τ of the function is given by

$$(2.6) \qquad \tau = \limsup_{r \to \infty} \frac{\log \log M[r]}{\log r} = \limsup_{(\mathbf{m}) \to \infty} \frac{(\mathbf{m}) \log (\mathbf{m})}{\log \dfrac{\sigma_{\mathbf{m}}}{|a_{\mathbf{m}}|}},$$

where

$$M[r] = \sup_{\mathbf{z} \in S_r} |f[\mathbf{z}]|.$$

3. Basic sets of polynomials of several complex variables. A sequence $\{p_j[\mathbf{z}]\}$ of polynomials of the complex variables $z_1, z_2, \ldots, z_k$ $(=\mathbf{z})$, is said to form a *basic set* if the different monomials $\mathbf{z}^{\mathbf{m}}$ admit finite unique representation of the form

$$(3.1) \qquad \mathbf{z}^{\mathbf{m}} = \sum_j \pi_{\mathbf{m}}^j p_j[\mathbf{z}].$$

Let $f[\mathbf{z}] = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ be any function regular about the origin $\mathbf{0}$; substituting for $\mathbf{z}^{\mathbf{m}}$ from (3. 1) and rearranging the terms we obtain the *basic series* associated with $f[\mathbf{z}]$ in the form

$$f[\mathbf{z}] \sim \sum_{j} c_j p_j[\mathbf{z}]; \quad c_j = \sum_{\mathbf{m}} \pi^j_{\mathbf{m}} a_{\mathbf{m}}. \tag{3. 2}$$

The basic set $\{p_j[\mathbf{z}]\}$ is said to *represent* the function $f[\mathbf{z}]$ in a sphere $\bar{S}_r$ if the basic series (3. 2) converges uniformly to $f[\mathbf{z}]$ in $\bar{S}_r$, and the set will be called *effective in* $\bar{S}_r$ if it represents in $\bar{S}_r$ every function which is regular there. It should be noted that effectiveness in a given sphere does not imply nor is implied by effectiveness in a larger or smaller sphere.

The basic set $\{p_j[\mathbf{z}]\}$ of polynomials will be a *Cannon set* if the number $N_{\mathbf{m}}$ of non-zero coefficients in (3. 1) is such that

$$\lim_{(\mathbf{m})\to\infty} N_{\mathbf{m}}^{\frac{1}{(\mathbf{m})}} = 1. \tag{3. 3}$$

To obtain a condition for effectiveness of the Cannon set $\{p_j[\mathbf{z}]\}$ in $\bar{S}_r$ we form the Cannon sum

$$\Omega_{\mathbf{m}}[r] = \sigma_{\mathbf{m}} \sum_{j} |\pi^j_{\mathbf{m}}| A_j[r], \tag{3. 4}$$

where

$$A_j[r] = \sup_{\mathbf{z}\in \bar{S}_r} |p_j[\mathbf{z}]|. \tag{3. 5}$$

It is easily seen from (3. 1) that

$$\Omega_{\mathbf{m}}[r] \geqq \sigma_{\mathbf{m}} \{\sup_{\mathbf{z}\in \bar{S}_r} |\mathbf{z}^{\mathbf{m}}|\} = r^{(\mathbf{m})}.$$

Thus, if the Cannon function for the set $\{p_j[\mathbf{z}]\}$ is given by

$$\Omega[r] = \limsup_{(\mathbf{m})\to\infty} \{\Omega_{\mathbf{m}}[r]\}^{\frac{1}{(\mathbf{m})}}, \tag{3. 6}$$

then

$$\Omega[r] \geqq r. \tag{3. 7}$$

The following theorem is the first to start with.

