Publ. Math. Debrecen
49 / 1-2 (1996), 127–134

On the norm of Jordan elementary operators in standard operator algebras

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Abstract. We establish a lower estimate for elementary operators of Jordan type in standard operator algebras.

1. Introduction

If $\mathcal{A}$ is an associative algebra, then given $a, b \in \mathcal{A}$ we define a *basic elementary operator* $M_{a,b} : \mathcal{A} \to \mathcal{A}$ by $M_{a,b}(x) = axb$. An *elementary operator* is a finite sum $E = \sum_{i=1}^{n} M_{a_i,b_i}$ of the basic ones. In the setting of Banach algebras and operator algebras in particular they were studied by many authors. Two recent papers on elementary operators are [6,7] where some older references can be found.

Many facts about the relation between the spectrum of E and spectrums of a_i, b_i are known. This is not the case with the relation between the operator norm of E and norms of a_i, b_i. This is in part a consequence of the fact that the representation of E with the above sum is not unique and in part due to the fact that $M_{a,b}$ can be zero with both a, b being nonzero. Thus most of the existing results concern the case of Banach

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Mathematics Subject Classification: 47B 47, 47A 30, 47D 30.
Key words and phrases: Hilbert space, standard operator algebra, elementary operator.
Authors were partially supported by grants from the Slovene Ministry of Science and Hungarian NFS, Grant No. 7292. This paper was written during the first author's visit at University of Maribor and the second author's visits at University of Szeged.
The final version was put together in London. The second author wishes to express his gratitude to Prof. Bunce and Prof. Chu for helpful conversations and the British Council for financial support.

algebras in which $M_{a,b} = 0$ implies $a = 0$ or $b = 0$. Such algebras are called *prime*. The problem here is of course a useful lower estimate for the norm of E because some upper estimates such as $\|E\| \leq \sum_{i=1}^{n} \|a_i\| \cdot \|b_i\|$ are trivial.

It was proved by Mathieu that in the case of prime C^*-algebras the norm of the basic elementary operator can not only be estimated but in fact computed precisely. The result is the best we can expect. Namely, $\|M_{a,b}\| = \|a\| \cdot \|b\|$. Mathieu also considered a problem for the operator $U_{a,b} = M_{a,b} + M_{b,a}$. He proved the following result, which was motivating for the present paper.

Theorem 1 (see [8]). *Let $\mathcal{A}$ be a prime C^*-algebra and a, b its elements. Then $\|U_{a,b}\| \geq \frac{2}{3}\|a\| \cdot \|b\|$.*

Remark. The notation $U_{a,b}$ for this sum of these two basic elementary operators is ours because this is Jacobson–McCrimmon notation from Jordan algebras (see below).

For us this result is interesting because the operator $U_{a,b}$ represents a Jordan triple structure of a C^*-algebra which is connected to the differential-geometric structure of its unit ball. The unit ball is a bounded symmetric domain and Jordan structure can be used to obtain nontrivial geometrical results. For the general theory of Jordan ternary structure and its applications to the geometry and analysis on symmetric spaces we refer to well-known books in this field [2–5] and [9–11]. Note that derivations and generalized derivations are also examples of elementary operators. For lower estimates concerning derivations we refer to [1].

If $\mathcal{H}$ is a Hilbert space, then the most obvious C^*-algebras, namely $B(\mathcal{H})$ consisting of all bounded operators and $C(\mathcal{H})$ consisting of all compact operators are prime which is quite easy to prove. In our present paper we are interested in a slightly different setting which still includes two above mentionded algebras. A *standard operator algebra* is a subalgebra of $B(\mathcal{H})$ containing all finite rank operators. To the contrast with Theorem 1, it is not assumed that $\mathcal{A}$ is selfadjoint or closed with respect to any topology. Important examples which are included in our result but not in Theorem 1 are Schatten p-classes. On the other hand, type II and type III von Neumann factors are included in Theorem 1 but not in our result. The algebras $B(\mathcal{H})$ and $C(\mathcal{H})$ however lie in the intersection of our work with the work of Mathieu.

In our main result we prove that for standard operator algebras it is possible to give a better lower bound $0,82\dots$ than in Theorem 1 where the bound is $0,66\dots$. This can be done by attaching a family of Hilbert spaces to a standard operator algebra and using inner products on them in order to obtain a supremum type estimate. On the other hand we make an obvious estimate using vectors ξ and η such that $\|a\xi\|$ is near $\|a\|$ and $\|b\eta\|$ is near $\|b\|$. Comparing both estimates we arrive at the lower bound $2(\sqrt{2}-1)$.

