Valued vector spaces of countable dimension

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The structure theory of valued vector spaces ([2], [5]) is especially simple for those of countable dimension; indeed, they are completely determined by their associated graded vector spaces. In this note[1]) we prove the above assertion, and illustrate it with applications to reduced p-primary abelian groups, linearly ordered abelian groups, and the square factor groups of certain valued fields.

Let G be an additively written abelian group, and let Λ be a linearly ordered set. Let $v: G \to \Lambda \cup \{\infty\}$ be a *valuation* on the group G; that is, for all $a, b \in G$ we have $v(a+b) \geqq \inf\{v(a), v(b)\}$, $v(a) = v(-a)$, and $v(a) = \infty$ if and only if $a=0$. (∞ is a formal symbol with $\infty > \lambda$ for all $\lambda \in \Lambda$.) v determines a Λ-filter on G

$$G_\lambda = \{a \in G \mid v(a) \geqq \lambda\} \qquad (\lambda \in \Lambda)$$

and thence a Λ-graded group (by which we simply mean a family of groups indexed by Λ)

$$\Delta_\lambda G = G_\lambda / \bigcup_{\beta > \lambda} G_\beta \qquad (\lambda \in \Lambda).$$

Finally, we let $\Sigma\Delta G$ denote the valued group

$$\Sigma_{\lambda \in \Lambda} \Delta_\lambda G$$

where the valuation maps each nonzero element of the direct sum to the least $\lambda \in \Lambda$ at which it is nonzero.

We say G is a *valued vector space* when in addition to being a valued group it is a vector space with $v(a) = v(ra)$ for all $a \in G$ and all nonzero scalars r. If G is a valued vector space, we may regard $\Sigma\Delta G$ as a valued vector space. Note that any valued group of prime exponent p may be regarded as a valued vector space over the field with p elements.

Proposition. *Suppose G is a valued vector space of countable dimension. Then G is isomorphic as a valued vector space to $\Sigma\Delta G$.*

PROOF. Let $b_1, b_2, \ldots, b_n, \ldots$ be a basis for G (possibly finite). One checks that a set of elements of G taking distinct values in Λ (that is, on which v is injective) is linearly independent (use that $v(a+b) = v(a)$ whenever $v(a) < v(b)$). Hence, for each n there is an element of maximal value among those elements in the span of $\{b_1, \ldots, b_n\}$

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which are not in the span of $\{b_1, \ldots, b_{n-1}\}$; pick such an element a_n. Then $a_1, a_2, \ldots, a_n, \ldots$ is a basis for G. Further, for any scalars r_i

$$v(\Sigma r_i a_i) = \min \{v(a_i) | r_i \neq 0\}.$$

For each $\lambda \in \Lambda$, let π_λ denote the composition of the canonical homomorphism $G_\lambda \to \Delta_\lambda G$ with the injection $\Delta_\lambda G \to \Sigma \Delta G$. We obtain the desired isomorphism by mapping each a_i to $\pi_{v(a_i)}(a_i)$, and extending this map linearly to all of G. The proposition is proved.

We now illustrate the proposition with several applications. We will denote by ΔG the graded group of the valued group G.

1. Suppose G is a reduced p-primary abelian group. The height function (defined into a suitably large set of ordinal numbers Λ) is a valuation on the group G (cf. [7], p. 28). Restricting the height function to the set of elements of G of exponent p, call it P, makes P into a valued vector space. Knowledge of the graded vector space $\Delta P = (\Delta_\lambda P)_{\lambda \in \Lambda}$ is equivalent to knowledge of the *Ulm invariants* of G (for each $\Delta_\lambda P$ is determined as a vector space by its dimension, which is simply the λ-th Ulm invariant of G).

It is interesting that in several cases in which G is known to be determined by its Ulm invariants, P is isomorphic as a valued vector space to $\Sigma \Delta P$. For example, if G is countable, then the proposition says P is isomorphic to $\Sigma \Delta P$, while Ulm's theorem says G is determined by its Ulm invariants. For other examples, consider the case when G is a direct sum of countable groups [8], or when P is the union of an ascending chain of groups of bounded height ([7], p. 25). In general, ΔP does not determine P as a valued vector space; this is one reason that G is not in general determined by its Ulm invariants. An unanswered question is whether it is the only reason; that is, whether the valued vector space P is a *determining* invariant for G.