Theorem 1. *If $\{p_j[\mathbf{z}]\}$ is a basic set of polynomials for which $\Omega[r]=\varrho\geqq r$, then the basic set will represent in $\bar{S}_r$ every function regular in $\bar{S}_\varrho$.*

In fact, if $f[\mathbf{z}] = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ is any function regular in $\bar{S}_\varrho$, then its radius of regularity ϱ_1 will be greater than ϱ. Hence, (2. 5), (3. 4) and (3. 6) together yield

$$\limsup_{(\mathbf{m})\to\infty} \{|a_{\mathbf{m}}| \sum_{j} |\pi^j_{\mathbf{m}}| A_j[r]\}^{\frac{1}{(\mathbf{m})}} =$$

$$= \limsup_{(\mathbf{m})\to\infty} \left\{\frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}} \Omega_{\mathbf{m}}[r]\right\}^{\frac{1}{(\mathbf{m})}} \leqq \limsup_{(\mathbf{m})\to\infty} \left\{\frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}}\right\}^{\frac{1}{(\mathbf{m})}} \Omega[r] = \frac{\varrho}{\varrho_1} < 1.$$

It follows that the series $\sum_{\mathbf{m}} a_{\mathbf{m}} \sum_{j} \pi_{\mathbf{m}}^{j} p_j[\mathbf{z}]$ is absolutely and uniformly convergent to $f[\mathbf{z}]$ in $\bar{S}_r$. Consequently, the basic series (3. 2) converges uniformly to $f[\mathbf{z}]$ in $\bar{S}_r$; as required.

The analogue of the fundamental theorem of Cannon[4]) is as follows.

Theorem 2. *Let $\{p_j[\mathbf{z}]\}$ be a Cannon set of polynomials and suppose that, for some value of $r>0$, $\Omega[r]>r$. Then there is a function of radius of regularity ϱ, where $r<\varrho<\Omega[r]$, which the basic set does not represent in $\bar{S}_r$.*

It can be easily verified that the proof of the corresponding Cannon's theorem, already referred to, can apply to this theorem with minor and obvious modifications if the coefficients $(\pi_{\mathbf{m}}^{j})$ of (3. 1) are arranged in the form of a two dimensional array Π.

To perform this arrangement a rule of ordering the vectors $\mathbf{m}=m_1, m_2, \ldots, m_k$ has to be followed[5]). In fact, the vectors are first ordered according to the increasing value of the sum $n = m_1+m_2+\cdots+m_k$ of their components. Then the vectors of the same sum n are arranged according to the usual way of arranging the components $m_1, m_2, \ldots, m_k$ as partitions of n, and finally the vectors whose components belong to a certain partition are arranged in a lexical order.

In this way each column of Π corresponds to a value of the index j of $\pi_{\mathbf{m}}^{j}$ and each row corresponds to a value of the vector $\mathbf{m}$ arranged in the above manner.

From theorems 1 and 2 and the inequality (3. 7) it is easy to deduce the following result.

Theorem 3. *A necessary and sufficient condition for the Cannon set $\{p_j[\mathbf{z}]\}$ of polynomials to be effective in $\bar{S}_r$ is that $\Omega[r]=r$; $r>0$.*

Now, the *order* Γ of the basic set $\{p_j[\mathbf{z}]\}$ of polynomials is given by

$$\Gamma = \lim_{r\to\infty} \limsup_{(\mathbf{m})\to\infty} \frac{\log \Omega_{\mathbf{m}}[r]}{(\mathbf{m}) \log (\mathbf{m})}. \tag{3. 8}$$

As in single variable case, the definition (3. 8) of order of basic sets provides a sufficient condition for representation of classes of integral functions in the form[6]).

Theorem 4. *If the basic set $\{p_j[\mathbf{z}]\}$ of polynomials is of finite order Γ, it will represent, in any finite sphere, every integral function of order less than $1/\Gamma$.*

For, let $f[\mathbf{z}] = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ be an integral function of order $\tau<1/\Gamma$, then the numbers η and γ can be chosen so that

$$\tau<\eta<1/\gamma<1/\Gamma. \tag{3. 9}$$

Applying the definitions (2. 6) of order of the integral function $f[\mathbf{z}]$ and (3. 8) of

[4]) cf. Whittaker [4; T_4, p. 9].
[5]) cf. Mursi and Makar [2; p. 61].
[6]) cf. Whittaker [3; L_{105}, p. 13].

