Local growth of the number of prime divisors of consecutive integers

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1. Let $\omega(n)$ denote the number of the different prime factors of n. Similarly, let $\Omega(n)$ be the number of all the prime factors of n each counted by ts multiplicity. Let $d(n)$ be the number of divisors of n.

As it is well known, $\omega(n) = (1+o(1))\cdot \log\log n$ holds for all n except a set of zero density.

We shall investigate now the maximal term of $\omega(n+1), \ldots, \omega(n+k)$.

Let

$$O_k(n) = \max_{j=1,\ldots,k} \omega(n+j), \tag{1.1}$$

$$o_k(n) = \min_{j=1,\ldots,k} \omega(n+j). \tag{1.2}$$

Let $\psi(z) = z \log \frac{z}{e} + 1$ be defined for $z \geqq 1$, and $\varrho(u)$ denote its inverse function ($n \geqq 0$).

I guess that the inequality

$$1-\varepsilon \leqq \frac{O_k(n)}{\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n} \leqq 1+\varepsilon \tag{1.3}$$

holds for almost all n, uniformly in $k(=1, 2, \ldots)$, ε being an arbitrary positive constant.

At this time, I can prove only the right hand side of (1. 3), i.e.

Theorem 1. *For every constant $\varepsilon<0$ and for all n except a set of zero density the inequalities*

$$O_k(n) \leqq (1+\varepsilon)\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n, \tag{1.4}$$

$$O_k(n) \leqq \left\{\varrho\left(\frac{\log k}{\log\log n}\right)+\varepsilon\right\}\log\log n \tag{4.5}$$

hold uniformly in k.

We shall prove only (1. 4), the proof of (1. 5) is similar. We use the notation $x_1=\log x$, $x_2=\log x_1$, $x_3=\log x_2$. The letters A, B, C denote suitable positive constants not necessarily the same at every occurrence; p will denote prime numbers.

Let

$$D(x,z)=\sum_{n\leqq x} z^{\omega(n)} \qquad (z\geqq 1).$$

We prove the inequality

$$D(x,z)\leqq B[(z-1)x_2]^A x_1^{z-1} \qquad (x>10) \tag{1.6}$$

Assume that x is large enough. Let A be such a large integer that

$$\sum_{p\leqq x^{1/A}} \frac{1}{p} \leqq x_2$$

holds. The symbol Q_r denotes a general square free integer having exactly r prime factors. Since for $Q_r\leqq x$, Q_r has at most A prime factors greater than $x^{1/A}$ we have

$$\sum_{Q_r\leqq x}\frac{1}{Q_r}\leqq \frac{1}{(r-A)!}\left(\sum_{p\leqq x^{1/A}}\frac{1}{p}\right)^{r-A}\left(\sum_{p\leqq x}\frac{1}{p}\right)^{A}<$$

$$<\frac{1}{(r-A)!}(x_2+B)^A x_1^{r-A}<\left(1+\frac{B}{x_2}\right)^A\frac{1}{(r-A)!}x_2^r.$$

Using the relation

$$z^{\omega(n)}=\sum_{r=0}^{\infty}\sum_{Q_r|n} w^r, \qquad w=z-1,$$

we have

$$D(x,z)=\sum_{r=0}^{\infty}\sum_{Q_r\leqq x} w^r\left[\frac{x}{Q_r}\right]\leqq x\sum_{r=0}^{\infty} w^r\sum_{Q_r\leqq x}\frac{1}{Q_r}.$$

So by (1. 7) we obtain

$$D(x,z)<Bxx_2^A w^A\sum_{r=A}^{\infty}\frac{w^{r-A}x_2^{r-A}}{(r-A)!}=Bxx_2^A w^A x_1^w,$$

which was desired.

Let $M(x,z)$ be the number of those n, $\leqq x$ for which $\omega(n)>zx_2$. Let $\eta>0$ be an arbitrary constant. By (1. 6) we have

$$M\big(x,z(1+\eta)\big)z^{z(1+\eta)x_2}\leqq Bx[(z-1)x_2]^A x_1^{z-1},$$

whence

$$M\big(x,z(1+\eta)\big)\leqq Bx[(z-1)x_2]^A\cdot x_1^{z-1-z(1+\eta)\log z(1+\eta)}. \tag{1.8}$$

Instead of (1. 4) we prove that

$$O_k(n)\leqq(1+\varepsilon)\varrho\left(\frac{\log k}{x_2}\right)x_2 \tag{1.9}$$

for all n, $\leqq x$ except at most $o(x)$ of n's.

Let $z = 1+\varepsilon'$, $\eta=\varepsilon'$, ε' be a small positive constant. By (1. 8) we have

$$M(x, (1+\varepsilon')^2) \leqq Bx[\varepsilon' x_2]^A x_1^{\varepsilon' - (1+\varepsilon')^2 \log (1+\varepsilon')^2} < Bxx_1^{-\varepsilon'/2}.$$

Let k_0 be such an integer that

$$\frac{\log k_0}{x_2} < \frac{\varepsilon'}{\varepsilon}.$$

Then

$$k_0 M(x, (1+\varepsilon')^2) < Bx \cdot x_1^{-\varepsilon'/\varepsilon}. \tag{1.10}$$

Hence it follows that $\max\limits_{k \leqq k_0} O_k(n) \leqq (1+\varepsilon')^2 x_2$ except $o(x)$ of n's. Choosing $(1+\varepsilon')^2 < (1+\varepsilon)$, we obtain that (1. 9) holds, if $\dfrac{\log k_0}{x_2} \leqq \dfrac{\varepsilon'}{3}\,(\varepsilon'>0)$.