Throughout this paper we use the quite customary notation $\alpha\otimes\beta$ for a rank one operator defined by $(\alpha\otimes\beta)(\xi)=\langle\xi,\beta\rangle\alpha$ where $\alpha,\beta,\xi\in\mathcal{H}$ and $\langle\ ,\ \rangle$ denotes the inner product. Recall again that $U_{a,b}(x)=axb+bxa$. Given $p\in\mathcal{A}$, we denote by L_p and R_p the left and the right multiplication operators induced by p. Then

$$\begin{aligned} U_{ap,bp}(x) &= (ap)x(bp)+(bp)x(ap)=[a(px)b]p+[b(px)a]p\\ &= U_{a,b}(px)p=R_pU_{a,b}L_p(x)\end{aligned}$$

holds for $a,b,x\in\mathcal{A}$ and since $\|L_p\|,\|R_p\|\le\|p\|$ is valid in normed algebras, we have the estimate

$$\|U_{ap,bp}\|\le\|U_{a,b}\|\cdot\|p\|^2.$$

This simple observation suggested the idea of reducing the lower estimates of $U_{a,b}$ in standard operator algebras to the finite rank part by taking p to be a minimal projection.

2. Proof of the main result

Let $\mathcal{K}$ be an inner product space and let $\langle x,y\rangle$ denote its inner product. We do not assume that $\mathcal{K}$ is complete. Fix $a,b\in\mathcal{K}$ and consider a real-linear operator $S_{a,b}:\mathcal{K}\to\mathcal{K}$ defined by

$$S_{a,b}(x)=\langle a,x\rangle b+\langle b,x\rangle a.$$

Proposition 2. *The estimate $\|S_{a,b}\|\ge\|a\|\cdot\|b\|+|\langle a,b\rangle|$ holds.*

Proof. We may assume that $\|a\|=\|b\|=1$. First we consider the case when $\langle a,b\rangle\in\mathbb{R}^+$. From

$$S_{a,b}(a+b)=(1+\langle a,b\rangle)(a+b)$$

we see that $1+\langle a,b\rangle$ is an eigenvalue which yields the result in this special case.

If $\langle a, b\rangle = re^{i\varphi}$, then $\langle a, e^{i\varphi}b\rangle = r \geq 0$ so

$$\|S_{a,b}\| = \|S_{a,e^{i\varphi}b}\| \geq 1 + r = 1 + |\langle a, b\rangle|. \qquad \square$$

Proposition 3. *Let $\mathcal{A}$ be a standard operator algebra acting on a Hilbert space $\mathcal{H}$. If $a, b \in \mathcal{A}$, then the estimate*

$$\|U_{a,b}\| \geq \sup_{\xi\in\mathcal{H}, \|\xi\|=1} \{\|a\xi\| \cdot \|b\xi\| + |\langle a\xi, b\xi\rangle|\}$$

holds.

Proof. Let $\xi \in \mathcal{H}$ be a unit vector. The rank one operator $p = \xi \otimes \xi$ is a selfadjoint projection.

Consider $\mathcal{K} = \mathcal{A}p$ and define the inner product on $\mathcal{K}$ by

$$\langle xp, yp\rangle = \langle x\xi, y\xi\rangle_{\mathcal{H}}.$$

It is not difficult to verify that this inner product is well-defined. We shall also define a ternary composition (the Jordan triple product) on $\mathcal{K}$ by

$$[(xp)(yp)(zp)] = xp(yp)^* zp + zp(yp)^* xp = xpy^* zp + zpy^* xp.$$

Despite the fact that $y \in \mathcal{A}$ does not imply $y^* \in \mathcal{A}$, this product is well-defined since py^* is a rank one operator and thus belongs to $\mathcal{A}$. A straightforward manipulation of rank one operators shows that

$$[(xp)(yp)(zp)] = \langle xp, yp\rangle zp + \langle zp, yp\rangle xp = S_{xp,zp}(yp)$$

holds. Therefore we can use Proposition 2 for $ap, bp \in \mathcal{K}$ which yields

$$\begin{aligned}\|S_{ap,bp}\| &\geq \sqrt{\langle ap, ap\rangle}\sqrt{\langle bp, bp\rangle} + |\langle ap, bp\rangle| \\ &= \|a\xi\| \cdot \|b\xi\| + |\langle a\xi, b\xi\rangle|.\end{aligned}$$

