Holomorphy theory of extra loops

By D. A. ROBINSON (Atlanta, Ga.)

§ 1. Introduction

Recently F. FENYVES [3] defined and studied extra loops (Definition 1). He showed that an extra loop is isomorphic to each of its loop isotopes and also that certain inner mappings of an extra loop are automorphisms.

The purpose of the present paper is to obtain a necessary and sufficient condition for a holomorph of a loop to be an extra loop. This condition (Main Theorem) is formulated in terms of nuclear automorphisms (Definition 3) and is analogous to a result of R. H. BRUCK [1] for Moufang loops. In § 4 we establish the existence of non-trivial nuclear automorphisms of extra loops and in § 5 we examine a special construction of extra loops.

§ 2. Preliminary information

Let $(G, \cdot)$ be a loop whose identity element is denoted by 1. For each $x \in G$, the mappings $R(x)$ and $L(x)$ are defined by $yR(x)=yx$ and $yL(x)=xy$ for all $y \in G$. It follows that $R(x)$ and $L(x)$ are one-to-one mappings of G onto G. If $(G, \cdot)$ is an inverse property loop, then corresponding to each $x \in G$ there is an $x^{-1} \in G$ so that $xx^{-1}=x^{-1}x=1$, $R(x)^{-1}=R(x^{-1})$ and $L(x)^{-1}=L(x^{-1})$.

Recall that an ordered triple $\langle U, V, W\rangle$ of one-to-one mappings U, V and W of G onto G is called an autotopism of the loop $(G, \cdot)$ if and only if $xU \cdot yV = (xy)W$ for all $x, y \in G$. It is well-known that the set of all autotopisms of a loop forms a group under the usual "componentwise multiplication". Recall also that a one-to-one mapping P of G onto G is called a pseudo-automorphism of the loop $(G, \cdot)$ if and only if there is an element $c \in G$ (called a companion of P) so that $\langle P, PR(c), PR(c)\rangle$ is an autotopism of $(G, \cdot)$.

For a detailed account of the loop theory concepts mentioned in the preceding paragraphs see Bruck [2]. Those basic results needed about Moufang loops and, in particular, about extra loops are now summarized as follows:

Theorem 1. *Let $(G, \cdot)$ be a Moufang loop. Then*

(i) $(G, \cdot)$ *is an inverse property loop* (*see* BRUCK [2], Ch. VII, Lemma 3. 1),

(ii) *the right, middle and left nuclei of* $(G, \cdot)$ *coincide with the nucleus of* $(G, \cdot)$ (*see* BRUCK [2], Ch, VII, Theorem 2. 1),

(iii) *the inner mapping* $R(x, y)$ *defined by* $R(x, y)=R(x)R(y)R(xy)^{-1}$ *is a pseudo-automorphism of* $(G, \cdot)$ *with the loop commutator* $[x, y]$ *as companion for each* $x, y\in G$ (see BRUCK [2], Ch. VII, Lemma 2. 2),

(iv) *the nucleus of* $(G, \cdot)$ *is a normal subgroup of* $(G, \cdot)$ (see BRUCK [2], Ch. VII, Theorem 2. 1).

Definition 1. A loop $(G, \cdot)$ is an *extra loop* if and only if

$$(xy\cdot z)x = x(y\cdot zx) \tag{1}$$

for all $x, y, z\in G$. (See FENYVES [3].)

Theorem 2. *Let* $(G, \cdot)$ *be an extra loop. Then*
(i) $(G, \cdot)$ *is Moufang* (see FENYVES [3], Theorem 3),
(ii) *the inner mappings* $R(x, y)$ *defined above are automorphisms of* $(G, \cdot)$ *for all* $x, y\in G$ (see FENYVES [3], Theorem 6),
(iii) $(G, \cdot)$ *is necessarily a group whenever* $(G, \cdot)$ *is commutative* (see FENYVES [3], Theorem 5).

Theorem 3. *A loop* $(G, \cdot)$ *is an extra loop if and only if*

$$A=\langle L(x), R(x)^{-1}, L(x)R(x)^{-1}\rangle \tag{2}$$

is an automorphism of $(G, \cdot)$ *for all* $x\in G$ (see FENYVES [3], Theorem 2).

