A note on compact objects

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1. $\mathcal{M}$-compact objects were introduced in [5] and investigated in [6]. In those papers it turned out that $\mathcal{M}$-compactness plays an important rôle in the characterization of $\mathcal{M}$-semi-simple objects, and $\mathcal{M}$-compact objects are closely related to $\mathcal{M}$-representable ideals introduced by SULIŃSKI [4].

The purpose of this brief note is to call the attention to the fact that $\mathcal{M}$-compactness means essentially a kind of the dual condition of Grothendieck's famous axiom AB5. This axiom is the following.

AB5. *The direct limit* $\varinjlim \{A_i\}_{i\in I}$ *of every direct family of subobjects* A_i, $i\in I$, *of an object* A *is the union* $\bigcup_{i\in I} A_i$.

In the algebra axiom AB5 is always fulfilled, but its dual condition has a very strong (topological) meaning.

Let us remark that originally GROTHENDIECK considered only complete abelian categories, and for such categories this formulation of AB5 is an equivalent version of the original one (cf. [1], Prop. 1. 8 or [3] III. Prop. 1. 2). Here we do not demand being the category complete abelian, but of course, it is supposed that the category considered admits the terms, for otherwise they make no sense.

For the definitions of the familiar category theoretical concepts we refer to KUROSCH—LIWSCHITZ—SCHULGEIFER—ZALENKO [2] and MITCHELL [3], and we shall adopt the notations of [3].

2. Concerning AB5 we present the following simple

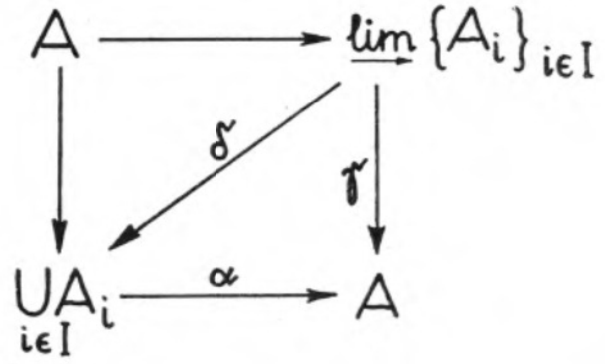
Proposition. *Suppose the category* $\mathcal{C}$ *has unions and direct limits, and consider an object* $A\in\mathcal{C}$ *and a direct family* $\{A_i\}_{i\in I}$ *consisting of subobject of* A. *The following conditions are equivalent:*

(C_1) $\varinjlim \{A_i\}_{i\in I}=\bigcup_{i\in I} A_i$;

(C_2) *The canonical map* $\gamma\colon \varinjlim \{A_i\}_{i\in I}\to A$ *is a monomorphism.*

PROOF. $(C_1)\Rightarrow(C_2)$. Since $\varinjlim \{A_i\}_{i\in I} = \bigcup_{i\in I} A_i$ is a subobject of A, so the canonical map γ to be a monomorphism.

$(C_2) \Rightarrow (C_1)$. Consider the commutative diagram where γ and δ are the canonical



maps. According to the assumption γ is a monomorphism. So by the definition of $\bigcup\limits_{i \in I} A_i$ there exists a monomorphism $\varphi: \bigcup\limits_{i \in I} A_i \to \varinjlim \{A_i\}_{i \in I}$ such that $\gamma\varphi = \alpha$ and the diagram remains commutative. In the view of the diagram we have $\alpha = \gamma\varphi = \alpha\delta\varphi$ and so $\delta\varphi$ is the identity map of $\bigcup\limits_{i \in I} A_i$. On the other hand $\gamma = \alpha\delta = \gamma\varphi\delta$ holds, and so $\varphi\delta$ is the identity map of $\varinjlim \{A_i\}_{i \in I}$. Thus δ is an isomorphism.

3. Let $\mathscr{C}$ be a category satisfying the following additional requirements:

$\mathscr{C}$ has zero objects;

the subobjects of any object of $\mathscr{C}$ form a complete lattice

the quotient objects of any object A form a complete lattice L_A, and the set of all normal quotient objects*) of A is a complete sublattice of L_A;

$\mathscr{C}$ has inverse limits.

Now we recall the definition of $\mathscr{M}$-compactness. In contrary to [5], defining $\mathscr{M}$-compactness, we shall use normal quotient objects instead of kernels.