order of the basic set $\{p_j[\mathbf{z}]\}$ it easily follows from (3. 9) that, corresponding to any finite number r, there exists an integer M such that

$$\Omega_{\mathbf{m}}[r] < \{(\mathbf{m})\}^{(\mathbf{m})\gamma}; \quad |a_{\mathbf{m}}| < \sigma_{\mathbf{m}}\{(\mathbf{m})\}^{-\frac{(\mathbf{m})}{\eta}}, \quad \text{for} \quad (\mathbf{m}) > M.$$

Whence, from (3. 4) we obtain

$$\limsup_{(\mathbf{m})\to\infty} \{|a_{\mathbf{m}}| \sum_j |\pi_{\mathbf{m}}^j| A_j[r]\}^{\frac{1}{(\mathbf{m})}} \leq \lim_{(\mathbf{m})\to\infty} \{(\mathbf{m})\}^{\gamma-\frac{1}{\eta}} = 0 < 1,$$

and the basic set $\{p_j[\mathbf{z}]\}$ represents $f[\mathbf{z}]$ in $\bar{S}_r$; as required.

4. Effectiveness of composite sets of polynomials. We are now in a position to study the composite sets $\{p_j[\mathbf{z}]\}$ already defined in the introductory section of this paper. We start this study by proving that the composite set is a basic set of polynomials of the complex variables $z_1, z_2, \ldots, z_k$.

First we show that the polynomials $\{p_j[\mathbf{z}]\}$ are linearly independent. In fact, assume that $\{p_j[\mathbf{z}]\}$ is the composite set of the constituent basic sets $\{q_j^{(i)}(z)\}$ and suppose, if possible, that there exists a finite linear relation of the form

$$\sum_{r=1}^{l} c_r p_{j_r}[\mathbf{z}] = 0, \tag{4. 1}$$

which holds for all values of $\mathbf{z} = z_1, z_2, \ldots, z_k$, where none of the coefficients (c_r) is zero. Applying the definition (1. 1) of the polynomials $\{p_j[\mathbf{z}]\}$, the relation (4. 1) can be written as a linear combination of the relevant polynomials $\{q_j^{(1)}(z)\}$ in the form

$$\sum_{s=1}^{m} C_s[\mathbf{z}'] q_{\mu_s}^{(1)}(z_1) = 0, \tag{4. 2}$$

where

$$C_s[\mathbf{z}'] = C_s(z_2, z_3, \ldots, z_k) = \sum_{t=1}^{n_s} a_{s,t} P_{\nu_t}[\mathbf{z}'], \tag{4. 3}$$

say, so that the sequence $\{P_\nu[\mathbf{z}']\}$ is a set of polynomials of the complex variables $z_2, z_3, \ldots, z_k$ and $a_{s,t}$, being equal to some $c_r, \neq 0$. If there is at least one value of $\mathbf{z}' = z_2, z_3, \ldots, z_k$ for which not all the coefficients $C_s[\mathbf{z}']$ are zeros then (4. 2) implies that the polynomials $\{q_j^{(1)}(z)\}$ are not linearly independent. If, on the other hand, $C_s[\mathbf{z}'] = 0$, for $s = 1, 2, \ldots, m$ and for all values of $\mathbf{z}'$ then (4. 3) shows that there is linear dependence among the polynomials $\{P_\nu[\mathbf{z}']\}$. Repeating this argument in a successive manner we arrive at the conclusion that the polynomials of at least one of the sets $\{q_j^{(i)}(z)\}$ are not linearly independent. This contradiction implies that the polynomials $\{p_j[\mathbf{z}]\}$ are linearly independent.

Furthermore, suppose that z^m admits the finite representation

$$z^m = \sum \pi_{m,j}^{(i)} q_j^{(i)}(z), \tag{4. 4}$$

then by the definition (1. 1) of composite sets we see that the monomial $\mathbf{z}^{\mathbf{m}} (= z_1^{m_1} \cdot z_2^{m_2} \ldots z_k^{m_k})$ admits the finite representation

$$\mathbf{z}^{\mathbf{m}} = \sum_j \pi_{\mathbf{m}}^j p_j[\mathbf{z}], \tag{4. 5}$$

where

(4. 6) $\pi_{\mathbf{m}}^{j}=\pi_{m_1,j_1}^{(1)}\cdot\pi_{m_2,j_2}^{(2)}\dots\pi_{m_k,j_k}^{(k)}$ when $p_j[\mathbf{z}]=q_{j_1}^{(1)}(z_1)\cdot q_{j_2}^{(2)}(z_2)\dots q_{j_k}^{(k)}(z_k)$

Since the polynomials $\{p_j[\mathbf{z}]\}$ are linearly independent then the representation (4. 5) is unique and the set $\{p_j[\mathbf{z}]\}$ will therefore be basic.