If $A(G)$ is a group of automorphisms of a loop $(G, \cdot)$, we shall let H be the Cartesian product $H = A(G)\times G$ and define

$$(\alpha, x)\circ(\beta, y) = (\alpha\beta, x\beta\cdot y) \tag{3}$$

for all (α, x) and (β, y) in H. One can then prove that $(H, \circ)$ is a loop and that $(H, \circ)$ has a subloop isomorphic to $(G, \cdot)$. This brings us to

Definition 2. If $A(G)$ is a group of automorphisms of a loop $(G, \cdot)$, then the loop $(H, \circ)$ constructed above is called the *$A(G)$-holomorph* of $(G, \cdot)$.
We also have need of

Definition 3. Let $(G, \cdot)$ be an inverse property loop with nucleus denoted by N. Then an automorphism α of $(G, \cdot)$ is *nuclear* if and only if $a^{-1}\cdot a\alpha\in N$ for all $a\in G$.

§ 3. Main theorem

Using Definitions 1 and 2 and expression (3), one obtains by direct computation the following

Lemma. *Let* $(G, \cdot)$ *be a loop and let* $A(G)$ *be a group of automorphisms of* $(G, \cdot)$. *Then the* $A(G)$*-holomorph* $(H, \circ)$ *of* $(G, \cdot)$ *is an extra loop if and only if*

$$(x\alpha\cdot y)z\cdot x = x\alpha(y\cdot zx) \tag{4}$$

for all $x, y, z\in G$ *and all* $\alpha\in A(G)$.

We are now in a position to present the

Main Theorem. *Let $(G, \cdot)$ be a loop and let $A(G)$ be a group of automorphisms of $(G, \cdot)$. Then the $A(G)$-holomorph $(H, \circ)$ of $(G, \cdot)$ is an extra loop if and only if $(G, \cdot)$ is an extra loop and each $\alpha \in A(G)$ is a nuclear automorphism of $(G, \cdot)$.*

PROOF. I. Let $(H, \circ)$ be an extra loop. Since $(G, \cdot)$ is isomorphic to a subloop of $(H, \circ)$ it follows that $(G, \cdot)$ must be an extra loop. By the preceding lemma, expression (4) holds for all $x, y, z \in G$ and all $\alpha \in A(G)$. Replacing z by zx^{-1} in (4) and appealing to the fact that an extra loop is an inverse property loop (see Theorems 1 and 2), we see that

$$(x\alpha \cdot y)(zx^{-1}) \cdot x = x\alpha \cdot yz \tag{5}$$

for all $x, y, z \in G$ and all $\alpha \in A(G)$. Applying the inverse property to (5), we obtain

$$(x\alpha \cdot y)(zx^{-1}) = (x\alpha \cdot yz)x^{-1}$$

for all $x, y, z \in G$ and all $\alpha \in A(G)$. Hence,

$$B = \langle L(x\alpha), R(x)^{-1}, L(x\alpha)R(x)^{-1} \rangle$$

is an automorphism of $(G, \cdot)$ for all $x \in G$ and all $\alpha \in A(G)$. But, in view of Theorem 3, we know that $A = \langle L(x), R(x)^{-1}, L(x)R(x)^{-1} \rangle$ is an autotopism of $(G, \cdot)$ for all $x \in G$. So it follows that $BA^{-1} = \langle L(x\alpha)L(x)^{-1}, I, L(x\alpha)L(x)^{-1} \rangle$ is an autotopism of $(G, \cdot)$ for all $x \in G$ and all $\alpha \in A(G)$ where I denotes the identity mapping on G. Hence, $1L(x\alpha)L(x)^{-1} \cdot b = (1 \cdot b)L(x\alpha)L(x)^{-1} = bL(x\alpha)L(x)^{-1}$ for all $b \in G$. So we have $L(x^{-1} \cdot x\alpha) = L(x\alpha)L(x)^{-1}$ for all $x \in G$ and all $\alpha \in A(G)$. The autotopism BA^{-1} can now be rewritten as $BA^{-1} = \langle L(x^{-1} \cdot x\alpha), I, L(x^{-1} \cdot x\alpha) \rangle$ for all $x \in G$ and all $\alpha \in A(G)$. Thus, $x^{-1} \cdot x\alpha$ is in the left nucleus of $(G, \cdot)$ and, by Theorem 1 (ii), also in the nucleus of $(G, \cdot)$ for all $x \in G$ and all $\alpha \in A(G)$. Hence, each $\alpha \in A(G)$ is nuclear.

