On various Boolean structures in a given Boolean algebra

By ABRAHAM GOETZ (Notre Dame, Indiana)

HANNA NEUMANN proved in [1] the conjecture of A. KERTÉSZ concerning various group operations in a given group. In this note we solve a similar problem for Boolean algebra.

1. Consider a Boolean algebra $\mathfrak{B}=(B, \vee, \wedge, ')$ with unity element I and zero element 0, join $x \vee y$, meet $x \wedge y$, and complement x'. Let a be an arbitrary element of B. Define two binary operations $x \cup y$ and $x \cap y$ by formulas

$$x \cup y = [a \wedge (x \vee y)] \vee [a' \wedge x \wedge y] \tag{1}$$

and

$$x \cap y = (x' \cup y')' = [a' \vee (x \wedge y)] \wedge [a \vee x \vee y]. \tag{2}$$

Theorem 1. *The algebra $\mathfrak{B}^a=(B, \cup, \cap, ')$ is a Boolean algebra with unity element a, zero a', join $x \cup y$, meet $x \cap y$ and complement x'. The mapping $\Phi: B \to B$ defined by formula*

$$\Phi(x) = (a \wedge x) \vee (a' \wedge x') = (I \cap x) \cup (0 \cap x') \tag{3}$$

is an involutory isomorphism $\Phi: \mathfrak{B} \to \mathfrak{B}^a$.

PROOF. We shall prove first that Φ is an involutory mapping, and therefore one-to-one and onto. Indeed, we have

$$\Phi[\Phi(x)] = [a \wedge \Phi(x)] \vee [a' \wedge \Phi(x)'] =$$
$$= (a \wedge [(a \wedge x) \vee (a' \wedge x')] \vee (a' \wedge [(a \wedge x) \vee (a' \wedge x')] =$$
$$= (a \wedge x) \vee [a' \wedge (a' \vee x) \wedge (a \vee x)] = (a \wedge x) \vee (a' \wedge x) = x.$$

Then we prove that

$$\Phi(x)' = \Phi(x') \tag{4}$$

and

$$\Phi(x) \cup \Phi(y) = \Phi(x \vee y). \tag{5}$$

Indeed

$$\Phi(x)' = [(a \wedge x) \vee (a' \wedge x')]' = (a' \vee x') \wedge (a \vee x) = (a \wedge x') \vee (a' \wedge x) = \Phi(x').$$

Further

$$\Phi(x)\cup\Phi(y) = \big(a\wedge[\Phi(x)\vee\Phi(y)]\big)\vee[a'\wedge\Phi(x)\wedge\Phi(y)] =$$
$$= \big(a\wedge[(a\wedge x)\vee(a'\wedge x')\vee(a\wedge y)\wedge(a'\wedge y')]\big)\vee$$
$$\vee\big(a'\wedge[(a\wedge x)\vee(a'\wedge x')]\wedge[(a\wedge y)\vee(a'\wedge y')]\big) =$$
$$= [a\wedge(x\vee y)]\vee(a'\wedge x'\wedge y') = \Phi(x\vee y).$$

Therefore, Φ is an isomorphism of the algebra $(B;\ \vee,\ ')$ onto $(B,\ \cup,\ ')$. This isomorphism sends the unit I into $(a\wedge I)\vee(a'\wedge I') = a$, and consequently $0\to a'$. Due to (2), we have also $\Phi(x\wedge y) = \Phi(x)\cap\Phi(y)$.

By a relatively simple computation one checks that $\Phi(x) = (I\cap x)\cup(0\cap x')$. Indeed,

$$I\cap x = [a'\vee x]\wedge[a\vee I\vee x] = a'\vee x,$$
$$0\cap x' = [a'\vee 0]\wedge(a\vee x') = a'\wedge x'.$$

Now

$$(I\cap x)\cup(0\cap x') = (a'\vee x)\cup(a'\wedge x') =$$
$$= \big(a\wedge[a'\vee x\vee(a'\wedge x')]\big)\vee[a'\wedge(a'\vee x)\wedge a'\wedge x'] = (a\wedge x)\vee(a'\wedge x') = \Phi(x).$$

Corollary. *The join and meet of the algebra $\mathfrak{B}$ can be expressed by the operations of the algebra $\mathfrak{B}^a$ by formulae*

$$x\vee y = [I\cap(a\cup y)]\cup[0\cap x\cap y] \tag{1'}$$
$$x\wedge y = (x'\vee y')' = [0\cup(a\cap y)]\cap[I\cup x\cup y] \tag{2'}$$

