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Theory of Finsler spaces with m-th root metric II

By MAKOTO MATSUMOTO (Kyoto)

Abstract. This is the second paper of a series concerned with Finsler spaces with m-th root metric. We consider mainly two- and three-dimensional Berwald spaces with cubic and quartic metrics.

Introduction

Recently we have several papers on Finsler spaces with m-th root metric [2], [5], [6], [7]. The theory of those spaces has been considerably developed by introducing the tensor field $a_{ij}(x, y)$ [5] and generalized Christoffel symbols [6].

In the early stage of the Finsler geometry, however, we have JOHANNES M. WEGENER's interesting paper [10] on Finsler spaces with cubic metric ($m = 3$) of dimension two and three. According to his paper [8], he submitted a thesis on Finsler spaces in March 1935 to the German University in Prague, the referee being Ludwig Berwald. His thesis consisted of three parts: (I) Two- and three-dimensional Finsler spaces, (II) Hypersurfaces as transversal surfaces of a family of extremals, and (III) Two- and three-dimensional Finsler spaces with cubic metric. His papers [9] and [10] are (II) and (III) of his thesis respectively. In 1986 the present author published the paper [4] which proposed an improvement of [9] based on the recent development of the notion of Finsler connections.

On the other hand, WEGENER's paper [10] is only an abstract of his (III) without almost all calculations. The present paper may be said as an improved version of [10] based on the results of a previous paper [6]. It must be reported that Wegener faild to find an interesting family of Berwald spaces of dimension three which is given in (I_2) of Proposition 4.

It is very sorry that J. M. WEGENER went out of the world of the Finsler geometry after submitted his thesis and published the three papers above. The author hopes to get intelligence about him.

§1. The Berwald connection

An n-dimensional Finsler space F^n with m-th root metric is by definition a Finsler structure $(M^n, L(x,y))$ on a differentiable n-manifold M^n equipped with the fundamental function $L(x,y)$ such that

$$L(x,y)^m = a_{i_1 \dots i_m}(x) y^{i_1} \cdots y^{i_m},$$

where $a_{i_1 \dots i_m}(x)$ are components of a symmetric covariant tensor field of order m. We suppose $m \geqq 3$ throughout the paper, because $m = 2$ gives merely a Riemannian metric.

From $L(x,y)$ we define Finslerian symmetric tensors of order r $(1 \leqq r \leqq m-1)$ with the components

$$a_{i_1 \dots i_r}(x,y) = \frac{1}{L^{m-r}} a_{i_1 \dots i_r j_1 \dots j_{m-r}}(x) y^{j_1} \cdots y^{j_{m-r}}.$$

Among these tensors we have three specially important tensors a_i, a_{ij} and a_{ijk}. In fact, the normalized supporting element $\ell_i = \dot\partial_i L$, the angular metric tensor $h_{ij} = L(\dot\partial_i \dot\partial_j L)$, the fundamental tensor g_{ij} and the C-tensor $C_{ijk} = (\dot\partial_k g_{ij})/2$ are written as

$$\tag{1.1} \begin{cases} \ell_i = a_i, \quad h_{ij} = (m-1)(a_{ij} - a_i a_j), \\ g_{ij} = (m-1)a_{ij} - (m-2)a_i a_j, \\ C_{ijk} = \dfrac{(m-1)(m-2)}{2L}(a_{ijk} - a_{ij}a_k - a_{jk}a_i - a_{ki}a_j + 2a_i a_j a_k). \end{cases}$$

Since $\det(g_{ij}) = (m-1)^{n-1} \det(a_{ij})$ as easily shown ([3], Proposition 30.1), the *regularity* of the m-th root metric is equivalent to $\det(a_{ij}) \neq 0$ ([5], [7]). Suppose, of course, the regularity throughout the paper. Then we have $(a^{ij}) = (a_{ij})^{-1}$ and

$$\ell^i = a^i \ (= a^{ir} a_r), \quad g^{ij} = \frac{1}{m-1}\{a^{ij} + (m-2)a^i a^j\}.$$

Next we define the *m-th Christoffel symbols* [6]

$$\tag{1.2} \begin{aligned} \{i_1 \dots i_m, j\} = \frac{1}{2(m-1)} (&\partial_{i_1} a_{i_2 \dots i_m j} + \partial_{i_2} a_{i_3 \dots i_m i_1 j} \\ &+ \cdots + \partial_{i_m} a_{i_1 \dots i_{m-1} j} - \partial_j a_{i_1 \dots i_m}), \end{aligned}$$

where the cyclic permutation is applied to $(i_1 \dots i_m)$ in the first m terms of the right-hand side. If we write the equations of geodesics in the usual form

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0,$$

then the quantities $G^i(x,y)$ are given ([6], (3.3)) by

$$a_{hr}G^r(x,y) = \frac{1}{mL^{m-2}}\{0\dots0,h\}, \tag{1.3}$$

where we denote by the index 0 the transvection by y^i as usual, that is, $\{0\dots0,h\} = \{i_1\dots i_m,h\}y^{i_1}\dots y^{i_m}$.

On account of the definition of a_{hr} we may write (1.3) in the form

$$a_{hr0\dots0}G^r = \frac{1}{m}\{0\dots0,h\}.$$

Differentiating this by y^i and then by y^j, we have

$$a_{hr0\dots0}G^r_i + (m-2)a_{hir0\dots0}G^r = \{i0\dots0,h\},$$

$$\begin{aligned} &a_{hr0\dots0}G_i{}^r{}_j + (m-2)(a_{hir0\dots0}G^r_j + a_{hjr0\dots0}G^r_i) \\ &\quad + (m-2)(m-3)a_{hijr0\dots0}G^r = (m-1)\{ij0\dots0,h\}, \end{aligned}$$

where $G^r_i = \dot\partial_i G^r$ and $G_i{}^r{}_j = \dot\partial_j G^r_i$ constitute the coefficients of the Berwald connection $B\Gamma = (G_i{}^r{}_j, G^r_i)$. These equations above may be written in the plainer form

$$L^{m-3}\{La_{hr}G^r_i + (m-2)a_{hir}G^r\} = \{i0\dots0,h\}, \tag{1.4}$$

$$\begin{aligned} &L^{m-4}\{L^2a_{hr}G_i{}^r{}_j + (m-2)L(a_{hir}G^r_j + a_{hjr}G^r_i) \\ &\quad + (m-2)(m-3)a_{hijr}G^r\} = (m-1)\{ij0\dots0,h\}. \end{aligned} \tag{1.5}$$

Further differentiation by y^k gives the hv-curvature tensor $G_i{}^h{}_{jk} = \dot\partial_k G_i{}^h{}_j$ of $B\Gamma$ as follows:

$$\begin{aligned} &L^{m-5}\Big[L^3a_{hr}G_i{}^r{}_{jk} + (m-2)L^2\{a_{hir}G_j{}^r{}_k + (i,j,k)\} + (m-2)(m-3) \\ &\quad \times L\{a_{hijr}G^r_k + (i,j,k)\} + (m-2)(m-3)(m-4)a_{hijkr}G^r\Big] \\ &\qquad = (m-1)(m-2)\{ijk0\dots0,h\}, \end{aligned} \tag{1.6}$$

where $\{\dots + (i,j,k)\}$ shows the cyclic permutation of the indices i, j, k and summation. Transvecting (1.6) by y^h we obtain

$$\begin{aligned} &L^{m-4}\Big[L^2y_rG_i{}^r{}_{jk} + (m-2)L^2\{a_{ir}G_j{}^r{}_k + (i,j,k)\} + (m-2)(m-3) \\ &\quad \times L\{a_{ijr}G^r_k + (i,j,k)\} + (m-2)(m-3)(m-4)a_{ijkr}G^r\Big] \\ &\qquad = (m-1)(m-2)\{ijk0\dots0,0\}. \end{aligned} \tag{1.7}$$

