On functions satisfying $\operatorname{Re}\left[\frac{f(z)}{z}\right] > \alpha$

By R. M. GOEL (Patiala)

Let S_α denote the class of functions

(1) $$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

which are regular analytic in the unit disc $|z| < 1$ and satisfy the condition

(2) $$\operatorname{Re}\left[\frac{f(z)}{z}\right] > \alpha \qquad (0 \leqq \alpha < 1) \quad \text{for} \quad |z| < 1.$$

In a recent paper [1] K. YAMAGUCHI has proved that if $f(z) \in S_0$, then

(3) $$\operatorname{Re}[f'(z)] \geqq \frac{1-2r-r^2}{(1+r)^2} \quad \text{for} \quad 0 \leqq r < \sqrt{2}-1, \; z = re^{i\theta}.$$

In the present paper, we increase the constant $\sqrt{2}-1$ to $\frac{1}{2}$. We shall however, consider a class T_α of functions belonging to the class S_α with the additional condition that the second coefficient a_2 is fixed. Without loss of generality we may assume $a_2 \geqq 0$ since $e^{-i\theta} f(ze^{i\theta})$ belongs to T_α whenever $f(z)$ belongs to T_α.

Let P denote the class of regular, analytic functions in the unit disc $|z| < 1$ whose real part is positive and which take the value 1 at the origin.

We need the following

Lemma. If $p(z) \in P$ and $|p'(0)| = 2b > 0$ then

(4) $$\operatorname{Re}[p(z) + zp'(z)] \geqq \frac{1-4r^2-4br^3-r^4}{(1+2br+r^2)^2}, \qquad |z| = r,$$

for $0 \leqq |z| < r_0$ where r_0 is the smallest positive root of

(5) $$1 - 3r^2 - 2br^3 = 0$$

The result is skarp.

PROOF. Since $p(z) \in P$, we can express it in the form

(6) $$p(z) = \frac{1-\omega(z)}{1+\omega(z)}$$

where $\omega(z)$ is regular analytic in $|z| < 1$ and satisfies the conditions $\omega(0) = 0$, $|\omega'(0)| = b$ and $|\omega(z)| < 1$.

Using (6) we obtain

$$p(z)+zp'(z) = \frac{1-\omega(z)}{1+\omega(z)} - \frac{2z\omega'(z)}{(1+\omega(z))^2}. \tag{7}$$

Such functions $\omega(z)$ are known [3, p. 167] to satisfy the inequalities

$$\frac{r(b-r)}{1-br} \leqq |\omega(z)| \leqq \frac{r(b+r)}{1+br}. \tag{8}$$

Let

$$\varphi(z) = \frac{\omega(z)}{z} \tag{9}$$

then $\varphi(z)$ is analytic in $|z|<1$ and satisfies the inequality [2, p. 18]

$$|\varphi'(z)| \leqq \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad \text{for} \quad |z| < 1. \tag{10}$$

From (9) and (10) we get

$$|z\omega'(z)-\omega(z)| \leqq \frac{r^2-|\omega(z)|^2}{1-r^2}, \qquad |z| = r. \tag{11}$$

(7) gives in conjunction with (11)

$$\operatorname{Re}[p(z)+zp'(z)] \geqq -1+2\operatorname{Re}\left[\frac{1}{(1+\omega(z))^2}\right] - \frac{2(r^2-|\omega(z)|^2}{(1-r^2)|1+\omega(z)|^2}, \tag{12}$$

which on substituting

$$\omega_1(z) = \frac{1}{1+\omega(z)} \tag{13}$$

reduces to

$$\operatorname{Re}[p(z)+zp'(z)] \geqq -1+2\operatorname{Re}\omega_1^2 - 2\frac{(r^2|\omega_1|^2-|1-\omega_1|^2)}{(1-r^2)}. \tag{14}$$

In view of the second inequality in (8) it is easy to see that

$$\left|\omega_1(z) - \frac{1}{1-r^2}\right| \leqq \frac{r}{1-r^2}. \tag{15}$$

However, because of the first inequality in (8) all values in the interior of the circle given by the (15) are not taken.