Consider a class $\mathscr{M}$ of objects of $\mathscr{C}$ having the following properties:

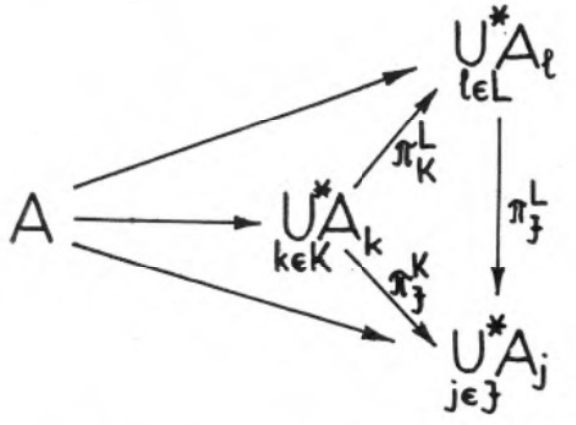
i) If $A \in \mathscr{M}$ and $A \approx B$, then $B \in \mathscr{M}$;

ii) If $A, B \in \mathscr{M}$ and $\alpha: A \to B$ is a normal epimorphism, then α is either a zero map or an isomorphism.

To any object A there belongs an inverse family Ω_A determined by $\mathscr{M}$ as follows. Let M_A denote the set of all normal quotient objects A_i, $i \in I$, of A with $A_i \in \mathscr{M}$. Form all finite counions $\bigcup\limits_{\text{finite}}^{*} A_i$ from the elements of M_A. If $J \subseteq K \subseteq L$ are finite subsets of the index set I, then there exist normal epimorphisms $\pi_J^K: \bigcup\limits_{k \in K}^{*} A_k \to \bigcup\limits_{j \in J}^{*} A_j$,

*) If $\alpha: A \to B$ is the cokernel of some map, then α and B will be called a normal epimorphism and a normal quotient object, respectively.

$\pi_K^L: \bigcup^*_{l \in L} A_l \to \bigcup^*_{k \in K} A_k$ and $\pi_J^L: \bigcup^*_{l \in L} A_l \to \bigcup^*_{j \in J} A_j$ such that the diagram is commutative.



Hence

$$\Omega_A = \{\bigcup^*_{j \in J} A_j \mid J \text{ runs over the finite subsets of } I\}$$

forms an inverse family of normal quotient objects of A. In such a way to A there belong an inverse limit $\varprojlim \Omega_A$ and a canonical map $\gamma: A \to \varprojlim \Omega_A$ (if M_A and so also Ω_A is empty, then $\varprojlim \Omega_A$ shall mean a zero object). *The object A is called $\mathcal{M}$-compact, if the canonical map γ is a normal epimorphism.*

Theoerm. *An object A is $\mathcal{M}$-compact if and only if*

$$\bigcup^*_{A_i \in M_A} A_i = \varprojlim \Omega_A.$$

The Theorem is a somewhat modified version of the dual statement of the Proposition, and so its proof is straightforward.

Consider the product $\bigtimes_{i \in I} A_i$ of a family of objects. Since the category has zero objects, so the objects A_i, $i \in I$, can be regarded as subobjects of $\bigtimes_{i \in I} A_i$ by the injections. The subobject $\bigcup_{i \in I} A_i$ of $\bigtimes_{i \in I} A_i$ is called the *discrete direct product* of the objects A_i, $i \in I$ and it will be denoted by $\underset{i \in I}{D} A_i$ (cf. [2]). An immediate consequence of the Theorem is

Corollary. *The discrete direct product $\underset{i \in I}{D} A_i$ of objects from $\mathcal{M}$ is $\mathcal{M}$-compact if and only if $\underset{i \in I}{D} A_i = \bigtimes_{i \in I} A_i$.*

At last, to illustrate the Corollary, we shall combine it with Theorem 5, 6 of [5]. Let us recall, that an object A is called *$\mathcal{M}$-semi-simple* if $\bigcup^*_{A_i \in M_A} A_i = A$. In a category, satisfying the requirements of [5], *the following conditions are equivalent:*

1) *every object is $\mathcal{M}$-compact*

2) $\underset{A_i \in \mathcal{M}}{D} A_i = \underset{A_i \in \mathcal{M}}{\times} A_i$

3) $A = \underset{A_i \in \mathcal{M}}{\times} A_i$ *for every* $\mathcal{M}$*-semisimple object* A.

The equivalence of 1) and 3) is just the statement of Theorem 5. 6 in [5], while the Corollary implies the further statements.

References

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