Let now $(\log k)/x_2 \geqq \varepsilon'/3$, $k=x_1^{\Psi(z)}$, i.e. $z = \varrho\left(\dfrac{\log k}{x_2}\right)$. Observing that

$$[x_2(z-1)]^A < x_1^{(\eta/2) z \log z} \text{ for all large } x,\ z > 1+\frac{\varepsilon'}{3},$$

we have

$$kM(x, z(1+\eta)) < Bx \cdot x_1^{-(\eta/2) z \log z} \tag{1.11}$$

Let

$$k_m = [m^{4/\eta}]+1 \quad z_m = \varrho\left(\frac{\log k_m}{x_2}\right), \quad \psi(z_m) = \frac{\log k_m}{x_2}.$$

Since

$$z \log z = \psi(z)+z-1 \geqq \psi(z) \qquad (z \geqq 1).$$

from (1. 11) we deduce

$$k_m M(x, z_m(1+\eta)) < Bx\, k_m^{-\eta/2} < Bx \cdot m^{-2}. \tag{1.12}$$

Hence

$$\sum_{\log k_m \geqq (\varepsilon'/3)x_2} k_m M(x, z_m(1+\eta)) = o(x).$$

Taking $\eta=\varepsilon$ we obtain that (1. 9) holds for all, $k=k_m$.

Let now $k_m<k<k_{m+1}$. It is obvious that $O_k(n) \leqq O_{k_m}(n)$. To finish the proof it is enough to show that

$$\varrho\left(\frac{\log k_{m+1}}{x_2}\right) \leqq (1+\eta)\varrho\left(\frac{\log k_m}{x_2}\right),$$

if m is large enough; η being an arbitrary positive constant. But this is an obvious consequence of the fact that $\varrho(u)$ has a bounded derivative in $u \geqq u_0(>0)$.

2. In a similar way we can prove the following assertion. Let $\psi(z) = z \log z - z+1$ be defined for $0<z \leqq 1$, and $\bar{\varrho}(u)$ its inverse function.

Theorem 2. *For every constant* $\varepsilon>0$ *and almost all* n, *the inequality*

$$o_k(n) \geqq (1-\varepsilon)\,\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)\log\log n \tag{2.1}$$

holds uniformly in k.

This is an easy consequence of the fact that the number of $n \leqq x$ having exactly r different prime factors is

$$(1+o(1))\frac{x}{x_1}\frac{x_2^{r-1}}{(r-1)!}.$$

I think that

$$o_k(n) \leqq (1+\varepsilon)\,\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)\log\log n$$

holds for almost all n and every k.

3. Finally we scetch the proof of the following

Theorem 3. *Let* $k = x_1^{2\log 2-1}\cdot e^{h(x)\sqrt{x_2}}$, *where* $h(x)\to\infty$, $h(x)=o(\sqrt{x_2})$. *Then*

$$O_k(n) = 2(1+o(1))x_2 \tag{3.1}$$

for all $n \leqq x$ *except* $o(x)$ *of them.*

We need only to prove that

$$O_k(n) \geqq 2(1-\varepsilon)x_2. \tag{3.2}$$

To prove this we use a result due to P. ERDŐS [1]. Let

$$A(n) = \sum_{i=1}^{k}{}_1\, d(n+i), \tag{3.3}$$

Σ_1 denotes that we sum over those integers for which $\Omega(n+i) \leqq 2x_2 + \frac{1}{2}h(x)\sqrt{x_2}$. Then

$$\sum_{n\leqq x}(A(n)-kx_1)^2 = o(xk^2x_1^2). \tag{3.4}$$

We split the sum $A(n)$ into two parts

$$A(n) = A_1(n)+A_2(n),$$

where in

$$A_1(n) \quad \Omega(n)-\omega(n) \leqq \eta x_2, \quad \text{and in} \quad A_2(n) \quad \Omega(n)-\omega(n) > \eta x_2.$$

We can prove easily that

$$\sum_{n\leqq x} A_2^2(n) = o(xk^2x_1^2),$$

whence by (3.4)

$$\sum_{n\leqq x}(A_1(n)-kx_1)^2 = o(xk^2x_1^2) \tag{3.5}$$

follows.

Let now

$$B(n) = \sum_{i=1}^{k}{}_1\, d(n+i), \quad \omega(n+i) < (2-\varepsilon)x_2.$$

Similarly, we can prove easily that

$$\sum_{n \leq x} B^2(n) = o(xk^2 x_1^2).$$

Hence by (3. 5) we have

$$A_1(n) - B(n) = \big(1+o(1)\big)kx_1 > 0$$

for almost all n.

Observing that

$$A_1(n) - B(n) = \sum_{i=1}^{k}{}_1\, d(n+i), \quad \omega(n+i) > (2-\varepsilon)x_2$$

our theorem follows immediately.

References

[1] P. ERDŐS, Asymtotische Untersuchungen über die Anzahl der Teiler von *n*., *Math. Ann.* **169** (1967), 230—238.

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