Putting $\omega_1(z) = u+iv$ and then denoting the right hand side of (14) by $S(u, v)$ we get

$$S(u,v) = \frac{1+r^2}{1-r^2} - \frac{4u}{1-r^2} + 4u^2, \tag{16}$$

which shows that $S(u, v)$ is a function of u only, say $M(u)$. The absolute minimum of $M(u)$ in $(0, \infty)$ is attained at $u_0 = \dfrac{1}{2(1-r^2)}$ and equals

$$M(u_0) = \frac{-r^4}{(1-r^2)^2}. \tag{17}$$

Since $\omega(z)$ is real whenever $\omega_1(z)$ is real, we have

$$u = \frac{1}{1+\operatorname{Re}\omega(z)} \leqq \frac{1+br}{1-r^2}$$

and

$$u = \frac{1}{1+\operatorname{Re}\,\omega(z)} \geqq \frac{1+br}{1+2br+r^2},$$

where we have used the inequalities

$$\frac{-r(b+r)}{1+br} \leqq \operatorname{Re}\omega(z) \leqq \frac{r(b+r)}{1+br}.$$

It is easy to see that $u_0 < \frac{1+br}{1-r^2}$ but u_0 is not always greater than $\frac{1+br}{1+2br+r^2}$. In the latter case the minimum of $M(u)$ in the segment $\left[\frac{1+br}{1+2br+r^2}, \frac{1+br}{1-r^2}\right]$ is attained at $u = \frac{1+br}{1+2br+r^2}$ and equals

$$M\left(\frac{1+br}{1+2br+r^2}\right) = \frac{1-4r^2-4br^3-r^4}{(1+2br+r^2)^2}. \tag{18}$$

(18) will hold for those values of r for which

$$\frac{1}{2(1-r^2)} \leqq \frac{1+br}{1+2br+r^2}$$

that is,

$$1-3r^2-2br^3 \geqq 0. \tag{19}$$

Since (19) holds at $r=0$ but it does not hold at $r=1$. Hence the minimum value (18) is acceptable for $0 \leqq r < r_0$ where r_0 is the smallest positive root of $1-3r^2-2br^3 = 0$.

Inequality (4) follows from (18).

The equality sign in (4) is attained for the function

$$P(z) = \frac{1-z^2}{1+2bz+z^2} \quad \text{at} \quad z = r. \tag{20}$$

This completes the proof of the lemma.

Remark. From (17) we find that

$$\operatorname{Re}[p(z)+zp'(z)] \geqq \frac{-r^4}{(1-r^2)^2} \quad \text{for} \quad r_0 \leqq r < 1,$$

but the bound is not sharp.

Theorem 1. *If $f(z)\in T_\alpha$, then*

$$\text{(21)}\qquad \operatorname{Re}[f'(z)] \geqq \frac{(2\alpha-1)r^4+4b(2\alpha-1)r^3+(-4+6\alpha+4\alpha b^2)r^2+4\alpha br+1}{(1+2br+r^2)^2}$$

for $0\leqq r<r_0$, $|z|=r$, $a_2=2b$ and r_0 is the smallest positive root of $1-3r^2-2br^3=0$. The bound is sharp.

PROOF. Since $f(z)\in T_\alpha$, $\operatorname{Re}\left[\frac{f(z)}{z}\right]>\alpha$ and hence

$$\text{(22)}\qquad \frac{f(z)/z-\alpha}{1-\alpha}=p(z)$$

where $p(z)\in P$. Differentiating (22) we obtain

$$f'(z)=\alpha+(1-\alpha)[p(z)+zp'(z)].$$

The theorem now follows easily on using the above lemma. The extremal function is given by

$$f(z)=\frac{z\left(1+2b\alpha z+(2\alpha-1)z^2\right)}{1+2bz+z^2}$$

Corollary 1. 1. If $f(z)\in S_\alpha$ then on putting $b=1$ in (21) we have

$$\text{(23)}\qquad \operatorname{Re}[f'(z)]\geqq\frac{(2\alpha-1)r^2+2(2\alpha-1)r+1}{(1+r)^2}\quad\text{for}\quad 0\leqq r<\frac{1}{2}.$$

The bound is sharp.

On putting $\alpha=0$ in (23) we get

$$\text{(24)}\qquad \operatorname{Re}f'(z)\geqq\frac{1-2r-r^2}{(1+r)^2}\quad\text{for}\quad 0\leqq r<\frac{1}{2}.$$

Corollary 1. 2. On putting $b=0$ in (21) we obtain

$$\text{(25)}\qquad \operatorname{Re}f'(z)\geqq\frac{(2\alpha-1)r^4+(6\alpha-4)r^2+1}{(1+r^2)^2}\quad\text{for}\quad 0\leqq r<\frac{1}{\sqrt{3}}.$$

This is the sharp estimate for odd functions $f(z)=z+a_3z^3+\dots$ such that $\operatorname{Re}\left[\frac{f(z)}{z}\right]>\alpha$ since an odd function has vanishing even coefficients and the extremal function in this case reduces to $f(z)=z\left(1+(2\alpha-1)z^2\right)/(1+z^2)$.