Concerning the effectiveness of the composite sets in a closed sphere the following result is established.

Theorem 5. *Let $\{q_j^{(i)}(z)\}$ be Cannon sets of polynomials and suppose that $\{p_j[\mathbf{z}]\}$ is their composite set. Then the set $\{p_j[\mathbf{z}]\}$ will be effective in the closed sphere $\bar{S}_R$; $R>0$, if, and only if, each of the sets $\{q_j^{(i)}(z)\}$ is effective in the circles $|z|\leqq r$, for $0<r\leqq R$.*

5. Proof of the "if" — statement of theorem 5. Mindful of the representation (4. 4), the Cannon sum for the set $\{q_j^{(i)}(z)\}$ will be

(5. 1) $$\omega_m^{(i)}(r)=\sum_j|\pi_{m,j}^{(i)}|\,B_j^{(i)}(r);\qquad(r\geqq 0),$$

where

(5. 2) $$B_j^{(i)}(r)=\sup_{|z|=r}|q_j^{(i)}(z)|;\quad(r>0),\qquad B_j^{(i)}(0)=|q_j^{(i)}(0)|.$$

The Cannon function for the same set will then be given by

(5. 3) $$\omega^{(i)}(r)=\limsup_{m\to\infty}\{\omega_m^{(i)}(r)\}^{\frac{1}{m}}.$$

If $N_m^{(i)}$ and $N_{\mathbf{m}}$ be the number of non-zero coefficients in (4. 4) and (4. 5) respectively, then

(5. 4) $$N_{\mathbf{m}}=N_{m_1}^{(1)}\cdot N_{m_2}^{(2)}\cdot\ldots\cdot N_{m_k}^{(k)}.$$

Hence, if each of the sets $\{q_j^{(i)}(z)\}$ is a Cannon set so also will be the composite set $\{p_j[\mathbf{z}]\}$, as it is seen from (3. 3).

In the notation (3. 5), (1. 1) and (5. 2) together give

(5. 5) $$A_j[r]=\sup_{\mathbf{z}\in S_r}|p_j[\mathbf{z}]|=\sup_{|t|=1}B_{j_1}^{(1)}(rt_1)\cdot B_{j_2}^{(2)}(rt_2)\dots B_{j_k}^{(k)}(rt_k).$$

Therefore, the Cannon sum for the set $\{p_j[\mathbf{z}]\}$ can be obtained by combination of (4. 6), (5. 1) and (5. 5), the following double inequality can be easily verified.

(5. 6) $$\sigma_{\mathbf{m}}\left[\sup_{|t|=1}\left\{\prod_{i=1}^{k}\omega_{m_i}^{(i)}(rt_i)\right\}\right]\leqq\Omega_{\mathbf{m}}[r]\leqq N_{\mathbf{m}}\sigma_{\mathbf{m}}\left[\sup_{|t|=1}\left\{\prod_{i=1}^{k}\omega_{m_i}^{(i)}(rt_i)\right\}\right].$$

To prove the "if"-statement of the theorem let α be any finite number greater than 1 and fix the positive number τ by

(5. 7) $$\tau^2=(1-\alpha^{-1})/k.$$

Suppose that each of the sets $\{q(_j^{(i)}z)\}$ is effective in $|z|\leqq r$ for $0<r\leqq R$, so that the Cannon function (5. 3) will be

$$\omega^{(i)}(r)=r;\qquad(0<r\leqq R).$$

Hence corresponding to the number α, there exists a finite number $K>1$ such that

$$\begin{cases}\omega_m^{(i)}(R\tau)<K(\sqrt{\alpha}R\tau)^m\\ \omega_m^{(i)}(R)<K(\sqrt{\alpha}R)^m; \qquad (m\geqq 0).\end{cases} \tag{5. 8}$$