Remark. In the equations (1.5), (1.6) and (1.7) we have some terms with coefficients $(m-3)$ and $(m-4)$. We shall be concerned mainly with cubic $(m=3)$ and quartic $(m=4)$ metrics

$$L^3 = a_{ijk}(x)y^i y^j y^k, \qquad L^4 = a_{hijk}(x)y^h y^i y^j y^k,$$

in the following. For these metrics it is supposed that the terms with $(m-3)$ and $(m-4)$ vanish respectively. For instance, (1.6) of a cubic metric is reduced to

$$La_{hr}G_i{}^r{}_{jk} + \{a_{hir}G_j{}^r{}_k + (i,j,k)\} = \{ijk,h\}.$$

§2. Landsberg spaces and Berwald spaces

We have two important families of special Finsler spaces. If $G_j{}^i{}_k$ are functions of position x alone, then the space is called a Berwald space ([1], [3]). As a consequence the space is a Berwald space, if and only if $G^i(x,y)$ are of quadratic forms $2G^i = G_j{}^i{}_k(x)y^j y^k$. Since G^i of a Finsler space F^n with m-th root metric are given by (1.3), we have

Theorem 1. *F^n with m-th root metric is a Berwald space, if and only if the homogeneous polynomial*

$$\{a_{hri_1\dots i_{m-2}}(x)y^{i_1}\dots y^{i_{m-2}}\}G_i{}^r{}_j(x)y^i y^j = \frac{2}{m}\{i_1\dots i_m,h\}y^{i_1}\dots y^{i_m}$$

in y^i is satisfied.

The condition for a Berwald space is obviously $G_i{}^h{}_{jk} = 0$. Next, if $G_i{}^h{}_{jk}$ satisfies $y_h G_i{}^h{}_{jk} = 0$, then the space is called a Lansdsberg space ([1], [3]). Therefore (1.6) and (1.7) lead to

Theorem 2. *F^n with m-th root metric is a Berwald space, if and only if we have*

$$\text{(Bm)} \qquad (m-1)\{ijk0\dots0,h\} = L^{m-5}\Big[L^2\{a_{hir}G_j{}^r{}_k + (i,j,k)\}$$
$$+(m-3)L\{a_{hijr}G^r_k + (i,j,k)\} + (m-3)(m-4)a_{hijkr}G^r\Big].$$

F^n is a Landsberg space, if and only if we have

$$\text{(Lm)} \qquad (m-1)\{ijk0\dots0,0\} = L^{m-4}\Big[L^2\{a_{ir}G_j{}^r{}_k + (i,j,k)\}$$
$$+(m-3)L\{a_{ijr}G^r_k + (i,j,k)\} + (m-3)(m-4)a_{ijkr}G^r\Big].$$

On the other hand, it is well-known ([1], [3]) that in the Cartan connection $C\Gamma = (\Gamma^{*}{}_{j}{}^{i}{}_{k},\ G^{i}_{j},\ C_{j}{}^{i}{}_{k})$ F^n is a Landsberg space and a Berwald space, if and only if $C_{hij|0} = 0$ and $C_{hij|k} = 0$ respectively. Since $C\Gamma$ satisfies $a_{i|j} = 0$ and $a_{ij|k} = 0$ [7], the third equation of (1.1) leads to Theorem of [7] as follows:

Proposition 1. *F^n with m-th root metric is a Landsberg space and a Berwald space, if and only if $a_{hij|0} = 0$ and $a_{hij|k} = 0$ respectively in the Cartan connection.*

The family of Berwald spaces is, of course, contained in the family of Landsberg spaces. We have, however, the interesting theorem on C-reducible Finsler spaces ([3], Theorem 30.4) as follows: *If a C-reducible Finsler space is a Landsberg space, then it is a Berwald space.* A Finsler space is called C-reducible, if the C-tensor is of the special form $C_{hij} = \{h_{hi}C_j + (h,i,j)\}/(n+1)$. We shall prove the following theorem which is similarly based on the special property of the C-tensor:

Theorem 3. *If a Finsler space with cubic metric is a Landsberg space, then it is a Berwald space.*

Proof. Suppose that F^n with cubic metric be a Lansberg space. Then we have $a_{hij|0} = 0$ from Proposition 1. By differentiating $a_{hij|0} = a_{hij|r}y^r = 0$ by y^k, we have

$$a_{hij|k} + a_{hij|r\cdot k}y^r = 0,$$

where $(\cdot)$ denotes the v-covariant differentiation in the Berwald connection $B\Gamma = (G_{j}{}^{i}{}_{k},\ G^{i}_{j})$, that is, $\dot\partial_k$. It is, however, well-known that $\Gamma^{*}{}_{j}{}^{i}{}_{k}$ of $C\Gamma$ coincides with $G_{j}{}^{i}{}_{k}$ of $B\Gamma$ for a Landsberg spece. Hence the equation above may be written as

$$a_{hij;k} + a_{hij;r\cdot k}y^r = 0, \tag{2.1}$$

in terms of the h-covariant differentiation $(;)$ in $B\Gamma$. We pay attention to one of the Ricci identities of $B\Gamma$:

$$a_{hij;r\cdot k} - a_{hij\cdot k;r} = -a_{his}G_{j}{}^{s}{}_{rk} - (h,i,j).$$

Here it is remarked that a_{hij} of a cubic metric are nothing but the functions of x alone. Thus we have $a_{hij\cdot k} = 0$ and $a_{hij;r\cdot k}y^r = 0$ from the identity $G_{j}{}^{s}{}_{rk}y^r = 0$. Consequently we get $a_{hij;k} = 0$ from (2.1), which is equivalent to $a_{hij|k} = 0$, so that the space is reduced to a Berwald space.

Remark. J. M. Wegener [10] has proved Theorem 3 only in the two- and three-dimensional cases. It seems that his proof is complicated.

§3. Cubic and quartic metrics of dimension two

In the paper [6] we dealt with the main scalars of two-dimensional Finsler spaces with cubic metrics and quartic metrics. The purpose of the present section is to give characteristic equations of such metrics in terms of the main scalar.

Let us recall the Berwald frame (ℓ, m) of a two-dimensional Finsler space F^2. $\ell = (\ell^i)$ is given by $\ell^i = y^i/L$ and (ℓ_i) is given by $\ell_i = \dot{\partial}_i L$. $m = (m_i)$ is found from the angular metric tensor h_{ij} by $h_{ij} = \varepsilon m_i m_j$ with the signature $\varepsilon = \pm 1$. Thus we get $g_{ij} = \ell_i \ell_j + \varepsilon m_i m_j$.