2. Now suppose G is a linearly ordered abelian group. For each $a \in G$ let $v(a)$ denote the convex (="isolated") subgroup of G generated by a (i.e. the set of $b \in G$ with $-na \leqq b \leqq na$ for some integer n). Then v may be regarded as a valuation on the group G (note that $\{v(a) | a \in G\}$ is linearly ordered by inclusion). If G is divisible (i.e. $nG = G$ for all natural numbers n), then G may be regarded as a valued vector space over the rational numbers.

Now assume G is countable. By the proposition there is a value preserving isomorphism $\Phi: G \to \Sigma \Delta G$. For each $\lambda \in \Lambda$, give $\Delta_\lambda G$ the Archimedean linear order induced by Φ and G (an element of $\Delta_\lambda G$ is positive if and only if it is the image of a positive element of G_λ under the composition of Φ with the projection $\Sigma \Delta G \to \Delta_\lambda G$). Then Φ is easily seen to be an order isomorphism, where we give $\Sigma \Delta G$ the lexicographic order (i.e. a formal sum

$$\Sigma_{\lambda \in \Lambda} a_\lambda \in \Sigma_{\lambda \in \Lambda} \Delta_\lambda G$$

is positive if a_μ is positive, where $\mu = \min \{\lambda | a_\lambda \neq 0\}$). Since by Holder's theorem $\Delta_\lambda G$ is order isomorphic to a subgroup of the additive group of real numbers, we have shown:

Let G be a divisible countable linearly ordered abelian group. Then G is order isomorphic to a lexicographic direct sum of subgroups of the real numbers.

A similar application of the proposition to ordered vector spaces over the real numbers yields Theorem 3. 5 of [3].

3. We mention an application of the proposition, appearing in [1]. Let v be a valuation on a field F (in the usual sense). We suppose F does not have characteristic two, and further, that the residue class field of v is perfect if it has characteristic two. Let $F^\times$ denote the multiplicative group of nonzero elements of F, and $F^\times/F^{\times 2}$ its square factor group.

If F with v has no immediate quadratic extensions (e.g. F is "algebraically complete" in the sense of Ersov [4]), then $F^\times/F^{\times 2}$ naturally (but nontrivially) takes the structure of a valued vector space (the valuation takes each coset $a \cdot F^{\times 2} \in F^\times/F^{\times 2}$ to the maximum of $\{v(1-b) | b \in a \cdot F^{\times 2}\}$). Thus if F has no immediate quadratic extensions and F (or even $F^\times/F^{\times 2}$) is countable, then $F^\times/F^{\times 2}$ is isomorphic as a valued vector space to $\Sigma\Delta(F^\times/F^{\times 2})$. Further, $\Delta(F^\times/F^{\times 2})$ can be *explicitly* calculated; it turns out to depend only on the value group of v, the image of 2 in the value group, and the residue class field of v. (For all details see [1], Remark (5. 6).)

4. A final example. Suppose G is any valued vector space. For simplicity, let us suppose that $\Sigma\Delta G$ has countable dimension. Denote by $\tilde{H}\Delta G$ the well-ordered product of the family ΔG (see [5]; briefly, $\tilde{H}\Delta G$ is the set of maps in the direct product $\Pi_{\lambda\in\Lambda}\Delta_\lambda G$ which are zero except on a well ordered subset of Λ. $\tilde{H}\Delta G$ is made into a valued vector space by assigning to each nonzero element the least $\lambda \in \Lambda$ at which it is nonzero). It is well known ([2], p. 11) that G can be embedded as a valued vector space into $\tilde{H}\Delta G$; further, various restrictions can be placed on this embedding and, in particular, on its image (see [6], Theorem (3. 1), which can be easily set into the language of valued vector spaces). Our proposition gives some idea of how arbitrary this image can be even with additional restrictions on the embedding. For if G has countable dimension, then it is isomorphic as a valued vector space to *every* subspace of $\tilde{H}\Delta G$ of countable dimension which contains $\Sigma\Delta G$!

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Added in proof: F. Richman has given. among other things, a negative answer to the question posed in the first application above (see Extensions of p-bounded groups, *Arch. Math.* **21** (1970), 449—454).

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