A generalization of the Laguerre polynomials

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1. Introduction. Some years back F. J. PALAS [1] studied the set of polynomials $\{T_{kn}(x)\}$, defined by the Rodrigues formula

$$T_{kn}(x) = \frac{1}{n!} e^{x^k} D^n[x^n e^{-x^k}].$$

Later L. CARLITZ [2] gave the following operational formula for the Laguerre polynomial:

$$\prod_{j=1}^{n} (xD - x + \alpha + j) = n! \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!} L_{n-\nu}^{(\alpha+\nu)}(x) D^\nu; \qquad \left(D = \frac{d}{dx}\right).$$

Following PALAS and CARLITZ, S. K. CHATTERJEA [3, 4] studied the set of polynomials $\{T_{kn}^{(\alpha)}(x, p)\}$ defined by the Rodrigues' formula

$$T_{kn}^{(\alpha)}(x, p) = \frac{1}{n!} x^{-\alpha} e^{px^k} D^n(x^{\alpha+n} e^{-px^k}) \tag{A}$$

and obtained some interesting operational formulae.

Observing the form of the Rodrigues formula in (A), one might enquire into the possibility of replacing px^k by an arbitrary polynomial of degree k. The present work is an answer to this question and it is shown that we can always replace px^k in (A) by an arbitrary polynomial in x of degree k to generate more general polynomial sets. In sequel, the generalizations of some of the results of Palas, Carlitz and Chatterjea have been obtained.

We shall make use of some of the results of STEPHENS [5] in deriving the operational formulae. The treatment being formal, we shall obtain our results quite heuristically.

2. Definition of the set $T_{kn}^{(\alpha)}(x)$. Let $p_k(x)$ be a polynomial in x of degree k, defined by

$$p_k(x) = \sum_{s=1}^{k} p_s x^s. \tag{1}$$

Let us consider the k-set $\{T_{kn}^{(\alpha)}(x)\}$ of polynomials, defined by the generating function

$$g(x, t) = (1-t)^{-\alpha-1} e^{p_k(x)} e^{-p_k[x(1-t)^{-1}]} = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n. \tag{2}$$

Let us define

$$[p_k(x)]^v = \left[\sum_{s=1}^{k} p_s x^s\right]^v = \sum_{s=v}^{kv} p_s^{(v)} x^s. \tag{3}$$

In order to obtain the explicit expression for the polynomials $T_{kn}^{(\alpha)}(x)$, defined in (2), we make use of the following rule [6], [7], for the nth derivative of a composite function:

$$D^n[(fg)(x)] = \sum_{v=0}^{n} (D^v f) g \cdot \sum_{s=0}^{v} \frac{(-1)^{v-s}}{s!(v-s)!} g^{v-s} D^n[g^s],$$

where $(fg)(x)=f[g(x)]$. We easily have

$$T_{kn}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} + \frac{1}{n!} \sum_{v=1}^{n} \sum_{l=1}^{v} \sum_{s=0}^{l} \sum_{m=s}^{ks} \binom{n}{v} \frac{(-1)^s}{s!(l-s)!} p_m^{(s)} (1+\alpha)_{n-v} (m) v \cdot \tag{4}$$
$$\cdot [p_k(x)]^{l-s} x^m \quad (n \geqq 1), \qquad T_0^{(\alpha)}(x) = 1,$$

which is the explicit expression for the polynomials $T_{kn}^{(\alpha)}(x)$. From (4), we see that, $T_{kn}^{(\alpha)}(x)$ is a polynomial in x of degree nk.

3. Rodrigues formula. By Maclaurins theorem, we have from (2)

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!}\left[\frac{d^n}{dt^n}\{(1-t)^{-\alpha-1} e^{p_k(x)} e^{-p_k[x(1-t)^{-1}]}\}\right]_{t=0} = \tag{5}$$
$$= \frac{1}{n!} e^{p_k(x)} \sum_{v=0}^{\infty} \sum_{s=v}^{kv} \frac{(-1)^v}{v!} p_s^{(v)} x^s (\alpha+s+1)_n,$$

where $(a)_n = a(a+1)\dots(a+n-1)$, $(a)_0 = 1$. Also,

$$D^n[x^{n+\alpha} e^{-p_k(x)}] = x^\alpha \sum_{v=0}^{\infty} \sum_{s=v}^{kv} \frac{(-1)^v}{v!} p_s^{(v)} x^s (\alpha+s+1)_n. \tag{6}$$

From (5) and (6), we have

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{p_k(x)} D^n[x^{n+\alpha} e^{-p_k(x)}]. \tag{7}$$

The relation (7) is the Rodrigues formula satisfied by the polynomials $T_{kn}^{(\alpha)}(x)$, defined in (2).