Note that since $xp = (x\xi) \otimes \xi$, we have $\|xp\|^2 = \|x\xi\|^2 = \langle xp, xp\rangle$ and so the operator norm induced from $B(\mathcal{H})$ and the Hilbert norm coincide on $\mathcal{K}$. Hence

$$\begin{aligned}\|S_{ap,bp}(yp)\| &= \|apy^* bp + bpy^* ap\| \leq \|apy^* b + bpy^* a\| \\ &= \|U_{a,b}(py^*)\| \leq \|U_{a,b}\| \cdot \|py^*\| = \|U_{a,b}\| \cdot \|yp\|\end{aligned}$$

and so $\|S_{ap,bp}\| \leq \|U_{a,b}\|$. Note that $py^* \in \mathcal{A}$ even if $y^* \notin \mathcal{A}$. If we now take into account the inequality from the previous paragraph and the supremum over all norm one elements in $\mathcal{H}$, we obtain the result. $\square$

Theorem 4. *Let $\mathcal{A}$ be a standard operator algebra acting on a Hilbert space $\mathcal{H}$. If $a, b \in \mathcal{A}$, then the uniform estimate*

$$\|U_{a,b}\| \geq 2(\sqrt{2}-1)\|a\| \cdot \|b\|$$

holds.

Proof. We may again suppose that $\|a\| = \|b\| = 1$. If $\varepsilon > 0$ is given, we can find vectors $\xi, \eta \in \mathcal{H}$ such that $\|\xi\| = \|\eta\| = 1$ and $\|a\xi\|, \|b^*\eta\| \geq 1-\varepsilon$. Form $x = \xi \otimes \eta \in \mathcal{A}$, denote $T = U_{a,b}(x)$ and consider $t = \langle T(b^*\eta), a\xi\rangle$. The obvious estimate is $|t| \leq \|T\|$. On the other hand

$$\begin{aligned} t &= \langle (a\xi \otimes b^*\eta)(b^*\eta), a\xi\rangle + \langle (b\xi \otimes a^*\eta)(b^*\eta), a\xi\rangle \\ &= \|b^*\eta\|^2\|a\xi\|^2 + \langle b^*\eta, a^*\eta\rangle\langle b\xi, a\xi\rangle \end{aligned}$$

and therefore

$$\begin{aligned} |t| &\geq \|b^*\eta\|^2\|a\xi\|^2 - |\langle b^*\eta, a^*\eta\rangle| \cdot |\langle b\xi, a\xi\rangle| \\ &\geq (1-\varepsilon)^4 - |\langle b^*\eta, a^*\eta\rangle| \cdot |\langle b\xi, a\xi\rangle|. \end{aligned}$$

Since $\|x\| = \|\xi\| \cdot \|\eta\| = 1$, it follows that

$$\|U_{a,b}\| \geq (1-\varepsilon)^4 - |\langle b^*\eta, a^*\eta\rangle| \cdot |\langle b\xi, a\xi\rangle| \tag{1}$$

Now we must combine this estimation with the estimation obtained in Proposition 3. It is obvious that $\mathcal{A}^*$ is also a standard operator algebra and $\|U^{\mathcal{A}}_{a,b}\| = \|U^{\mathcal{A}^*}_{a^*,b^*}\|$. Now Proposition 3 yields

$$\|U_{a,b}\| \geq 2|\langle b\xi, a\xi\rangle|, \qquad \|U_{a,b}\| \geq 2|\langle b^*\eta, a^*\eta\rangle|.$$

This gives $\|U_{a,b}\|^2 \geq 4|\langle b\xi, a\xi\rangle| \cdot |\langle b^*\eta, a^*\eta\rangle|$ which combined with (1) implies

$$4\|U_{a,b}\| + \|U_{a,b}\|^2 \geq 4(1-\varepsilon)^4$$

for all positive ε. From this the result follows easily. $\square$

3. Some remarks

We feel that the estimate in Theorem 4 is not the best possible. A new inequality builded on yet another approach should perhaps be added. Estimations of large sums or rank one operators are very complicated so this is probably not leading towards a solution unless a good guess is possible. Anyway, we believe that the following is true.

Conjecture 5. *Let $\mathcal{A}$ be a standard operator algebra acting on a Hilbert space $\mathcal{H}$. If $a, b \in \mathcal{A}$, then the estimate*

$$\|a\| \cdot \|b\| \leq \|U_{a,b}\| \leq 2\|a\| \cdot \|b\|$$

holds.