We shall now consider some applications of Theorem 1.

Theorem 2. *If $f(z)\in T_\alpha$, then*

$$\text{(26)}\qquad f(z)\ \text{ is univalent in }\ |z|<r_0\ \text{ for }\ \frac{1}{2}\leqq\alpha<1;$$

and

$$\text{(27)}\qquad f(z)\ \text{ is univalent in }\ |z|<\min(r_0,r_1)\ \text{ for }\ 0\leqq\alpha<\frac{1}{2},$$

where r_0 *and* r_1 *are the smallest positive roots of* $1-3r^2-2br^3=0$ *and*

$$(2\alpha-1)r^4+4b(2\alpha-1)r^3+(-4+6\alpha+4\alpha b^2)r^2+4\alpha br+1=0$$

respectively.

PROOF. It is easily seen that for $\frac{1}{2}\leqq\alpha<1$, right hand side of (21) is always positive and consequently $\operatorname{Re} f'(z)<0$ for $|z|<r_0$. For $0\leqq\alpha<\frac{1}{2}$, $\operatorname{Re} f'(z)>0$ provided

$$K(r)=(2\alpha-1)r^4+4b(2\alpha-1)r^3+(-4+6\alpha+4\alpha b^2)r^2+4\alpha br+1>0.$$

Let r_1 denote the smallest positive root of $K(r)=0$. Then $\operatorname{Re} f'(z)>0$ for $|z|<\min(r_0, r_1)$.

Theorem 2. now follows immediately from the following well known *Wolff—Noshiro Lemma:* If $f(z)$ is analytic in $|z|<R$ and $\operatorname{Re} f'(z)>0$ in $|z|<R$, then $f(z)$ is univalent in $|z|<R$.

Corollary 2. 1. If $f(z)\in S_\alpha$, then $f(z)$ is univalent in $|z|<\sqrt{\frac{2-2\alpha}{1-2\alpha}}-1$ for $0\leqq\alpha\leqq\frac{1}{10}$.

The result follows by combining Theorem 2 with Corollary 1. 1.

Theorem 3. *If* $f(z)=z+a_2z^2+\dots+a_nz^n+\cdots$ *belongs to* S_α, *then the partial sums*

$$S_n(z)=z+a_2z^2+\dots+a_nz^n \qquad (n=2, 3, \dots)$$

are univalent in $|z|<\frac{1}{4(1-\alpha)}$ *for* $0\leqq\alpha\leqq\frac{1}{10}$.

PROOF. The function

$$p(z)=\frac{f(z)/z-\alpha}{1-\alpha}=1+\frac{1}{1-\alpha}(a_2z+a_3z^2+\dots) \tag{28}$$

belongs to P. Hence, by Caratheodory – Toeplitz's theorem

$$|a_n|\leqq 2(1-\alpha) \qquad (n=2, 3, \dots). \tag{29}$$

Now

$$\operatorname{Re}[S_n'(z)]=\operatorname{Re} f(z)-\operatorname{Re}\left[\sum_{m=n+1}^{\infty} ma_mz^{m-1}\right],$$

which gives together with (23) and (29)

$$\operatorname{Re}[S_n'(z)]\geqq\frac{(2\alpha-1)r^2+2(2\alpha-1)r+1}{(1+r)^2}-\frac{2(1-\alpha)(n+1-nr)r^n}{(1-r)^2}, \qquad |z|=r.$$

If $r=\frac{1}{4(1-\alpha)}\leqq\left(\sqrt{\frac{2-2\alpha}{1-2\alpha}}-1\right)$, then

$$(n+1-nr)r^n\leqq(5-4r)r^4 \qquad (n=4, 5, \dots),$$

and hence for $r = \dfrac{1}{4(1-\alpha)}$,

$$\operatorname{Re} S_n'(z) \geqq \frac{7-6\alpha}{(5-4\alpha)^2} - \frac{(4-5\alpha)}{8(1-\alpha)^2(3-4\alpha)^2} > 0 \quad \text{for} \quad 0 \leqq \alpha \leqq \frac{1}{10} \quad \text{and} \quad n \geqq 4.$$

The real part of $S_n'(2)$ is harmonic function and therefore by the principle of minimum modulus we have

$$\operatorname{Re} S_n'(z) > 0 \quad \text{for} \quad |z| \leqq \frac{1}{4(1-\alpha)}, \qquad n \geqq 4, \quad 0 \leqq \alpha \leqq \frac{1}{10}.$$

The theorem is thus proved for $n \geqq 4$.