Putting $F = L^2/2$ and $\dot{\partial}_{i_1} \dots \dot{\partial}_{i_r} F = F_{i_1 \dots i_r}$, we have

$$\tag{3.1} \begin{cases} F_h = L\ell_h, \quad F_{hi} = g_{hi} = \ell_h \ell_i + \varepsilon m_h m_i, \\ F_{hij} = 2C_{hij} = \frac{2}{L} I m_h m_i m_j, \end{cases}$$

where I is the main scalar. We have the well-known differentiation formulae

$$L\dot{\partial}_j \ell_i = \varepsilon m_i m_j, \quad L\dot{\partial}_j m_i = (-\ell_i + \varepsilon I m_i) m_j,$$

and the notation $L\dot{\partial}_j S = S_{;2} m_j$ for a $(0)p$-homogeneous scalar field S [3]. Then long but straightforward calculations lead to

$$\tag{3.2} \begin{cases} L^2 F_{hijk} = 2(I_{;2} + 3\varepsilon I^2) m_h m_i m_j m_k - 2I\{\ell_h m_i m_j m_k + (4)\}, \\ L^3 F_{hijk\ell} = 2(I_{;2;2} + 10\varepsilon I I_{;2} + 12 I^3 - 4\varepsilon I) m_h m_i m_j m_k m_\ell \\ \qquad - 4(I_{;2} + 3\varepsilon I^2)\{\ell_h m_i m_j m_k m_\ell + (5)\} \\ \qquad + 4I\{\ell_h \ell_i m_j m_k m_\ell + (10)\}, \end{cases}$$

where by the abbreviation $\{\cdots + (\cdot\cdot)\}$ we denote the cyclic permutation of indices and summation such that $\{\cdots + (\cdot\cdot)\}$ becomes completely symmetric in all the indices. For instance, provided that a_{ij} and b_{ijk} be symmetric quantities,

$$\{a_{hi} b_{jk\ell} + (10)\} = a_{hi} b_{jk\ell} + a_{hj} b_{ik\ell} + \cdots + a_{k\ell} b_{hij},$$

consisting of ten terms.

Now a Finsler metric $L(x, y)$ is a cubic metric, if and only if $\dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k (F^{3/2}) = 0$, which is written as

$$\tag{3.3} \begin{aligned} &8F^3 F_{hijk} + 4F^2\{F_h F_{ijk} + (4)\} + 4F^2\{F_{hi} F_{jk} + (3)\} \\ &\quad -2F\{F_{hi} F_j F_k + (6)\} + 3F_h F_i F_j F_k = 0. \end{aligned}$$

Next $L(x,y)$ is a quartic metric, if and only if $\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k\dot\partial_\ell(F^2)=0$, which is written as

$$FF_{hijk\ell}+\{F_hF_{ijk\ell}+(5)\}+\{F_{hi}F_{jk\ell}+(10)\}=0. \tag{3.4}$$

These equations (3.3) and (3.4) can be written in terms of I and its derivatives by y^i in virtue of (3.1) and (3.2). Therefore we have

Theorem 4. (1) *A two-dimensional Finsler space F^2 is with cubic metric, if and only if the main scalar I satisfies*

$$2I_{;2}+6\varepsilon I^2+3=0.$$

(2) *F^2 is with quartic metric, if and only if I satisfies*

$$I_{;2;2}+10\varepsilon II_{;2}+4I(3I^2+4\varepsilon)=0.$$

Remark. In the paper [5] the former equation of Theorem 4 was written in the form without ε, because we were concerned with positive-definite Finsler metrics alone. The similar remark is also necessary to the following Berwald's theorem and so on. Cf. [1], 3.5.

Next the h-scalar curvature or the Gauss curvature $R(x,y)$ of a two-dimensional Finsler space is defined by the h-curvature tensor $R_h{}^i{}_{jk}$ of $C\Gamma$ as follows:

$$R_h{}^j{}_{jk}=\varepsilon R(\ell_hm^i-\ell^im_h)(\ell_jm_k-\ell_km_j),$$

or the (v)h-torsion tensor $R^i{}_{jk}=y^hR_h{}^i{}_{jk}$ as follows:

$$R^i{}_{jk}(=\partial_kG^i_j-\partial_jG^i_k-G_j{}^i{}_rG^r_k+G_k{}^i{}_rG^r_j)=\varepsilon LRm^i(\ell_jm_k-\ell_km_j). \tag{3.5}$$

A Berwald space having $R=0$ is a locally Minkowski space ([1], [3]). We have the well-known *Berwald's theorem* ([1], 3.5; [3], §28): All Berwald spaces of dimension two are divided into three classes as follows:

$$\begin{aligned}&(1)\quad I=\text{const. and } R\neq 0,\\ &\left.\begin{aligned}&(2)\quad I=\text{const. and } R=0,\\ &(3)\quad I\neq\text{const. and } R=0.\end{aligned}\right\}\ldots\text{ locally Minkowski,}\end{aligned}$$

The fundamental function $L(x,y)$ of spaces belonging to (1) and (2) are written in the following four kinds:

$$\text{(i)}\quad \varepsilon=+1,\ I^2<4: L^2=(\beta^2+\gamma^2)\exp\left\{\frac{2I}{J}\tan^{-1}\left(\frac{\gamma}{\beta}\right)\right\},$$
$$J=\sqrt{4-I^2},$$

(ii) $\varepsilon = +1,\ I^2 = 4 : L^2 = \beta^2 \exp\left(I\dfrac{\gamma}{\beta}\right),$

(iii) $\varepsilon = +1,\ I^2 > 4 : L^2 = \beta^{1-I/J}\gamma^{1+I/J},\ J = \sqrt{I^2 - 4},$

(iv) $\varepsilon = -1 : L^2 = \beta^{1-I/J}\gamma^{1+I/J},\ J = \sqrt{I^2 + 4},$

where β and γ are 1-forms in (y^i).

From (1) of Theorem 4 it follows that $I =$ const. implies $\varepsilon = -1$ and $I^2 = 1/2$, so that (iv) leads to $L^3 = \beta\gamma^2$. Next (2) of Theorem 4 shows that $I =$ const. implies $I = 0$, or $\varepsilon = -1$, $I^2 = 4/3$ and $L^4 = \beta\gamma^3$ from (iv). Therefore we have

Theorem 5. (1) *All Berwald spaces of dimension two with cubic metric are divided into two classes as follows:*

(i) *locally Minkowski spaces,*

(ii) $\varepsilon = -1,\ I^2 = \dfrac{1}{2},\ L^3 = \beta\gamma^2.$

(2) *All Berwald spaces of dimension two with quartic metric are divided into three classes as follows:*

(i) *locally Minkowski spaces,*

(ii) *Riemannian spaces,*

(iii) $\varepsilon = -1,\ I^2 = \dfrac{4}{3},\ L^4 = \beta\gamma^3.$

In each case β and γ are 1-forms in (y^i).