Proof. Applying theorem 1 with I taking the place of a to the algebra $\mathfrak{B}^a$ we obtain another algebra with join and meet

$$x\cup y = [I\cap(x\cup y)]\cup[0\cap x\cap y],$$
$$x\cap y = [I\cup x\cup y]\cap[0\cup(x\cap y)],$$

which is an isomorphic image of the algebra $\mathfrak{B}^a$ under the isomorphism $x\to(a\cap x)\cup\cup(a'\cup x')$. But this isomorphism is nothing else than $\Phi=\Phi^{-1}$. Therefore, the isomorphic image of $\mathfrak{B}^a$ coincides with $\mathfrak{B}$ and $x\cup y=x\vee y$, $x\cap y=x\wedge y$, and so formulas (1′), (2′) hold.

Theorem 2. *The binary operations $\vee$, $\wedge$, $\cup$, $\cap$ are distributive with respect to each other.*

The proof is a matter of simple computations.

2. The question arises whether or not the Boolean algebras $\mathfrak{B}^a$ are the only Boolean algebras on the set B whose operations can be expressed in terms of the operations of $\mathfrak{B}$ and constants. We shall give an affirmative answer to this question.

We start with the following lemmas.

Lemma 1. *Any unary operation $\bar{x}$ in B which can be expressed in terms of the original operations of $\mathfrak{B}$ and constants, and which satisfies the identity $\bar{\bar{x}}=x$ must be of the form*

$$\bar{x} = (v'\wedge x)\vee(v\wedge x'), \tag{6}$$

where $v\in B$ is some constant.

PROOF. The most general word in one variable with constants is

$$\bar{x} = (u \wedge x) \vee (v \wedge x') \vee c$$

(u, v, c — constants). The condition $\bar{\bar{x}}=y$ yields

$$x = \bar{\bar{x}} = \big(u \wedge [(u \wedge x) \vee (v \vee x') \vee c]\big) \vee [v \wedge (u' \vee x') \wedge (v' \vee x) \wedge c'] \vee c =$$
$$= (u \wedge x) \vee (u \wedge v \wedge x') \vee (u \wedge c) \vee (v \wedge u' \wedge x \wedge c') \vee c =$$
$$= \big([u \vee (v \wedge c')] \wedge x\big) \vee (u \wedge v \wedge x') \vee c.$$

Setting $x=0$ we obtain $0 = (u \wedge v) \vee c$, which implies $c=0$ and $u \wedge v = 0$. Setting $x=I$, $c=0$, we obtain $u \vee v = I$. Consequently $u=v'$ which proves (6).

Lemma 2. *Any binary operation $x \cup y$ in B, which can be expressed in terms of x, y, the operations of $\mathfrak{B}$ and constants, and which satisfies the conditions*

(7) $$0 \cup 0 = 0, \quad 0 \cup I = I, \quad I \cup 0 = I \quad \text{and} \quad I \cup I = I,$$

coincides with the operation $x \vee y$.

PROOF. The most general binary operation which can be so expressed is

$$x \cup y = a \vee (b \wedge x) \vee (c \wedge y) \vee (d \wedge x') \vee (e \wedge y') \vee (f \wedge x \wedge y) \vee$$
$$\vee (g \wedge x' \wedge y) \vee (h \wedge x \wedge y') \vee (j \wedge x' \wedge y')$$

with constants a, b, c, d, e, f, g, h, j.

The condition $0 \cup 0 = 0$ implies that

$$a=d=e=j=0,$$

whence

$$x \cup y = (b \wedge x) \vee (x \wedge y) \vee (f \wedge x \wedge y) \vee (g \wedge x \wedge y') \vee (h \wedge x' \wedge y).$$

One can easily check by computations that

$$(b \wedge x) \vee (c \wedge y) \vee (f \wedge x \wedge y) = (b \wedge x) \vee (c \wedge y) \vee (f \wedge b' \wedge c' \wedge x \wedge y),$$
$$(b \wedge x) \vee (g \wedge x \wedge y') = (b \wedge x) \vee (g \wedge b' \wedge x \wedge y'),$$
$$(c \wedge x) \vee (h \wedge x' \wedge y) = (c \wedge x) \vee (h \wedge c' \wedge x' \wedge y).$$

Therefore the constant f can be replaced by $f \wedge b' \wedge c'$ disjoint with $b \vee c$, g can be replaced by $g \wedge b'$ disjoint with b, and h can be replaced by $h \wedge c'$ disjoint with c. Thus we can assume that

(8) $$f \wedge (b \vee c) = 0, \quad g \wedge b = 0, \quad h \wedge c = 0$$

without loss of generality.