4. Recurrence relations. From (2), we have

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n = (1-t)^{-(\alpha-\beta)} (1-t)^{-\beta-1} e^{p_k(x)} e^{-p_k[x(1-t)^{-1}]} =$$
$$= (1-t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x) t^n.$$

Therefore, we have

$$T_{kn}^{(\alpha)}(x) = \sum_{v=0}^{n} \frac{(\alpha-\beta)v}{v!} T_{k(n-v)}^{(\beta)}(x). \tag{8}$$

Differentiating (2) with respect to x, we get

$$\frac{d}{dx}g(x,t)=\{p_k'(x)-p_k'[x(1-t)^{-1}]\}g(x,t), \tag{9}$$

from which, we get

$$T_{kn}^{(\alpha)\prime}(x)=p_k'(x)T_{kn}^{(\alpha)}(x)-\sum_{s=1}^{k}\sum_{v=0}^{n}T_{k(n-v)}^{(\alpha)}(x)\binom{s+v-1}{v}sp_sx^{s-1}. \tag{10}$$

Multiplying (9) by $(1-t)^k$, and equating the coefficients of t^n, we get

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^iT_{k(n-i)}^{(\alpha)\prime}(x)= \tag{11}$$

$$=\sum_{s=1}^{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^isp_sx^{s-1}T_{k(n-i)}^{(\alpha)}(x)-\sum_{s=1}^{k}\sum_{i=0}^{k-s}\binom{k-s}{i}(-1)^isp_sx^{s-1}T_{k(n-i)}^{(\alpha)}(x).$$

Again, from Rodrigues formula, we have

$$(v+1)!\,x^\alpha e^{-p_k(x)}T_{k(v+1)}^{(\alpha)}(x)=D^{v+1}[x\cdot x^{v+\alpha}e^{-p_k(x)}]$$

from which we get

$$(v+1)T_{k(v+1)}^{(\alpha)}(x)=xT_{kv}^{(\alpha)\prime}(x)+\big(v+1+\alpha-xp_k'(x)\big)T_{kv}^{(\alpha)}(x) \tag{12}$$

Differentiating (12), m times with respect to x, we get

$$(v+1)D^mT_{k(v+1)}^{(\alpha)}(x)= \tag{13}$$

$$=xD^{m+1}T_{kv}^{(\alpha)}(x)+\big(m+v+1+\alpha-xp_k'(x)\big)D^mT_{kv}^{(\alpha)}(x)-\sum_{s=1}^{m}\binom{m}{s}D^s[xp_k'(x)]\cdot D^{m-s}T_{kv}^{(\alpha)}(x)$$

The relations (8), (10), (11), (12) and (13) are precisely the recurrence relations satisfied by the polynomials $T_{kn}^{(\alpha)}(x)$.

5. The moment problem. We now consider the moment problem pertaining to the polynomials $T_{nk}^{(\alpha)}(x)$. Let the relevant weight function be $x^\alpha w(x)$, where $w(x)=$ $=e^{-p_k(x)}$. Let

$$M(s,n)=\int_0^\infty x^sx^\alpha w(x)T_{kn}^{(\alpha)}(x)\,dx. \tag{14}$$

Assuming $\alpha>-1$ and $p_k>0$, we have from (14), by making use of the Rodrigues formula, that

$$M(s,n)=\frac{1}{n!}(-1)^v\binom{s}{v}v!\int_0^\infty x^{s-v}D^{n-v}[x^{n+\alpha}w(x)]\,dx, \tag{15}$$

since,

$$\frac{1}{n!}\sum_{i=0}^{v-1}(-1)^i\binom{s}{i}i!\,x^{s-i}D^{n-1-s}[x^{n+\alpha}w(x)]\Big|_0^\infty=0$$

for all r and s.

Now, if $s=0, 1, 2, \ldots, n-1$, we have for $r=s$ in (13), $M(s, n)=0$. Next for $s \geqq n$ and $r=n$, (15) becomes

$$M(s, n) = (-1)^n \binom{s}{n} \int_0^\infty x^{s+\alpha} e^{-p_k(x)}\, dx,$$

and we have the theorem:

Theorem. *The moments of* $\{T_{kn}^{(\alpha)}(x)\}$ *with respect to the weight function* $x^\alpha e^{-p_k(x)}$, *have values*

$$M(s, n) = \begin{cases} 0 & s = 0, 1, \ldots, n-1. \\ (-1)^n \binom{s}{n} \int_0^\infty x^{s+\alpha} e^{-p_k(x)}\, dx, & s = n, n+1, \ldots, \end{cases}$$

where

$$M(s, n) = \int_0^\infty x^{s+\alpha} e^{-p_k(x)}\, T_{kn}^{(\alpha)}(x)\, dx.$$