Since the function $q_j^{(i)}(z)/z^m$ is regular in the annulus $R\tau\leqq|z|\leqq R$ then in the notations (5. 1) and (5. 2), we should have from (5. 8)

$$|\pi_{m,j}^{(i)}|\,B_j^{(i)}(Rt)/(Rt)^m\leqq\sup\{\omega_m^{(i)}(R\tau)/(R\tau)^m,\ \omega_m^{(i)}(R)/R^m\}<K\alpha^{\frac{1}{2}m},$$

for $\tau\leqq t\leqq 1$. Therefore, it follows that

$$\omega_m^{(i)}(Rt)<KN_m^{(i)}(\sqrt{\alpha}Rt)^m \qquad (\tau\leqq t\leqq 1;\ m\geqq 0). \tag{5. 9}$$

Now, considering the parameters (t_i) in (5. 6), we see that some of these parameters may lie in the interval $(0,\tau)$ and the others in $(\tau,1)$. Since the order of the polynomials $\{q_{j_i}^{(i)}(z_i)\}$ forming the polynomial $p_j[\mathbf{z}]$ is immaterial to our argument, we may assume that

$$\tau\leqq t_s\leqq 1;\quad (1\leqq s\leqq l),\quad 0\leqq t_u<\tau;\quad (l+1\leqq u\leqq k), \tag{5. 10}$$

where l is any integer not exceeding k. Some of the notations (2. 1) have to be modified to suit the integer l, thus we may write

$$m_1, m_2, \ldots, m_l=\mathbf{m}';\quad m_1+m_2+\cdots+m_l=(\mathbf{m}');$$

$$t_1^{m_1}\cdot t_2^{m_2}\cdot\ldots\cdot t_l^{m_l}=\mathbf{t}'^{\mathbf{m}'};\quad \mathbf{t}'=(t_1, t_2, \ldots, t_l).$$

With these notations it is easily seen that

$$\sup_{|\mathbf{t}'|\leqq 1}\mathbf{t}'^{\mathbf{m}'}\leqq\frac{1}{\sigma_{\mathbf{m}'}}. \tag{5. 11}$$

Moreover, in view of the definition (2. 4) of $\sigma_{\mathbf{m}}$, and taking in (5. 10) $t_u=\tau$, $(l+1\leqq u\leqq k,)$ it can be verified that

$$\tau^{(\mathbf{m})-(\mathbf{m}')}\{\sup_{|t'|^2=1-(k-l)\tau^2}\mathbf{t}'^{\mathbf{m}'}\}\leqq\frac{1}{\sigma_{\mathbf{m}}}.$$

Evaluating the supremum inside the brackets in this relation we obtain

$$\sigma_{\mathbf{m}}^{-1}\tau^{(\mathbf{m})-(\mathbf{m}')}\leqq\sigma_{\mathbf{m}'}(1-k\tau^2)^{-\frac{1}{2}(\mathbf{m}')}. \tag{5. 12}$$

Furthermore in view of (5. 10), we may take $\omega_{m_u}^{(u)}(Rt_u)\leqq\omega_{m_u}^{(u)}(R\tau)$, for the elements of the product on the right hand side of (5. 6) for which $l+1\leqq u\leqq k$. Hence application of (5. 4), (5. 7), (5. 8), (5. 9), (5. 10), (5. 11) and (5. 12) easily leads to the following relations.