Moreover, we feel that the number $\|U_{a,b}\|$ measures some sort of "angle" between the operators a and b. The case $\|U_{a,b}\| = \|a\| \cdot \|b\|$ should correspond to "orthogonality" while $\|U_{a,b}\| = 2\|a\| \cdot \|b\|$ should correspond to "being parallel". There is some evidence to that in the below observations.

Observation 6. *Let $\mathcal{A}, a, b$ be as in Conjecture 5. If $a = b$, then $\|U_{a,b}\| = 2\|a\| \cdot \|b\|$.*

Proof. Suppose that $\|a\| = 1$. Given a positive ε, there exist $\xi, \eta \in \mathcal{H}$ such that $\|\xi\| = \|\eta\| = 1$ and $\|a\xi\|, \|a^*\eta\| \geq 1 - \varepsilon$. Then

$$\|U_{a,a}(\xi \otimes \eta)\| = 2\|a\xi \otimes a^*\eta\| = 2\|a\xi\| \cdot \|a^*\eta\| \geq 2(1-\varepsilon)^2$$

and the result is now obvious. $\square$

Observation 7. *Let $\mathcal{A}$, a, b be as above. If $b = a^*$, then $\|U_{a,b}\| \geq \|a\| \cdot \|b\|$.*

Proof. Suppose that $\|a\| = 1$. Again, given a positive ε, there exists $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$ and $\|a\xi\| \geq 1 - \varepsilon$. Take $x = \xi \otimes \xi$ and denote

$$T = axa^* + a^*xa = a\xi \otimes a\xi + a^*\xi \otimes a^*\xi.$$

Then we have

$$\|T\| \geq |\langle T(a\xi), a\xi \rangle| = \|a\xi\|^4 + |\langle a\xi, a^*\xi \rangle|^2 \geq \|a\xi\|^4 \geq (1-\varepsilon)^4$$

and hence the result. $\square$

Observation 8. *Let $\mathcal{A}$ be as above and let ξ, η be orthogonal unit vectors from $\mathcal{H}$. Define $a = \xi \otimes \xi$ and $b = \eta \otimes \eta$. Then $\|U_{a,b}\| = \|a\| \cdot \|b\|$.*

Proof. Obviously $\|a\| = \|b\| = 1$. Further

$$U_{a,b}(\xi \otimes \eta) = (\xi \otimes \xi)(\xi \otimes \eta)(\eta \otimes \eta) + (\eta \otimes \eta)(\xi \otimes \eta)(\xi \otimes \xi) = \xi \otimes \eta$$

and so $\|U_{a,b}\| \geq 1$. On the other hand, for every $x \in \mathcal{A}$ and $\rho \in \mathcal{H}$, we have

$$\begin{aligned}\|U_{a,b}(x)\rho\|^2 &= \|\langle x\eta, \xi\rangle\langle\rho, \eta\rangle\xi + \langle x\xi, \eta\rangle\langle\rho, \xi\rangle\eta\|^2 \\ &= |\langle x\eta, \xi\rangle|^2|\langle\rho, \eta\rangle|^2 + |\langle x\xi, \eta\rangle|^2|\langle\rho, \xi\rangle|^2 \\ &\leq \|x\|^2(|\langle\rho, \eta\rangle|^2 + |\langle\rho, \xi\rangle|^2) \leq \|\rho\|^2\|x\|^2\end{aligned}$$

and so $\|U_{a,b}\| \leq 1$. $\square$

We conclude with two problems whose solution might cast some light on the relation of $\|U_{a,b}\|$ and the "angle" between a, b.

Problem 9. Let $\mathcal{A}$ be as above. Suppose that $\|U_{a,b}\| = 2\|a\| \cdot \|b\|$. What can we say about a and b?

Problem 10. Let $\mathcal{A}$ be as above. Suppose that $\|U_{a,b}\| = \|a\| \cdot \|b\|$. What can we say about a and b? Is it true that $ab^* = a^*b = 0$?

Added in proof. After the submission of this paper a highly interesting work of Cabrera and Rodríguez, Proc. London Math. Soc. 69 (1994), 576–604, came to our attention. Among other things authors showed that for much more general class of algebras it is possible to give universal estimate $\|U_{a,b}\| \geq \frac{1}{10206}\|a\| \cdot \|b\|$.

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(Received September 21, 1995)