II. Now let $(G, \cdot)$ be an extra loop and let each $\alpha \in A(G)$ be nuclear. Then, for each $\alpha \in A(G)$ and each $x \in G$, the element $x^{-1} \cdot x\alpha$ is in the middle nucleus of $(G, \cdot)$ and it follows that

$$x\alpha \cdot y = [x(x^{-1} \cdot x\alpha)]y = x[(x^{-1} \cdot x\alpha)y]$$

for all $y \in G$. Hence, we see that

$$L(x^{-1} \cdot x\alpha) = L(x\alpha)L(x)^{-1}$$

for all $x \in G$ and all $\alpha \in A(G)$. But, for all $x \in G$ and all $\alpha \in A(G)$, we know that $x^{-1} \cdot x\alpha$ is in the left nucleus of $(G, \cdot)$. Hence,

$$C = \langle L(x^{-1} \cdot x\alpha), I, L(x^{-1} \cdot x\alpha) \rangle = \langle L(x\alpha)L(x)^{-1}, I, L(x\alpha)L(x)^{-1} \rangle$$

is an autotopism of $(G, \cdot)$ for all $x \in G$ and all $\alpha \in A(G)$. But again, by Theorem 3, we know that $A = \langle L(x), R(x)^{-1}, L(x)R(x)^{-1} \rangle$ is an autotopism of $(G, \cdot)$ for all $x \in G$. Therefore,

$$CA = \langle L(x\alpha), R(x)^{-1}, L(x\alpha)R(x)^{-1} \rangle$$

is an autotopism of $(G, \cdot)$ for all $x \in G$ and all $\alpha \in A(G)$. Thus, we get

$$(x\alpha \cdot y)(zx^{-1}) = (x\alpha \cdot yz)x^{-1} \tag{6}$$

for all $x, y, z \in G$ and all $\alpha \in A(G)$. Replacing z by zx in (6) and using the inverse property, we see that (4) holds for all $x, y, z \in G$ and all $\alpha \in A(G)$. Hence, $(H, \circ)$ is an extra loop and the proof is complete.

§ 4. Nuclear automorphisms of extra loops

As for the existence of nuclear automorphisms of extra loops, we offer the following

Theorem 4. *If* $(G, \cdot)$ *is an extra loop, then each inner mapping* $R(x, y)$ (see Theorem 1 (iii)) *is a nuclear automorphism of* $(G, \cdot)$.

PROOF. For $x, y, z, w \in G$ it is clear that $w = z^{-1} \cdot zR(x, y)$ if and only if $zx \cdot y = zw \cdot xy$. Since $R(x, y)$ is an automorphism of $(G, \cdot)$ (see Theorem 2 (ii)) and also a pseudo-automorphism of $(G, \cdot)$ with companion $[x, y]$ (see Theorem 1 (iii)), we see that

$$D = \langle R(x, y),\ R(x, y),\ R(x, y)\rangle$$

and

$$E = \langle R(x, y),\ R(x, y)R([x, y]),\ R(x, y)R([x, y])\rangle$$

are autotopisms of $(G, \cdot)$ for all $x, y \in G$. Hence, $D^{-1}E = \langle I, R([x, y]), R([x, y])\rangle$ is an autotopism of $(G, \cdot)$ for all $x, y \in G$. Therefore, all the loop commutators $[x, y]$ of $(G, \cdot)$ are in the right nucleus of $(G, \cdot)$ and, by Theorem 1 (ii), also in the nucleus N of $(G, \cdot)$. This information together with Theorem 1 (iv) indicates that the quotient loop G/N exists and is commutative. But the loop G/N is a homomorphic image of the extra loop $(G, \cdot)$ and, hence G/N is also an extra loop. But a commutative extra loop is a group (see Theorem 2 (iii)), so G/N is associative. Let θ be the natural homomorphism of $(G, \cdot)$ onto G/N and note that N is the kernel of θ. Then it follows that $z\theta \cdot x\theta \cdot y\theta = z\theta \cdot w\theta \cdot x\theta \cdot y\theta$ in the group G/N. Hence, $w\theta$ is the identity element of G/N and so w is in the kernel of θ. That is, $w = z^{-1} \cdot zR(x, y)$ is in N for all $x, y, z \in G$. Hence, each $R(x, y)$ is nuclear and the proof is complete.