6. Certain operational formulae. Let y be a differentiable function of x. Let

$$\Omega_n y = x^{-\alpha} e^{p_k(x)} D^n[x^{\alpha+n} e^{-p_k(x)} y]. \tag{16}$$

Since,

$$\Omega_{n+1} y = x^{-\alpha} e^{p_k(x)} D^{n+1}[x^{\alpha+n+1} e^{-p_k(x)} y] = [xD + \alpha - xp_k'(x) + (n+1)]\, \Omega_n y,$$

we have

$$\Omega_n y = \prod_{j=1}^{n} \big(xD + \alpha - xp_k'(x) + j\big) y. \tag{17}$$

Again, it can be easily seen that

$$D^n[x^{\alpha+n} e^{-p_k(x)} y] = n!\, x^\alpha e^{-p_k(x)} \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!}\, T_{k(n-\nu)}^{(\alpha+\nu)}(x)\, D^\nu y,$$

and therefore, we have

$$\Omega_n y = n! \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!}\, T_{k(n-\nu)}^{(\alpha+\nu)}(x)\, D^\nu y. \tag{18}$$

Hence, we have

$$\prod_{j=1}^{n} \big(xD - xp_k'(x) + \alpha + j\big) y = n! \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!}\, T_{k(n-\nu)}^{(\alpha+\nu)}(x)\, D^\nu y, \tag{19}$$

which is the operational representation for the polynomials $T_{kn}^{(\alpha)}(x)$. Again, from (16),

$$\Omega_n y = x^{-\alpha} e^{p_k(x)} D^n[x^{n-k} \cdot x^{\alpha+k} e^{-p_k(x)} y] = x^{-\alpha-n} e^{p_k(x)} x^n D^n[x^{n-\nu} \cdot x^{\alpha+\nu} e^{-p_k(x)} y] =$$

$$= x^{-\alpha-n} e^{p_k(x)} \prod_{j=0}^{n-1} (\delta - j) x^{n-\nu} (x^{\alpha+\nu} e^{-p_k(x)} y) =$$

$$= x^{-\alpha-\nu} e^{p_k(x)} \prod_{j=0}^{n-1} (\delta - \nu + 1 + j)(x^{\alpha+\nu} e^{-p_k(x)} y) =$$

$$= x^{-\alpha-\nu} e^{p_k(x)} x^{-n} [x(\delta - \nu + 1)]^n (x^{\alpha+\nu} e^{-p_k(x)} y).$$

Hence, using (18), we get the operational formula

$$(20)\qquad [x(\delta-v+1)]^n(x^{\alpha+v}e^{-p_k(x)}y) = n!\,x^{\alpha+v+n}e^{-p_k(x)}\sum_{s=0}^{n}\frac{x^s}{s!}T^{(\alpha+s)}_{k(n-s)}(x)D^s y$$

Giving different integral values to r, we get different operational representation for the relevant polynomial.

From (19), we have by taking $y=1$,

$$(21)\qquad n!\,T^{(\alpha)}_{kn}(x) = \prod_{j=1}^{n}\big(xD-xp_k'(x)+\alpha+j\big)\cdot 1$$

and hence

$$(22)\qquad nT^{(\alpha)}_{kn}(x) = \big(xD-xp_k'(x)+\alpha+n\big)T^{(\alpha)}_{k(n-1)}(x).$$

Putting $y=1$ in (20), we get

$$(23)\qquad n!\,T^{(\alpha)}_{kn}(x) = x^{-\alpha-m-n}e^{p_k(x)}[x(\delta-m+1)]^n(x^{\alpha+m}e^{-p_k(x)}).$$

From (23), by giving different integral values to v, we get different operational formulae for the polynomial $T^{(\alpha)}_{kn}(x)$.