A locally Minkowski space is by definition a Finsler space such that there exists a covering by local coordinate neighborhoods in each of which L does not depend on the point variables (x^i), so that all $G_j{}^i{}_k$ vanish. Such (x^i) is called adapted. For a locally Minkowski space with m-th root metric, in an adapted coordinate system (x^i), the equation $(L^m)_{;j} = a_{i_1 \dots i_m ; j} y^{i_1} \dots y^{i_m} = 0$ in $B\Gamma$ implies $\partial_j a_{i_1 \dots i_m} = 0$. Therefore we have

Proposition 2. *A Finsler space with m-th root metric is a locally Minkowski space, if and only if there exists a covering by local coordinate neighborhoods in each of which all coefficients $a_{i_1 \dots i_m}$ of L^m are reduced to constants.*

§4. Examples of dimension two

Let us recall some results of the paper [6]. For a two-dimensional cubic metric we used the notation: $(x^i) = (x, y)$, $(y^i) = (p, q)$ and $(a_{111}, a_{112}, a_{122}, a_{222}) = (c_0, c_1, c_2, c_3)$. Thus the metric is written in the form

$$L^3 = c_0p^3 + 3c_1p^2q + 3c_2pq^2 + c_3q^3.$$

We get a_i and a_{ij} as follows:

$$(4.1) \qquad \begin{cases} L^2(a_1, a_2) = (c_0p^2 + 2c_1pq + c_2q^2,\ c_1p^2 + 2c_2pq + c_3q^2), \\ L(a_{11}, a_{12}, a_{22}) = (c_0p + c_1q, c_1p + c_2q, c_2p + c_3q). \end{cases}$$

We introduced the quantities

$$(4.2) \qquad \begin{cases} H = H_{ij}y^iy^j, \\ (H_{11}, 2H_{12}, H_{22}) = (H_0, 2H_1, H_2) \\ \quad = (c_0c_2 - (c_1)^2, c_0c_3 - c_1c_2, c_1c_3 - (c_2)^2). \end{cases}$$

Then, since $h_{ij} = 2(a_{ij} - a_ia_j)$ from (1.1), (4.1) yields $L^4(h_{11}, h_{12}, h_{22}) = 2H(q^2, -pq, p^2)$. From $h_{ij} = \varepsilon m_im_j$ and $m_1p + m_2q = 0$ we get

$$(4.3) \qquad (m_i) = m(-q, p), \quad m^2 = \frac{2\varepsilon H}{L^4}.$$

Then, from $m^i\ell_i = 0$, $m^im_i = \varepsilon$ and $\ell_i = a_i$ we obtain

$$(4.3') \qquad (m^i) = \bar{m}(-a_2, a_1), \quad \bar{m} = \frac{\varepsilon}{mL}.$$

Further, to find the main scalar we introduced the quantities

$$(4.4) \qquad \begin{cases} G = G_{ijk}y^iy^jy^k, \\ (G_{111}, 3G_{112}, 3G_{122}, G_{222}) = (H_0c_1 - H_1c_0, \\ \ 2H_0c_2 - H_1c_1 - H_2c_0, H_0c_3 + H_1c_2 - 2H_2c_1, H_1c_3 - H_2c_2). \end{cases}$$

Then the main scalar I was given by

$$(4.5) \qquad I^2 = \varepsilon\frac{G^2}{2H^3}.$$

Now, to find G^i, G^i_j and $G_j{}^i{}_k$ we shall consider the second Christoffel symbols $\{hij, k\}$. In the two-dimensional case the symbols are divided into four types as follows:

$$(4.6)\qquad \begin{cases} (1) \quad 4\{iii,i\} = 2\partial_i a_{iii}, \\ (2) \quad 4\{iii,j\} = 3\partial_i a_{iij} - \partial_j a_{iii}, \\ (3) \quad 4\{iiij,i\} = \partial_i a_{iij} + \partial_j a_{iii}, \\ (4) \quad 4\{iij,j\} = 2\partial_i a_{ijj}, \end{cases} \qquad i,j = 1,2, \neq .$$

Example 1 ([10], (7)). We are first concerned with a typical cubic metric with $c_1 = c_2 = 0$:

$$L^3 = c_0(x,y)p^3 + c_3(x,y)q^3.$$

Then we have $H = c_0c_3pq$ and $G = -c_0c_3(c_0p^3 - c_3q^3)/2$. Consequently we get the main scalar:

$$I^2 = \varepsilon \frac{(c_0p^3 - c_3q^3)^2}{8c_0c_3(pq)^3}.$$

As a consequence I *can not be constant*, as indicated by Wegener.

Next, putting $\partial_j c_i = c_{ij}$, $i = 0,3$; $j = 1,2$, we have from (4.6)

$$\{111,1\} = \frac{1}{2}c_{01}, \quad \{112,1\} = -\{111,2\} = \frac{1}{4}c_{02},$$
$$\{221,2\} = -\{222,1\} = \frac{1}{4}c_{31}, \quad \{222,2\} = \frac{1}{2}c_{32},$$

and $\{112,2\} = \{221,1\} = 0$. Hence we have

$$\{000,1\} = \frac{1}{4}(2c_{01}p^3 + 3c_{02}p^2q - c_{31}q^3),$$
$$\{000,2\} = \frac{1}{4}(2c_{32}q^3 + 3c_{31}pq^2 - c_{02}p^3),$$

and (1.3) yields

$$2G^1 = \frac{1}{6c_0}\left(2c_{01}p^2 + 3c_{02}pq - c_{31}\frac{q^3}{p}\right),$$
$$2G^2 = \frac{1}{6c_3}\left(2c_{32}q^2 + 3c_{31}pq - c_{02}\frac{p^3}{q}\right).$$

Consequently it is obvious that the space is a *Berwald space*, if and only if the last terms of the above vanish, that is, $c_0 = c_0(x)$ and $c_3 = c_3(y)$. Therefore we have such a coordinate transformation $(x,y) \to (\bar{x},\bar{y})$ that we have $L^3 = \bar{p}^3 + \bar{q}^3$ and the space is reduced to a *locally Minkowski space*.

Next we shall find the condition for the space to be a *Landsberg space*; $a_{hij|0} = 0$ from Proposition 1. The equation $G^i_j = \dot{\partial}_j G^i$ yields immediately $G^1_2 = (c_{02}p - c_{31}q^2/p)4c_0$. From $y^h a_{hij|0} = y^h(LC_{hij|0}) = 0$ it follows that it is sufficient for us to observe

$$a_{112|0} = -a_{111}G^1_2 = \frac{1}{4p}(c_{31}q^2 - c_{02}p^2) = 0,$$

which implies $c_{31} = c_{02} = 0$ is necessary and sufficient for the space to be a Landsberg space. This coincides with the condition for a Berwald space. Cf. Theorem 3 and [10].

Remark. WEGENER [10] was concerned with the *stretch curvature tensor* of the above space. But it seems to the author that to find this tensor, introduced by Berwald in 1927, needs long and complicated calculations.

Example 2 ([10], (9)). We consider the quite special cubic metric with $c_0 = c_2 = c_3 = 0$:

$$L^3 = 3c(x,y)p^2q.$$

This is expressed as $L^3 = \beta\gamma^2$, $(\beta,\gamma) = (3cq, p)$. Thus Theorem 5 shows that $\varepsilon = -1$, $I^2 = 1/2$ and the space is a Berwald space.

We consider the condition for the space to be a locally Minkowski space, that is, to have the vanishing Gauss curvature R.