$$\sigma_{\mathbf{m}}\left[\sup_{|\mathbf{t}|=1}\left\{\prod_{i=1}^{k}\omega_{m_i}^{(i)}(Rt_i)\right\}\right]=\sigma_{\mathbf{m}}\left[\sup_{|t|=1:0\leqq t_u<\tau\leqq t_s\leqq 1}\left\{\prod_{s=1}^{l}\omega_{m_s}^{(s)}(Rt_s)\prod_{u=l+1}^{k}\omega_{m_u}^{(u)}(Rt_u)\right\}\right]\leqq$$

$$(5.13)\qquad \leqq\sigma_{\mathbf{m}}\left[\sup_{|t'|\leqq 1}\left\{\prod_{s=1}^{l}KN_{m_s}^{(s)}(\sqrt{\alpha}Rt_s)^{m_s}\right\}\left\{\prod_{u=l+1}^{k}K(\sqrt{\alpha}R\tau)^{m_u}\right\}\right]\leqq$$

$$\leqq\sigma_{\mathbf{m}}N_{\mathbf{m}}K^k(\sqrt{\alpha}R)^{(\mathbf{m})}\tau^{(\mathbf{m})-(\mathbf{m}')}[\sup_{|t'|\leqq 1}\mathbf{t}'^{\mathbf{m}'}]\leqq$$

$$\leqq N_{\mathbf{m}}K^k\alpha^{\frac{1}{2}(\mathbf{m})+\frac{1}{2}(\mathbf{m}')}R^{(\mathbf{m})}\leqq N_{\mathbf{m}}K^k(\alpha R)^{(\mathbf{m})}.$$

It should be observed that the last inequality in (5. 13) still holds in the special cases when (i) $l=0$, (ii) $l=k$. In fact, in case (i) we have, from (2. 4), (5. 8) and (5. 10)

$$\sigma_{\mathbf{m}}\left[\sup_{|t|\leqq 1}\left\{\prod_{u=1}^{k}\omega_{m_u}^{(u)}(Rt_u)\right\}\right]\leqq\sigma_{\mathbf{m}}\left\{\prod_{u=1}^{k}\omega_{m_u}^{(u)}(R\tau)\right\}\leqq\sigma_{\mathbf{m}}\tau^{(\mathbf{m})}K^k(\sqrt{\alpha}R)^{(\mathbf{m})}<K^k(\alpha R)^{(\mathbf{m})}.$$

In case (ii) we appeal to the relations (2. 4), (5. 4), (5. 9) and (5. 10) to obtain

$$\sigma_{\mathbf{m}}\left[\sup_{|t|\leqq 1}\left\{\prod_{s=1}^{k}\omega_{m_s}^{(s)}(Rt_s)\right\}\right]<\sigma_{\mathbf{m}}K^k(\sqrt{\alpha}R)^{(\mathbf{m})}[\sup_{|t|\leqq 1}\mathbf{t}^{\mathbf{m}}]<N_{\mathbf{m}}K^k(\alpha R)^{(\mathbf{m})}.$$

Introducing now (5. 13) in the right hand side of (5. 6) we deduce that

$$\Omega_{\mathbf{m}}[R]<N_{\mathbf{m}}^2K^k(\alpha R)^{(\mathbf{m})},$$

and noting that the set $\{p_j[\mathbf{z}\}$ is a Cannon set it follows that

$$(5.14)\qquad \Omega[R]=\limsup_{(\mathbf{m})\to\infty}\{\Omega_{\mathbf{m}}[R]\}^{\frac{1}{(\mathbf{m})}}\leqq\alpha R.$$

Finally since α is arbitrary chosen near to 1, (5. 14) implies in view of (3. 7), that $\Omega[R]=R$, and the set $\{p_j[\mathbf{z}]\}$ will be effective in $\bar{S}_R$; as required.

6. Continuation of the Proof of Theorem 5. To complete the proof of the theorem we suppose that the set $\{q_j^{(1)}(z)\}$, for example, is not effective in $|z|\leqq R$. Then taking $m_1=m$, $t_1=1$, $m_i=t_i=0$; $2\leqq i\leqq k$ in the left hand side of (5. 6), then (2. 4), (3. 6) and (5. 3) yield

$$(6.1)\qquad \Omega[R]\geqq\limsup_{m\to\infty}\left[\sigma_{m,0,0,\dots,0}\,\omega_m^{(1)}(R)\prod_{i=2}^{k}\omega_0^{(i)}(0)\right]^{\frac{1}{m}}=\omega^{(1)}(R)>R,$$

and the set $\{p_j[\mathbf{z}]\}$ will not be effective in $\bar{S}_R$. Moreover, let μ be any positive integer and suppose that the set $\{q_j^{(1)}(z)\}$ is not effective in $|z|\leqq R(1+\mu)^{-\frac{1}{2}}$. Then there are a number $\beta<1$ and a sequence (n_j) of positive integers such that

$$(6.2)\qquad \omega_{n_j}^{(1)}\{R(1+\mu)^{-\frac{1}{2}}\}>\{\beta R(1+\mu)^{-\frac{1}{2}}\}^{n_j};\qquad (j\geqq 1).$$