Generating Functions: From (21), we have

$$T^{(\alpha)}_{kn}(x) = \frac{1}{n!}e^{p_k(x)}\prod_{j=1}^{n}(\delta+\alpha+j)e^{-p_k(x)}.$$

Hence,

$$\sum_{n=0}^{\infty}T^{(\alpha)}_{kn}(x)t^n = e^{p_k(x)}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\delta+\alpha+1)_n e^{-p_k(x)}\cdot(1-t)^{-\alpha-1}e^{p_k(x)}(1-t)^{-\delta}e^{-p_k(x)}$$

and since $a^\delta f(x)=f(ax)$, we have

$$\sum_{n=0}^{\infty}T^{(\alpha)}_{kn}(x)t^n = (1-t)^{-\alpha-1}e^{p_k(x)}e^{-p_k[x(1-t)^{-1}]}$$

which is the same as (2). Again,

$$\sum_{n=0}^{\infty}T^{(\alpha-n)}_{kn}(x)t^n = e^{p_k(x)}\sum_{n=0}^{\infty}\binom{\delta+\alpha}{n}t^n e^{-p^k(x)} = (1+t)^\alpha e^{p_k(x)}(1+t)^\delta e^{-p_k(x)}.$$

Hence, we have

$$(24)\qquad \sum_{n=0}^{\infty}T^{(\alpha-n)}_{kn}(x)t^n = (1+t)^\alpha e^{p_k(x)}e^{-p_k[x(1+t)]}.$$

Again from (21), we have

$$(m+n)!\,T^{(\alpha)}_{k(m+n)}(x) = \prod_{j=1}^{m}\big(xD-xp_k'(x)+\alpha+n+j\big)\prod_{k'=1}^{n}\big(xD-xp_k'(x)+\alpha+k'\big)\cdot 1 =$$

$$= n!\prod_{j=1}^{m}\big(xD-xp_k'(x)+\alpha+n+j\big)T^{(\alpha)}_{kn}(x).$$

Hence we have

$$(25)\qquad (m+n)!\,T^{(\alpha)}_{k(m+n)}(x) = n!\,m!\sum_{v=0}^{m}\frac{x^v}{v!}T^{(\alpha+n+v)}_{k(m-v)}(x)D^v T^{(\alpha)}_{kn}(x).$$

Using (25), we consider the sum

$$\sum_{m=0}^{\infty}\binom{m+n}{n} T_{k(m+n)}^{(\alpha)}(x)\,t^m = \sum_{m=0}^{\infty}\sum_{v=0}^{\infty}\frac{x^v}{v!}\,T_{km}^{(\alpha+n+v)}(x)\,D^v T_{kn}^{(\alpha)}(x)\,t^{m+v} =$$

$$= \sum_{v=0}^{\infty}\frac{(xt)^v}{v!}\,D^v T_{kn}^{(\alpha)}(x)(1-t)^{-\alpha-n-v-1}e^{p_k(x)}e^{-p_k[x(1-t)^{-1}]} =$$

$$= e^{p_k(x)}e^{-p_k[x(1-t)^{-1}]}(1-t)^{-\alpha-n-1}e^{\frac{xt}{1-t}D}\,T_{kn}^{(\alpha)}(x),$$

and we have

$$(26)\qquad \sum_{m=0}^{\infty}\binom{m+n}{n} T_{k(m+n)}^{(\alpha)}(x)\,t^m = (1-t)^{-\alpha-n-1}e^{p_k(x)}e^{-p_k[x(-t)^{-1}]}\,T_{kn}^{(\alpha)}\big(x(1-t)^{-1}\big).$$

Again

$$\sum_{n=0}^{\infty}\binom{m+n}{n} T_{k(m+n)}^{(\alpha-n)}(x)\,t^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,e^{p_k(x)}\binom{\delta+\alpha}{n}\prod_{j=1}^{m}(\delta+\alpha+j)e^{-p_k(x)} =$$

$$= (1+t)^{\alpha}e^{p_k(x)}(1+t)^{\delta}e^{-p_k(x)}\,T_{km}^{(\alpha)}(x).$$

Hence, we have

$$(27)\qquad \sum_{n=0}^{\infty}\binom{m+n}{n} T_{k(m+n)}^{(\alpha-n)}(x)\,t^n = (1+t)^{\alpha}e^{p_k(x)}e^{-p_k[x(1+t)]}\,T_{km}^{(\alpha)}\big(x(1+t)\big).$$

Further, if we replace t by $-t/x$ and a by $a-m$ in (27), we find the interesting formula:

$$(28)\qquad \sum_{n=0}^{\infty}\frac{(-t)^n(m+n)!}{n!\,x^{n+m}}\,T_{k(m+n)}^{a-n-m}(x) = x^{-a}(x-t)^a\frac{m!}{(x-t)^m}\,e^{p_k(x)}e^{-p_k(x-t)}\,T_{km}^{(a-m)}(x-t).$$

It is interesting to note, that if we take $p_k(x)=x$, the above results reduce to those for Laguerre polynomial. Further, if we take $p_k(x)=px^k$, we get results for the polynomial considered by Chatterjea, and if we also take $a=0$, we get results for the polynomial considered by Palas.

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