First, putting $\partial_i c = c_i$, $i = 1, 2$, we get

$$(G^1, G^2) = \left(\frac{c_1}{4c}p^2, \frac{c_2}{2c}q^2\right), \quad (G^1_1, G^2_2) = \left(\frac{c_1}{2c}p, \frac{c_2}{c}q\right),$$
$$G_1{}^1{}_1 = \frac{c_1}{2c}, \quad G_2{}^2{}_2 = \frac{c_2}{c}, \quad \text{other } G^i_j,\ G_j{}^i{}_k = 0.$$

Consequently we get

$$R^1{}_{12} = \frac{p}{2}\frac{\partial^2}{\partial x\partial y}\log|c|, \quad R^2{}_{12} = -q\frac{\partial^2}{\partial x\partial y}\log|c|.$$

On the other hand, we have $H = -(cp)^2$ and

$$(\ell_i) = (2cpq, cp^2)/L^2, \quad (m_i) = m(-q,p),\ m^2 = \frac{2c}{3qL},$$
$$(m^i) = \bar{m}(-cp^2, 2cpq)/L^2, \quad \bar{m} = \frac{-1}{mL}.$$

Thus (3.5) leads to

$$R = -\frac{3pq}{2L^2}\,\frac{\partial^2}{\partial x \partial y}\log|c|.$$

Therefore the space is a *locally Minkowski space*, if and only if $\partial^2(\log|c|)/\partial x\partial y = 0$, which shows that $c(x,y)$ is of the separate form $c = h(x)k(y)$. Then L can be transformed into the form $L^3 = 3\bar{p}^2\bar{q}$.

Remark. The coefficient $2/3$ of R given in [10] must be corrected to $3/2$ as above.

Example 3 ([6]). The *strongly spherically symmetric metric* was considered as Example 2 and 5 in [6]; it is a quartic metric of the form

$$L^4 = c_0 p^4 + 6c_2 p^2 q^2 + c_4 q^4.$$

Let us deal with this metric again. From $(a_{1111}, a_{1122}, a_{2222}) = (c_0, c_2, c_4)$ we have a_{ij} as follows:

$$L^2(a_{11}, a_{12}, a_{22}) = (c_0 p^2 + c_2 q^2, 2c_2 pq, c_2 p^2 + c_4 q^2).$$

The second Christoffel symbols $\{hijk, \ell\}$ have been given in [6].

Suppose that the space be a Berwald space. Putting $\partial_j c_i = c_{ij}$, $i = 0, 2, 4$; $j = 1, 2$, Theorem 1 yields

$$\begin{aligned}
&4(c_0 p^2 + c_2 q^2)G^1 + 8c_2 pq G^2 \\
&\qquad = \frac{1}{2}c_{01}p^4 + \frac{2}{3}c_{02}p^3 q + c_{21}p^2 q^2 + 2c_{22}pq^3 - \frac{1}{6}c_{41}q^4, \\
&8c_2 pq G^1 + 4(c_2 p^2 + c_4 q^2)G^2 \\
&\qquad = -\frac{1}{6}c_{02}p^4 + 2c_{21}p^3 q + c_{22}p^2 q^2 + \frac{2}{3}c_{41}pq^3 + \frac{1}{2}c_{42}q^4.
\end{aligned}$$

Substituting $2G^i = G_1{}^i{}_1 p^2 + 2G_1{}^i{}_2 pq + G_2{}^i{}_2 q^2$, $i = 1, 2$, with $G_j{}^i{}_k(x,y)$ and comparing the coefficients of p^4 $p^3 q$, $p^2 q^2$, pq^3 and q^3, we obtain ten equations as follows:

$$(1) \qquad \begin{cases} 4c_0 G_1{}^1{}_1 = c_{01}, \quad 12c_2 G_2{}^1{}_2 = -c_{41}, \\ 4c_4 G_2{}^2{}_2 = c_{42}, \quad 12c_2 G_1{}^2{}_1 = -c_{02}, \end{cases}$$

$$(2) \qquad 6(c_0 G_1{}^1{}_2 + c_2 G_1{}^2{}_1) = c_{02}, \quad 6(c_2 G_2{}^1{}_2 + c_4 G_1{}^2{}_2) = c_{41},$$

$$(3) \qquad 2c_2(G_1{}^1{}_1 + G_1{}^2{}_2) = 2c_0 G_2{}^1{}_2 + 2c_2(G_1{}^1{}_1 + 4G_1{}^2{}_2) = c_{21},$$

$$(4) \qquad 2c_2(G_1{}^1{}_2 + G_2{}^2{}_2) = 2c_4 G_1{}^2{}_1 + 2c_2(G_2{}^2{}_2 + 4G_1{}^1{}_2) = c_{22}.$$

(i) Suppose $c_0c_2c_4 \neq 0$: Then (1) and (2) give

$$G_1{}^1{}_1 = \frac{c_{01}}{4c_0}, \qquad G_1{}^1{}_2 = \frac{c_{02}}{4c_0}, \quad G_2{}^1{}_2 = -\frac{c_{41}}{12c_2},$$
$$G_1{}^2{}_1 = -\frac{c_{02}}{12c_2}, \quad G_1{}^2{}_2 = \frac{c_{41}}{4c_4}, \quad G_2{}^2{}_2 = \frac{c_{42}}{4c_4}.$$

Then (3) and (4) can be written as

(3') $$\frac{1}{2}\left(\frac{c_{01}}{c_0} + \frac{c_{41}}{c_4}\right) = \frac{c_{01}}{2c_0} + \frac{2c_{41}}{c_4} - \frac{c_0c_{41}}{6(c_2)^2} = \frac{c_{21}}{c_2},$$

(4') $$\frac{1}{2}\left(\frac{c_{02}}{c_0} + \frac{c_{42}}{c_4}\right) = \frac{c_{42}}{2c_4} + \frac{2c_{02}}{c_0} - \frac{c_4c_{02}}{6(c_2)^2} = \frac{c_{22}}{c_2}.$$

The first equations of (3') and (4') yield respectively

$$c_{41}\left\{9(c_2)^2 - c_0c_4\right\} = 0, \quad c_{02}\left\{9(c_2)^2 - c_0c_4\right\} = 0.$$

(i-1) If $9(c_2)^2 - c_0c_4 = 0$, then the right-hand side of L^4 becomes a perfect square and the metric is reduced to a *Riemannian* metric obviously. Cf. Theorem 5; [6], Example 5, $I = 0$.

(i-2) If $c_{41} = c_{02} = 0$, then (3') and (4') are reduced to $c_{01}/c_0 = 2c_{21}/c_2$ and $c_{42}/c_4 = 2c_{22}/c_2$ respectively. Consequently we have $c_0 = c_0(x)$, $c_4 = c_4(y)$ and $(c_2)^2 = kc_0c_4$ with a constant $k \neq 0$. Then L^4 can be written in a coordinate system $(\bar{x}, \bar{y})$ as $L^4 = \bar{p}^4 + \bar{c}\bar{p}^2\bar{q}^2 + \bar{q}^4$ with a non-zero constant $\bar{c}$ and the space is a *locally Minkowski space*. Cf. Theorem 5.