Here we take $m_1=m$, $m_2=\mu m$, $t_1=(1+\mu)^{-\frac{1}{2}}$, $t_2=\{\mu/(1+\mu)\}^{\frac{1}{2}}$, $m_i=t_i=0$; $3\leqq i\leqq k$ in (5. 6). This is always possible since $k\geqq 2$. In this case (2. 4) implies that

$$(6.3)\qquad \sigma_{\mathbf{m}}={}_{m,\mu m,0,\dots,0}=\frac{(1+\mu)^{\frac{1}{2}(1+\mu)m}}{\mu^{\frac{1}{2}\mu m}}.$$

A combination of (3. 6), (5. 6), (6. 2) and (6. 3) gives

$$\Omega[R] \geqq \limsup_{m\to\infty} \{\Omega_{m,\mu m,0,\dots,0}[R]\}^{\frac{1}{(1+\mu)m}} \geqq$$

$$\geqq \limsup_{m\to\infty} \left[\sigma_{m,\mu m,0,\dots,0}\,\omega_m^{(1)}\left(\frac{R}{\sqrt{1+\mu}}\right)\omega_{\mu m}^{(2)}\left(R\sqrt{\frac{\mu}{1+\mu}}\right)\prod_{i=3}^{k}\omega_0^{(i)}(0)\right]^{\frac{1}{(1+\mu)m}} \geqq$$

$$\geqq R^{\frac{\mu}{1+\mu}}(1+\mu)^{\frac{1}{2}(1+\mu)^{-1}} \limsup_{n_j\to\infty}\left\{\omega_{n_j}^{(1)}\left(\frac{R}{\sqrt{1+\mu}}\right)\right\}^{\frac{1}{(1+\mu)n_j}} \geqq \beta^{\frac{1}{1+\mu}} R > R.$$

Hence the set $\{p_j[\mathbf{z}]\}$ will not be effective in $\bar{S}_R$ either. Therefore, in order that the set $\{p_j[\mathbf{z}]\}$ may be effective in the sphere $\bar{S}_R$, the set $\{q_j^{(1)}(z)\}$ should be effective in the circles $|z| \leqq R$ and $|z| \leqq R(1+\mu)^{-\frac{1}{2}}$. By the familiar properties of the Cannon function $\omega^{(1)}(r)$ we infer that the set $\{q_j^{(1)}(z)$ should be effective in $|z| \leqq r$ for $R(1+\mu)^{-\frac{1}{2}} \leqq r \leqq R$. Since μ can be taken arbitrary large we conclude that the set $\{q_j^{(1)}(z)\}$ should be effective in $|z| \leqq r$ for $0 < r \leqq R$. In the same way it can be proved that to ensure the effectiveness of the composite set $\{p_j[\mathbf{z}]\}$ in the sphere $\bar{S}_R$, each of the constituent sets $\{q_j^{(i)}(z)\}$ should be effective in $|z| \leqq r$ for $0 < r \leqq R$. Theorem 5 is therefore established.

It should be noted that theorem 5 implies also the following result.

Corollary. *Let $\{q_j^{(i)}(z)\}$ be Cannon sets of polynomials and suppose that $\{p_j[\mathbf{z}]\}$ is their composite set. Then the set $\{p_j[\mathbf{z}]\}$ will be effective in the spheres $\bar{S}_r$ for $0 < r \leqq R$ if, and only if, each of the sets $\{q_j^{(i)}(z)\}$ is effective in the circles $|z| \leqq r$ for $0 < r \leqq R$.*

7. Order of composite sets. We consider in what follows, the representation of classes of integral functions. After theorem 4, this representation is governed by the order of the basic set considered. The following theorem relates the order of the composite set with those of its constituent sets.