The conditions $I \cup 0 = I$, $0 \cup I = I$ and $I \cup I = I$ imply

$$b \vee g = I, \quad c \vee h = I, \quad b \vee c \vee f = I$$

which together with (8) yields

$$g=b', \quad h=c', \quad f= b' \wedge c'.$$

Consequently we have

$$x \cup y = (b \wedge x) \vee (c \wedge y) \vee (b' \wedge c' \wedge x \wedge y) \vee (b' \wedge x \wedge y') \vee (c' \wedge x' \wedge y).$$

But

$$(b' \wedge c' \wedge x \wedge y) \vee (b' \wedge x \wedge y') = b' \wedge x \wedge [(c' \wedge y) \vee y'] =$$

$$= (b' \wedge x) \wedge (c' \vee y') = (b' \wedge x) \wedge (c \wedge y)',$$

and

$$(b' \wedge c' \wedge x \wedge y) \vee (c' \wedge x' \wedge y) = (c' \wedge y) \wedge (b \wedge x)'.$$

Further

$$(b \wedge x) \vee [(x' \wedge y) \wedge (b \wedge x)'] = (b \wedge x) \vee (c' \wedge y),$$

$$(c \wedge y) \vee [(b' \wedge x) \wedge (c \wedge y)'] = (c \wedge y) \vee (b' \wedge x).$$

Consequently,

$$x \cup y = (b \wedge x) \vee (c' \wedge y) \vee (c \wedge y) \vee (b' \wedge x) = x \vee y,$$

Theorem 3. *Any two Boolean structures on B, with coinciding zeros, and such that the Boolean operations of one of them can be expressed in terms of the Boolean operations of the other and constants, coincide.*

Proof. We can assume without loss of generality that one of the Boolean structures is the original algebra $\mathfrak{B}$. Let the other algebra be $\overline{\mathfrak{B}}=(B, \cup, \cap, ^-)$ with join $x \cup y$, meet $x \cap y$ and complement $\bar{x}$. Since 0 is assumed to be the zero of $\overline{\mathfrak{B}}$, the identities (7) will be satisfied, and therefore, by lemma 2,

$$x \cup y = x \vee y.$$

Now, by lemma 1,

$$\bar{x} = (v' \wedge x) \vee (v \wedge x').$$

Since 0 is the zero of the algebra $\overline{\mathfrak{B}}$, its unity will be $\bar{0} = (v' \wedge 0) \vee (v \wedge 0') = v$. Consequently v must satisfy the condition $v \cup x = v$ for every x in particular

$$v = v \cup v' = v \vee v' = I.$$

Thus

$$\bar{x} = (I' \wedge x) \vee (I \wedge x') = x'.$$

This proves the theorem.

Now we are ready to prove the main result of the paper:

Theorem 4. *Every Boolean algebra $\mathfrak{B}^*$ with the underlying set B whose operations can be expressed in terms of the operations of the Boolean algebra $\mathfrak{B}$ and constants coincide with one of the algebras $\mathfrak{B}^a$ of theorem 1.*

Proof. Consider the algebra $\mathfrak{B}^*$ and let a' be its zero element. By theorem 1, the algebra $\mathfrak{B}^a$ is then another Boolean algebra on B with the same zero element a'. Since the operations of $\mathfrak{B}$ can be expressed in terms of the operations of $\mathfrak{B}^a$ by the corollary to theorem 1, also the operations of $\mathfrak{B}^*$ are expressible in terms of the operations of $\mathfrak{B}^a$ and constants. By theorem 3 the two algebras $\mathfrak{B}^a$ and $\mathfrak{B}^*$ with coinciding zero element a' must coincide. This proves the theorem.

Remark. Theorem 3 is a strengthening of Theorem 1 of T. TRACZYK [2] in which the assertion of theorem 3 is obtained under the additional assumptions that the expressions for the operations do not involve constants other than I and 0, and that I is the unity of both algebras.

References

[1] HANNA NEUMANN, On a question of Kertész, *Publ. Math. Debrecen* **8** (1961), 75—78.
[2] T. TRACZYK, Weak isomorphisms of Boolean and Post algebras, *Colloquium Math.* **13** (1965), (1965), 159—164.

(Received July 9, 1969.)