(ii) Suppose $c_2 = 0$ and $c_0c_4 \neq 0$: Then (1) leads to $c_0 = c_0(x)$ and $c_4 = c_4(y)$. Thus $L^4 = c_0(x)p^4 + c_4(y)q^4$, which is obviously a *locally Minkowski metric*.

(iii) Suppose $c_4 = 0$ and $c_0c_2 \neq 0$: Then (1) and (2) give

$$G_1{}^1{}_1 = \frac{c_{01}}{4c_0}, \quad G_1{}^1{}_2 = \frac{c_{02}}{4c_0}, \quad G_2{}^1{}_2 = 0, \quad G_1{}^2{}_1 = -\frac{c_{02}}{12c_2},$$

and (3) and (4) are written respectively as

$$G_1{}^2{}_2 = \frac{1}{2}\left(\frac{c_{21}}{c_2} - \frac{c_{01}}{2c_0}\right) = \frac{1}{8}\left(\frac{c_{21}}{c_2} - \frac{c_{01}}{2c_0}\right),$$
$$G_2{}^2{}_2 = \frac{c_{22}}{2c_2} - \frac{c_{02}}{4c_0} = \frac{c_{22}}{2c_2} - \frac{c_{02}}{c_0}.$$

Consequently we have $c_{02} = 0$ and $2c_0c_{21} - c_2c_{01} = 0$, which lead to $c_0 = c_0(x)$ and $(c_2)^2 = c_0g(y)$. Therefore we have

$$L^4 = c_0(x)p^4 + 6\{c_0(x)g(y)\}^{\frac{1}{2}}p^2q^2,$$

which is obviously transformed into a *locally Minkowski metric*: $L^4 = \bar{p}^4 + 6\bar{p}^2\bar{q}^2$.

(iv) Finally we suppose $c_0 = c_4 = 0$ and $c_2 \neq 0$: Then we have a (quasi-)*Riemannian metric*: $L^4 = 6c_2p^2q^2$. Cf. [6], Example 5, $I = 0$.

Summarizing all the above we have

Proposition 3. *If the strongly spherically symmetric Finsler space of dimension two is a Berwald space, then it is a Riemannian space, or a locally Minkowski space:*

Riemannian space:

(1) $9(c_2)^2 = c_0c_4$, *or*

(2) $c_0 = c_4 = 0$,

Locally Minkowski space:

(1) $c_0 = c_0(x)$, $c_4 = c_4(y)$, $(c_2)^2 = kc_0c_4$, $k = 0$, *or non-zero constant,*

(2) $c_0 = 0$, $c_4 = c_4(y)$, $(c_2)^2 = c_4f(x)$,

(3) $c_4 = 0$, $c_0 = c_0(x)$, $(c_2)^2 = c_0g(y)$.

§5. Three-dimensional Berwald spaces with cubic metric of the normal form

The first half of the third section of Wegener's paper [10] is devoted to making a list of Berwald spaces and locally Minkowski spaces with cubic metric of the normal form. We again consider this subject throughly and show that Berwald spaces with important and typical metric are omitted from his list.

All cubic metrics of dimension three are divided into the following six classes of the normal forms: In the abbreviations $(x^i) = (x, y, z)$ and $(y^i) = (p, q, r)$

(I) $L^3 = c_1p^3 + c_2q^3 + c_3r^3 + 6bpqr, \quad c_1c_2c_3b \neq 0,$

(II) $L^3 = c_1p^3 + c_2q^3 + c_3r^3, \quad c_1c_2c_3 \neq 0,$

(III $L^3 = c_1p^3 + c_2q^3 + 6bpqr, \quad c_1c_2b \neq 0,$

(IV) $L^3 = c_1p^3 + 6bpqr, \quad c_1b \neq 0,$

(V) $L^3 = 6bpqr, \quad b \neq 0,$

(VI) $L^3 = 3apr^2 + bq^3, \quad ab \neq 0,$

where c_1, c_2, c_3, a and b are functions of (x, y, z).

The metrics belonging to (I)–(V) can be written together in the form

$$L^3 = c_1 p^3 + c_2 q^3 + c_3 r^3 + 6bpqr, \tag{5.1}$$

where some of the coefficients may vanish, but they must satisfy the regularity condition $\det(a_{ij}) \neq 0$; we have $La_{ii} = c_i y^i$ and $La_{ij} = by^k$, $i, j, k = 1, 2, 3, \neq$, so that

$$\det(a_{ij}) = \big(c_1 c_2 c_3 + 8b^3\big)\frac{pqr}{L^3} - b^2 \neq 0. \tag{5.2}$$

The second Christoffel symbols of three dimensions are of the following six types:

$$\begin{cases} 4\{iii, i\} = 2\partial_i a_{iii}, \quad 4\{iii, j\} = 3\partial_i a_{iij} - \partial_j a_{iii}, \\ 4\{ijj, i\} = 2\partial_j a_{iij}, \quad 4\{ijj, j\} = \partial_i a_{jjj} + \partial_j a_{ijj}, \\ 4\{ijj, k\} = \partial_i a_{jjk} + 2\partial_j a_{ijk} - \partial_k a_{ijj}, \\ 4\{ijk, k\} = \partial_i a_{jkk} + \partial_j a_{ikk}, \quad i, j, k = 1, 2, 3, \neq . \end{cases} \tag{5.3}$$

We shall research the condition for the Finsler spaces with cubic metric above to be a Berwald space. We have already the condition in the form

$$2\{ijk, h\} = a_{hir} G_j{}^r{}_k + (i, j, k), \quad i, j, k, h = 1, 2, 3. \tag{5.4}$$

We shall write down (5.4) for the metric (5.1): If we put $\partial_j c_i = c_{ij}$ and $\partial_j b = b_j$, then (5.3) gives

$$\begin{gathered} \{iii, i\} = \frac{1}{2}c_{ii}, \quad \{iii, j\} = -\frac{1}{4}c_{ij}, \quad \{ijj, i\} = 0, \\ \{ijj, j\} = \frac{1}{4}c_{ji}, \quad \{ijj, k\} = \frac{1}{2}b_j, \quad \{ijk, k\} = 0, \\ i, j, k = 1, 2, 3, \neq . \end{gathered}$$

Consequently (5.4) for (5.1) is written as

$$\begin{cases} (1)\ 3c_i G_i{}^i{}_i = c_{ii}, & (4)\ 2bG_j{}^k{}_j + 4c_j G_i{}^j{}_j = c_{ji}, \\ (2)\ 6bG_i{}^k{}_i = -c_{ij}, & (5)\ b(G_j{}^j{}_j + 2G_i{}^i{}_j) = b_j, \\ (3)\ c_i G_j{}^i{}_j + 2bG_i{}^k{}_j = 0, & (6)\ b(G_i{}^i{}_k + G_j{}^j{}_k) + c_k G_i{}^k{}_j = 0, \\ \qquad i, j, k = 1, 2, 3, \neq . \end{cases} \tag{5.5}$$

(**I**) We are concerned with the metric (I). Then (1)–(4) of (5.5) give immediately

$$G_i{}^i{}_i = \frac{c_{ii}}{3c_i}, \quad G_i{}^k{}_i = -\frac{c_{ij}}{6b}, \quad G_i{}^j{}_j = \frac{c_{ji}}{3c_j}, \quad G_i{}^k{}_j = \frac{c_i c_{jk}}{12b^2}.$$