Theorem 6. *Let $\{q_j^{(i)}(z)\}$ be Cannon sets of polynomials of respective orders (γ_i), then the order of the composite set $\{p_j[\mathbf{z}]\}$ is $\Gamma = \sup_i(\gamma_i)$.*

Proof. First of all, as in (6. 1), we observe from (5. 6) that

$$\Omega_{m,0,\dots,0}[r] \geqq \sigma_{m,0,\dots,0}\,\omega_m^{(1)}(r)\prod_{i=2}^{k}\omega_0^{(i)}(0).$$

Hence

$$\limsup_{(\mathbf{m})\to\infty}\frac{\log\Omega_{\mathbf{m}}[r]}{(\mathbf{m})\log(\mathbf{m})} \geqq \limsup_{m\to\infty}\frac{\log\Omega_{m,0,\dots,0}[r]}{m\log m} \geqq \limsup_{m\to\infty}\frac{\log\omega_m^{(1)}(r)}{m\log m}.$$

Therefore, as r tends to infinity, (3. 8) implies that

$$\Gamma \geqq \gamma_1.$$

In the same way, it can be proved that

$$\Gamma \geqq \gamma_1, \gamma_2, \dots, \gamma_k,$$

so that

(7. 1) $$\Gamma \geqq \sup_i(\gamma_i).$$

Moreover, since the definition (2. 4) of the number $\sigma_{\mathbf{m}}$ is equivalent to

$$\sup_{|t|=1} \mathbf{t}^{\mathbf{m}} = \frac{1}{\sigma_{\mathbf{m}}},$$

then by taking $t_i = k^{-\frac{1}{2}}$, we deduce that

$$\sigma_{\mathbf{m}} \leqq k^{\frac{1}{2}(\mathbf{m})}. \tag{7. 2}$$

Now, if one of the orders (γ_i) is infinite, then (7. 1) implies that $\Gamma = \infty$, and the theorem is proved in this case.

Suppose therefore that all the orders (γ_i) are finite and choose the finite number γ such that

$$\gamma > \gamma_1, \gamma_2, \dots, \gamma_k.$$

Then corresponding to any finite number r there exists a positive number $L = L(r)$ such that

$$\omega_m^{(i)}(r) < L m^{\gamma m}, \qquad (m \geqq 1). \tag{7. 3}$$

Also, since each of the sets $\{q_j^{(i)}(z)\}$ is a Cannon set, so also will be the composite set $\{p_j[\mathbf{z}]\}$. Hence, given any finite number $a > 1$, there is a positive constant M so that

$$N_{\mathbf{m}} < M a^{(\mathbf{m})}, \tag{7. 4}$$

where $N_{\mathbf{m}}$ is, as before, the number of non-zero coefficients in (4. 5).

The relations (7. 2), (7. 3) and (7. 4) can be introduced in the right hand side of (5. 6) to yield

$$\Omega_{\mathbf{m}}[r] \leqq N_{\mathbf{m}} \sigma_{\mathbf{m}} \prod_{i=1}^{k} \omega_{m_i}^{(i)}(r) < M L^k a^{(\mathbf{m})} k^{\frac{1}{2}(\mathbf{m})} \{(\mathbf{m})\}^{\gamma(\mathbf{m})}.$$

Proceeding to the limit as the sum $(\mathbf{m})$ tends to infinity we obtain

$$\limsup_{(\mathbf{m}) \to \infty} \frac{\log \Omega_{\mathbf{m}}[r]}{(\mathbf{m}) \log (\mathbf{m})} \leqq \gamma,$$

so that $\Gamma \leqq \gamma$. Since γ can be chosen arbitrary near to the greatest of the orders (γ_i), it can be inferred that

$$\Gamma \leqq \sup_i (\gamma_i). \tag{7. 5}$$

Finally, the relations (7. 1) and (7. 5) imply that $\Gamma = \sup_i (\gamma_i)$, and the proof of theorem 6 is therefore complete.

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