Let us now consider the case $n=2$.

We have by (29)

$$\operatorname{Re} S_2'(z) = \operatorname{Re}(1+2a_2 z) \geqq 1-4(1-\alpha)|z| > 0 \quad \text{for} \quad |z| < \frac{1}{4(1-\alpha)}.$$

Finally we consider the case $n=3$.

Let $p(z) = 1+p_1 z+p_2 z^2+\dots$ belong to P, then by Caratheodory—Toeplitz's theorem we have

$$|2p_2-p_1^2| \leqq 4-|p_1|^2. \tag{30}$$

Let

$$c = 2p_2-p_1^2, \quad p_1 z = \alpha'+i\beta', \quad \sqrt{c}\, z = \gamma'+i\delta', \quad |z| \leqq \frac{1}{4(1-\alpha)}.$$

Then on substituting the power series expansion for $p(z)$ in (28) we obtain

$$S_3(z) = z+a_2 z^2+a_3 z^3 = z+p_1(1-\alpha)z^2+p_2(1-\alpha)z^3.$$

Therefore,

$$S_3'(z) = 1+(1-\alpha)(2p_1 z+3p_2 z^2) = 1+(1-\alpha)\left[2p_1 z+\frac{3}{2}(c+p_1^2)z^2\right].$$

$$\operatorname{Re} S_3'(z) = 1+(1-\alpha)\left[2\alpha'+\frac{3}{2}(\gamma'^2-\delta'^2)+\frac{3}{2}(\alpha'^2-\beta'^2)\right] =$$

$$= 1+(1-\alpha)\left[2\alpha'+\frac{3}{2}\alpha'^2-\frac{3}{2}\beta'^2+\frac{3}{2}\gamma'^2-\frac{3}{2}(|cz^2|-\gamma'^2)\right] \geqq$$

$$\geqq 1+(1-\alpha)\left[2\alpha'+\frac{3}{2}\alpha'^2-\frac{3}{2}\beta'^2+\frac{3}{2}\gamma'^2-\frac{3}{2}\left(\frac{4-|p_1|^2}{16(1-\alpha)^2}-\gamma'^2\right)\right] \geqq$$

$$\geqq 1+(1-\alpha)\left[2\alpha'+\frac{3}{2}\alpha'^2-\frac{3}{2}\beta'^2+\frac{3}{2}\gamma'^2-\frac{3}{2}\left(\frac{1}{4(1-\alpha)^2}-\alpha'^2-\beta'^2-\gamma'^2\right)\right] =$$

$$= 1+(1-\alpha)\left[2\alpha'+3\alpha'^2+3\gamma'^2-\frac{3}{8(1-\alpha)^2}\right] \geqq 1-\frac{3}{8(1-\alpha)}+(1-\alpha)[2\alpha'+3\alpha'^2] \geqq$$

$$\geqq \frac{7-8\alpha-8\alpha^2}{24(1-\alpha)} > 0 \quad \text{for} \quad 0 \leqq \alpha \leqq \frac{1}{10}.$$

To show that the result is sharp, consider the function

$$f(z) = z\left[\frac{1+(1-2\alpha)z}{1-z}\right] = z+2(1-\alpha)z^2+\dots$$

which belongs to S_α. For this function

$$S_2(z) = z+2(1-\alpha)z^2,$$

and $$S_2'(z) = 0 \quad \text{when} \quad z = -\frac{1}{4(1-\alpha)}.$$

This completes the proof of the theorem.

References

[1] KUNIO YAMAGUCHI, On functions satisfying Re $[f(z)/z] > 0$, *Proc. Amer. Math. Soc.* **17** (1966), 588—591.

[2] C. CARATHEODORY, Theory of functions, Vol. 2 *Chelsea, New York*, 1954.

[3] Z. NEHARI, Conformal Mapping, McGraw-Hill Book Co., Inc. 1952.

(Received 28, November 1968.)