From $G_i{}^k{}_j = G_j{}^k{}_i$ it follows from the last one that $c_i c_{jk} = c_j c_{ik}$. Hence we must have quantities d_k such that $c_{jk} = c_j d_k$ Thus we have

$$\text{(5.6-I)} \qquad \begin{cases} G_i{}^i{}_i = \dfrac{c_{ii}}{3c_i}, \quad G_i{}^k{}_i = -\dfrac{c_i d_j}{6b}, \\ G_i{}^j{}_j = \dfrac{d_i}{3}, \quad G_i{}^k{}_j = \dfrac{c_i c_j}{12b^2} d_k, \end{cases} \qquad i, j, k = 1, 2, 3, \neq .$$

The remaining equations (5) and (6) of (5.5) are written as

$$\text{(5')} \qquad \frac{c_{jj}}{c_j} + 2d_j = \frac{3b_j}{b},$$

$$\text{(6')} \qquad (c_i c_j c_k + 8b^3) d_k = 0.$$

(**I**$_1$) We treat of the simple condition $d_k = 0$ from (6'). Then c_{ik} ($= \partial_k c_i$) $= 0$, so that $c_i = c_i(x^i)$, $i = 1, 2, 3$, and (5') shows $b^3/c_1 c_2 c_3 = k^3$ (const.). Consequently L^3 is written as

$$L^3 = c_1(x)\dot{x}^3 + c_2(y)\dot{y}^3 + c_3(z)\dot{z}^3 + 6k(c_1 c_2 c_3)^{1/3}\dot{x}\dot{y}\dot{z},$$

which can be transformed into the form $L^3 = \bar{p}^3 + \bar{q}^3 + \bar{r}^3 + 6k\bar{p}\bar{q}\bar{r}$ in a coordinate system $(\bar{x}^i)$. Therefore the space is reduced to a *locally Minkowski space*.

(**I**$_2$) We treat of the remarkable condition $c_i c_j c_k + 8b^3 = 0$, $i, j, k = 1, 2, 3, \neq$, from (6'). If we differentiate this by y^i, then $c_{ik} = c_i d_k$ and $c_i c_j c_k = -8b^3$ yield (5') immediately.

Now $c_{ik} = c_i d_k$ shows that d_k is a gradient vector which may be written as $d_k = \partial_k d$, and, as a consequence, $\partial_k(\log|c_i| - d) = 0$, $i, j, k \neq$. Thus we have $\log|c_i| - d = g_i(x^i)$. Therefore, putting $e^{g_i} = f_i(x^i)$, we obtain

$$\text{(5.7)} \qquad \begin{cases} c_i = e^d f_i(x^i), \quad i = 1, 2, 3; \quad d = d(x^1, x^2, x^3), \\ 8b^3 = -e^{3d} f_1 f_1 f_3. \end{cases}$$

We have now L of the form

$$L^3 = e^d \left\{ f_1(x)\dot{x}^3 + f_2(y)\dot{y}^3 + f_3(z)\dot{z}^3 - 3(f_1 f_2 f_3)^{1/3}\dot{x}\dot{y}\dot{z} \right\}.$$

Hence there exists a coordinate system, which is written as (x^i) again, such that L is of the form

(5.7') $$L^3 = e^\sigma(\dot{x}^3 + \dot{y}^3 + \dot{z}^3 - 3\dot{x}\dot{y}\dot{z}), \quad \sigma = \sigma(x, y, z).$$

Thus this L is conformal to the typical Minkowski metric

(5.8) $$(L_0)^3 = \dot{x}^3 + \dot{y}^3 + \dot{z}^3 - 3\dot{x}\dot{y}\dot{z}.$$

It is noted that $G_j{}^i{}_k$ of (5.7') are written as

(5.6-I$_2$) $$G_i{}^i{}_i = G_j{}^k{}_j = G_i{}^j{}_j = G_j{}^i{}_k = \frac{\sigma_i}{3}, \\ i, j, k = 1, 2, 3, \neq; \quad \sigma_i = \partial_i \sigma.$$

(**II**) It follows from (2) of (5.5) that we have $c_{ij} = 0$, $i, j = 1, 2, 3, \neq$. Hence the metric is written in the form $L^3 = c_1(x)\dot{x}^3 + c_2(y)\dot{y}^3 + c_3(z)\dot{z}^3$, which is clearly a *locally Minkowski metric*.

(**III**) In this case (5.5) yields

$$G_i{}^i{}_i = \frac{b_i}{b}, \quad i = 1, 2, 3; \quad \text{other } G_j{}^i{}_k = 0, \\ c_{ij} = 0, \quad \frac{3b_i}{b} = \frac{c_{ii}}{c_i}, \quad i, j = 1, 2, \neq .$$

Consequently we have $c_1 = c_1(x)$, $c_2 = c_2(y)$ and $b^3 = c_1 c_2 w(z)$ with a function $w(z)$. Therefore the space is reduced obviously to a *locally Minkowski space*.

(**IV**) Similarly we get

$$G_i{}^i{}_i = \frac{b_i}{b}, \quad i = 1, 2, 3; \quad \text{other } G_j{}^i{}_k = 0, \\ c_{12} = c_{13} = 0, \quad \frac{c_{11}}{c_1} = \frac{3b_1}{b}.$$

Consequently we have $c_1 = c_1(x)$ and $b^3 = c_1 g(y, z)$ with a function $g(y, z)$. Therefore we obtain Berwald spaces with L such that

(5.9) $$L^3 = c_1(x)\dot{x}^3 + 6b\dot{x}\dot{y}\dot{z}, \quad b^3 = c_1(x)g(y, z).$$

(**V**) In this case (5.5) yields only

$$G_i{}^i{}_i = \frac{b_i}{b}; \quad \text{other } G_j{}^i{}_k = 0.$$

Therefore the spaces with the metric (V) are Berwald spaces without any condition.

(**VI**) For this exceptional case we put $a_{133} = a$ and $a_{222} = b$. Denoting $\partial_i a = a_i$ and $\partial_i b = b_i$, the surviving second Christoffel symbols are as follows:

$$2\{113,3\} = a_1, \qquad 4\{122,2\} = -4\{222,1\} = b_1,$$
$$4\{233,1\} = 4\{123,3\} = -4\{133,2\} = a_2, \qquad 2\{222,2\} = b_2,$$
$$4\{133,3\} = \frac{4}{3}\{333,1\} = a_3, \qquad 4\{223,2\} = -4\{222,3\} = b_3.$$

Then the condition (5.4) gives only

$$a_2 = b_1 = b_3 = 0; \quad G_1{}^1{}_1 = \frac{a_1}{2a}, \quad G_2{}^2{}_2 = \frac{b_2}{6b}, \quad G_3{}^3{}_3 = \frac{a_3}{4a},$$

and other $G_j{}^i{}_k = 0$. Therefore the spaces are Berwald spaces, if and only if $a = a(x,z)$ and $b = b(y)$.