If $(G, \cdot)$ is an extra loop which is not a group, then not all of the inner mappings $R(x, y)$ can be the identity mapping on G. Hence, in view of the preceding theorem, an extra loop possesses non-trivial nuclear automorphisms. We now prove

Theorem 5. *The set* $S(G)$ *of all nuclear automorphisms of an extra loop* $(G, \cdot)$ *is a normal subgroup of the automorphism group of* $(G, \cdot)$.

PROOF. Clearly $S(G)$ is not empty. Just as in Part II of the proof of our Main Theorem we see that

$$L(a^{-1} \cdot a\alpha) = L(a\alpha)L(a)^{-1}$$

for all $a \in G$ and all $\alpha \in S(G)$. If $\alpha \in S(G)$, then $a^{-1} \cdot a\alpha$ must be in the left nucleus of $(G, \cdot)$ for all $a \in G$. It follows then that

$$A(\alpha, a) = \langle L(a^{-1} \cdot a\alpha), I, L(a^{-1} \cdot a\alpha)\rangle = \langle L(a\alpha)L(a)^{-1}, I, L(a\alpha)L(a)^{-1}\rangle$$

is an autotopism of $(G, \cdot)$ for all $a \in G$ and all $\alpha \in S(G)$. Hence, if $\alpha, \beta \in S(G)$, we see that

$$A(\beta, a\alpha)A(\alpha, a) = \langle L(a\alpha\beta)L(a)^{-1}, I, L(a\alpha\beta)L(a)^{-1}\rangle$$

is an autotopism of $(G, \cdot)$ for all $a \in G$. Hence, we have

$$1L(a\alpha\beta)L(a)^{-1} \cdot yI = (1 \cdot y)L(a\alpha\beta)L(a)^{-1}$$

for all $y \in G$ and so $L(a\alpha\beta)L(a)^{-1} = L(a^{-1} \cdot a\alpha\beta)$. Hence, the autotopism $A(\beta, a\alpha)A(\alpha, a)$ exhibited above can now be rewritten as

$$A(\beta, a\alpha)A(\alpha, a) = \langle L(a^{-1} \cdot a\alpha\beta), I, L(a^{-1} \cdot a\alpha\beta)\rangle.$$

Thus, $a^{-1} \cdot a\alpha\beta$ is in the left nucleus of $(G, \cdot)$ for all $a \in G$. But then, by Theorem 1 (ii), we see that $a^{-1} \cdot a\alpha\beta \in N$ for all $a \in G$ where N is the nucleus of $(G, \cdot)$. Therefore, $\alpha\beta$ is in $S(G)$. If $\alpha \in S(G)$, $A(\alpha, a)$ is an autotopism of $(G, \cdot)$ for all $a \in G$ and so $A(\alpha, (a\alpha^{-1}))^{-1}$ is an autotopism of $(G, \cdot)$ for all $a \in G$. But

$$A(\alpha, (a\alpha^{-1}))^{-1} = \langle L(a^{-1} \cdot a\alpha^{-1}), I, L(a^{-1} \cdot a\alpha^{-1})\rangle.$$

Hence, if $\alpha \in S(G)$, it follows that $\alpha^{-1} \in S(G)$. Thus, $S(G)$ is a subgroup of the automorphism group of $(G, \cdot)$.