Thus we have found all Berwald spaces with the metric belonging to (I)–(VI). Summarizing up we have

Proposition 4. *The three-dimensional Finsler spaces with cubic metric of the normal forms* (I)–(VI) *are Berwald spaces, if and only if*

(I_1) $c_1 = c_1(x)$, $c_2 = c_2(y)$, $c_3 = c_3(z)$, $b^3 = kc_1c_2c_3$, $k = const. \neq 0$. *The spaces are locally Minkowski.*

(I_2) $c_1 = e^d f_1(x)$, $c_2 = e^d f_2(y)$, $c_3 = e^d f_3(z)$, $d = d(x,y,z)$, $8b^3 = -e^{3d} f_1 f_2 f_3$. *The metrics are conformal to a Minkowski metric* $(\dot{x}^3 + \dot{y}^3 + \dot{z}^3 - 3\dot{x}\dot{y}\dot{z})^{1/3}$.

(II) $c_1 = c_1(x)$, $c_2 = c_2(y)$, $c_3 = c_3(z)$. *The spaces are locally Minkowski.*

(III) $c_1 = c_1(x)$, $c_2 = c_2(y)$, $b^3 = c_1c_2w(z)$. *The spaces are locally Minkowski.*

(IV) $c_1 = c_1(x)$, $b^3 = c_1 g(y,z)$.

(V) *The spaces are Berwald spaces with the metric conformal to a Minkowski metric* $(\dot{x}\dot{y}\dot{z})^{1/3}$.

(VI) $a = a(x,z)$, $b = b(y)$.

Remark. (I_2) and (V) give interesting Berwald spaces, because *they are conformal to typical Minkowski metrics*

$$(\dot{x}^3+\dot{y}^3+\dot{z}^3-3\dot{x}\dot{y}\dot{z})^{1/3}, \quad (\dot{x}\dot{y}\dot{z})^{1/3},$$

respectively. In particular, the former is interesting, though WEGENER [10] *failed to find it*. In the first half of 1940's J. Devisme and P. Humbert considered the geometry based on this metric [5].

We shall find the condition for the spaces above to be a locally Minkowski space. It is well-known ([3], Theorem 24.5; [1], Corollary 3.1.3.1.) that a Berwald space is a locally Minkowski space, if and only if the (v)h-torsion tensor $R^i{}_{jk}$ vanishes.

($\mathbf{I}_2$) It is sufficient to be concerned with the metric (5.7') with $G_j{}^i{}_k$ given by (5.6-I_2). Then we get

$$G^i_i=\frac{1}{3}\left(\sigma_i y^i+\sigma_j y^j+\sigma_k y^k\right), \quad G^i_j=\frac{1}{3}\left(\sigma_j y^i+\sigma_k y^j+\sigma_i y^k\right),$$
$$i,j,k=1,2,3,\neq .$$

Hence $G_i{}^i{}_r G^r_j-G_j{}^i{}_r G^r_i=0$ and $G_i{}^k{}_r G^r_j-G_j{}^k{}_r G^r_i=0$ are easily shown. Then we get

$$R^i{}_{ij}=\partial_j G^i_i-\partial_i G^i_j=\frac{1}{3}\left\{(\sigma_{jj}-\sigma_{ki})y^j-(\sigma_{ii}-\sigma_{jk})y^k\right\},$$
$$R^i{}_{jk}=\partial_k G^i_j-\partial_j G^i_k=\frac{1}{3}\left\{(\sigma_{kk}-\sigma_{ij})y^j-(\sigma_{jj}-\sigma_{ik})y^k\right\}.$$

Consequently the space is locally Minkowski, if and only if $\sigma(x,y,z)$ satisfies

$$\partial_i\partial_i\sigma=\partial_j\partial_k\sigma, \quad i,j,k=1,2,3,\neq . \tag{5.10}$$

We consider this condition (5.10) in detail. It is first remarked that

$$p^3+q^3+r^3-3pqr=(p+q+r)(p+\omega q+\omega^2 r)(p+\omega^2 q+\omega r),$$

where $\omega=\left(-1+\sqrt{3}\,i\right)/2$. Thus we have

$$(p+\omega q+\omega^2 r)(p+\omega^2 q+\omega r)=\{p-(q+r)/2\}^2+(3/4)(q-r)^2.$$

This suggests that we should consider the coordinate transformation $(x,y,z)\to(u,v,w)$ such that

$$u=x+y+z, \quad v=x-\frac{1}{2}(y+z), \quad w=\left(\sqrt{3}/2\right)(y-z).$$

In (u, v, w) we have the metric L under consideration of the form

$$L^3 = e^s \left\{\dot{u}(\dot{v}^2 + \dot{w}^2)\right\}, \quad s = s(u, v, w), \tag{5.7”}$$

and it is easy to show that (5.10) is written in the form $s_{uv} = s_{uw} = 0$ and $s_{vv} + s_{ww} = 0$. Thus we have

$$s = f(u) + g(v, w), \quad g_{vv} + g_{ww} = 0. \tag{5.10’}$$

Hence $g(v, w)$ is a harmonic function. Consequently the metric is of the form

$$L^3 = \left\{e^{f(u)}\dot{u}\right\}\left\{e^{g(v,w)}(\dot{v}^2 + \dot{w}^2)\right\}. \tag{5.11}$$

Since $g(v, w)$ is harmonic, the curvature of the two-dimensional Riemannian space with $ds^2 = e^g(dv^2 + dw^2)$ vanishes. Therefore we have a coordinate system $(\underline{v}, \underline{w})$ in which $e^g(\dot{v}^2 + \dot{w}^2) = \dot{\underline{v}}^2 + \dot{\underline{w}}^2$, and it is obvious that the metric (5.11) is certainly locally Minkowski.

We shall turn to the discussion of the Berwald spaces belonging to (IV), (V) and (VI). They have such a similar property as follows: The surviving components of ${G_j}^i{}_k$ are ${G_i}^i{}_i$, $i = 1, 2, 3$, only. Thus the surviving components of $R^i{}_{jk}$ are $R^i{}_{ij} = (\partial_j {G_i}^i{}_i)y^i$. Consequently the conditions under consideration are easily given as follows: (IV) $b^3 = c_1(x)v(y)w(z)$, (V) $b = u(x)v(y)w(z)$, (VI) $a = u(x)w(z)$.

Summarizing up we have

Proposition 5. *The conditions for the Berwald spaces with cubic metric belonging to* (I_2), (IV), (V) *and* (VI) *to be locally Minkowski are as follows:*

(I_2) $L^3 = e^\sigma(\dot{x}^3 + \dot{y}^3 + \dot{z}^3 - 3\dot{x}\dot{y}\dot{z})$, $\partial_x^2\sigma = \partial_y\partial_z\sigma$, $\partial_y^2\sigma = \partial_z\partial_x\sigma$, $\partial_z^2\sigma = \partial_x\partial_y\sigma$. *Then the metric can be written in a coordinate system* (u, v, w) *as* $L^3 = \left\{e^{f(u)}\dot{u}\right\}\left\{e^{g(v,w)}(\dot{v}^2 + \dot{w}^2)\right\}$, $g(v, w)$ *being a harmonic function.*

(IV) $b^3 = c_1(x)v(y)w(z)$,

(V) $b = u(x)v(y)w(z)$,

(VI) $a = u(x)w(z)$.

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MAKOTO MATSUMOTO
15, ZENBU–CHO, SHIMOGAMO
SAKYO–KU, KYOTO
606, JAPAN

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