Let $\alpha \in S(G)$. Then $a^{-1} \cdot a\alpha$ is in the left nucleus of $(G, \cdot)$ for all $a \in G$ and so $(a^{-1} \cdot a\alpha)(xy) = (a^{-1} \cdot a\alpha)x \cdot y$ for all $a, x, y \in G$. If γ is an automorphism of $(G, \cdot)$, then we have

$$[(a\gamma)^{-1} \cdot a\alpha\gamma](x\gamma \cdot y\gamma) = [(a\gamma)^{-1} \cdot a\alpha\gamma]x\gamma \cdot y\gamma$$

for all $a, x, y \in G$ and, replacing a by $a\gamma^{-1}$, we are able to conclude that $a^{-1} \cdot a\gamma^{-1}\alpha\gamma$ is in the left nucleus of $(G, \cdot)$. Again, in view of Theorem 1 (ii), we see that $a^{-1} \cdot \cdot a\gamma^{-1}\alpha\gamma \in N$ for all $a \in G$ and all automorphisms γ of $(G, \cdot)$. Hence, $\gamma^{-1}\alpha\gamma \in S(G)$ for all $\alpha \in S(G)$ and all automorphisms γ of $(G, \cdot)$. So $S(G)$ is, indeed, normal in the automorphism group of $(G, \cdot)$.

Fenyves [3] indicates that all of the inner mappings $L(x, y) = L(x)L(y)L(yx)^{-1}$ of an extra loop $(G, \cdot)$ are automorphisms of $(G, \cdot)$. Since, for any Moufang loop, $R(x, y) = L(x^{-1}, y^{-1})$ (see Bruck [2], Ch. VII, Lemma 2. 2), the inner mappings $L(x, y)$ have automatically been included in our foregoing discussion. The inner mappings $T(x) = R(x)L(x)^{-1}$ for an extra loop are generally not automorphisms of $(G, \cdot)$. In fact, it is not difficult to show that an extra loop $(G, \cdot)$ is a group if and only if the mappings $T(x)$ are automorphisms of $(G, \cdot)$ for all $x \in G$. However, if N is the nucleus of an extra loop $(G, \cdot)$, one can prove (by adopting the techniques used in the proof of Theorem 4) that $z^{-1} \cdot zT(x) \in N$ for all $x, z \in G$.

§ 5. An example

We now show how one can modify an earlier construction of D. A. Robinson [4] to obtain examples of extra loops.

Let R be an alternative division ring which is not associative but which is of characteristic 2. Let G be the Cartesian product $G = R \times R \times R \times R$. If $x = (a_1, b_1, c_1, d_1)$

and $y=(a_2, b_2, c_2, d_2)$ are elements of G, define

$$xy = (a_1+a_2, b_1+b_2+a_1a_2, c_1+c_2+a_1a_2, d_1+d_2+a_1c_2+b_1a_2) \tag{7}$$

and

$$x-y = (a_1-a_2, b_1-b_2, c_1-c_2, d_1-d_2). \tag{8}$$

Clearly $(G, \cdot)$ is a loop whose identity is $(0, 0, 0, 0)$. If $x=(a_1, b_1, c_1, d_1)$, $y=(a_2, b_2, c_2, d_2)$ and $z=(a_3, b_3, c_3, d_3)$ are in G, then by a direct but somewhat tedious computation we see that

$$(xy\cdot z)x-x(y\cdot zx) = (0, 0, 0, [a_1, a_2, a_3]+[a_1, a_2, a_1]+[a_1, a_3, a_1]+[a_2, a_3, a_1]) \tag{9}$$

and

$$xy\cdot z-x\cdot yz = (0, 0, 0, [a_1, a_2, a_3]) \tag{10}$$

where $[r, s, t]$ denotes the ring associator $[r, s, t] = rs\cdot t-r\cdot st$ for all $r, s, t\in R$. But it is well-known (see R. D. SCHAFER [5], pp. 27, 28) that the associator $[r, s, t]$ for an alternative ring is skew-symmetric in r, s, t and is 0 whenever any two of r, s, t are equal. This, together with the fact that R has characteristic 2, shows that

$$[a_1, a_2, a_3]+[a_1, a_2, a_1]+[a_1, a_3, a_1]+[a_2, a_3, a_1] = 2[a_1, a_2, a_3] = 0$$

for all $a_1, a_2, a_3\in R$. Hence, from (9), we now conclude that $(xy\cdot z)x = x(y\cdot zx)$ for all $x, y, z\in G$ and the loop $(G, \cdot)$ is an extra loop. Since R is not associative, it is clear from (10) that $(G, \cdot)$ is not associative. Thus, in view of § 4, this extra loop has non-trivial nuclear automorphisms and, consequently, has non-trivial holomorphs which are also extra loops.

References

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