A note on an ideal quasi-order in semigroups

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Let S be a semigroup. Denote by $L(x)$, $R(x)$ and $M(x)$ the principal left, right and two-sided ideal of S generated by $x \in S$, respectively. We shall say that S *has property* (L), (R) and (M) if the relation

(L) $$L(x)=L(y) \Rightarrow x=y,$$

(R) $$R(x)=R(y) \Rightarrow x=y \quad \text{and}$$

(M) $$M(x)=M(y) \Rightarrow x=y$$

holds for every $x, y \in S$, respectively. In [1] G. SZÁSZ studies semigroups having property (M). In this paper we shall consider semigroups having property (L) and (R), respectively. In fact, the second half of the paper deals mostly with semigroups having property (M).

Let E denote the set of all idempotents of S. If $e \in E$, then G_e denotes the maximal subgroup of S containing e, and K_e denotes the set of all elements x of S such that $x^n=e$ for some positive integer n. If S is a periodic semigroup, then $S = \bigcup_{e \in E} K_e$, $G_e \subset K_e$ and $eK_e=G_e=K_e e$. (See [2].)

I

Theorem 1. *A semigroup S has property* (L) *if and only if the following implication*

(1) $$x=abx \Rightarrow x=bx$$
holds for every $x, a, b \in S$.

PROOF. 1. Let S have property (L). If $x=abx$ for x, $a, b \in S$, then $x \in L(bx)$. Since $bx \in L(x)$, hence $L(x)=L(bx)$. Thus we have, by (L), $x=bx$.

2. Let the implication (1) hold for every $x, a, b \in S$. Suppose that S does not fulfill property (L). Then there exist $x, y \in S$ such that $L(x)=L(y)$ and $x \neq y$. Evidently $x \in L(y)$ and $y \in L(x)$. Then there exist $a, b \in S$ such that $x=ay$ and $y=bx$. From this it follows that $x=abx$. By (1) we obtain $x=bx=y$ which is a contradiction. Hence S has property (L).

Corollary. *If a semigroup S has property* (L), *then every subgroup of S consists of a single element.*

PROOF. Let G be a subgroup of S and let e be the identity of G. If $x \in G$, then $e = x^{-1}xe$. According to (1) we have $e = xe = x$.

Theorem 2. *The following conditions on a semigroup S are equivalent:*

1. *S is periodic with property* (L);
2. *S is periodic, every subgroup of S consists of a single element and for each $a, b \in S$ the implication*

$$ab \in K_e, \quad b \in K_f \ (e, f \in E) \Rightarrow e = fe \tag{2}$$

holds;

3. *for each $a, b \in S$ there exists a positive integer n such that*

$$(ab)^n = b(ab)^n. \tag{3}$$

PROOF. $1 \Rightarrow 2$. Let S be a periodic semigroup with property (L). It follows from the Corollary of Theorem 1 that every subgroup of S consists of a single element. Let $a, b \in S$ and $ab \in K_e$, $b \in K_f$ $(e, f \in E)$. Since card $G_e = 1$, hence $e = abe$. It follows from Theorem 1 that $e = be$. Since $f = b^n$ for some positive integer n, hence $e = be = b^2e = \dots = b^ne = fe$.

$2 \Rightarrow 3$. Let S be a periodic semigroup. Let every subgroup of S consist of a single element and let the implication (2) hold for each $a, b, \in S$. If $a, b \in S$, then $ab \in K_e$, $b \in K_f$ for some $e, f \in E$. There exists a positive integer n such that $(ab)^n = e$. Since card $G_f = 1$, hence $f = bf$ and thus, by (2), we have $(ab)^n = e = fe = bfe = be = b(ab)^n$.

$3 \Rightarrow 1$. Suppose that for each $a, b \in S$ there exists a positive integer n such that (3) holds. If $x \in S$, then it follows from (3) that $x^{2n} = x^{2n+1}$ for some positive integer n. This implies that S is a periodic semigroup. We shall prove that S has property (L). Let $x = abx$ for $x, a, b \in S$. By (3) there exists a positive integer n such that $(ab)^n = b(ab)^n$ and thus we have $x = abx = (ab)^2x = \dots = (ab)^nx = b(ab)^nx = bx$. From this and from Theorem 1 it follows that S satisfies property (L).

Corollary 1. *Let S be a periodic semigroup and let every subgroup of S consist of a single element. If $e = fe$ for each $e, f \in E$, then S has property* (L).

Corollary 2. *If S is a nilpotent semigroup, then S has property* (L).

A semigroup S is called *left weakly commutative* if for every $a, b \in S$ there exist $x \in S$ and a positive integer k such that $(ab)^k = bx$. (See [3].)

Corollary 3. *If S is a periodic semigroup having property* (L), *then S is left weakly commutative.*

Theorem 3. *Let S be an idempotent semigroup, i.e. $S = E$. Then the following conditions on S are equivalent:*

1. *S has property* (L);
2. *S is left weakly commutative;*
3. *for every $e, f \in S$, $ef = fef$.*

PROOF. $1 \Rightarrow 2$. This follows from the Corollary 3 of Theorem 2.

$2 \Rightarrow 3$. Let $e, f \in S$. Since S is left weakly commutative, hence $ef = fx = f^2x = fef$ for some $x \in S$.

$3 \Rightarrow 1$. This follows from Theorem 2.

Theorem 4. *Let S be a semigroup having property* (L). *Then the following conditions on S are equivalent:*

1. *S is left regular;*
2. *S is regular;*
3. *S is idempotent.*

PROOF. $1 \Rightarrow 3$. Let S be left regular and $x \in S$. Then there exists an $a \in S$ such that $x = ax^2$. By (1) we have $x = x^2$. This implies that S is idempotent.

$2 \Rightarrow 3$. Let S be regular and $x \in S$. Then there exists an $a \in S$ such that $x = xax$. By (1) we have $x = ax$, and thus $x = x^2$. This implies that S is idempotent.

$3 \Rightarrow 1$ and $3 \Rightarrow 2$. Evident.

Example 1. The following example shows that a right regular semigroup S having property (L) need not be necessarily idempotent.

Let N be a set with card $N = \aleph_0$. Denote by H the set of all mappings of N into itself and define a multiplication in H by the following rule: if $x, y \in H$, then $z = x \circ y$, where $z(n) = y(x(n))$ for all $n \in N$.

Let S be the set of all $x \in H$ such that

(i) $x(n) \neq x(m)$ for $n, m \in N$ $(n \neq m)$;

(ii) card $(N - x(N)) = \aleph_0$.

It is clear that S is a subsemigroup of H.

Lemma 1a. *If $x = y \circ x$ for $x \in S$, $y \in H$, then $y \notin S$.*

PROOF. Suppose that $x = y \circ x$ for $x, y \in S$. Then $N - y(N) \neq \emptyset$. If $k \in N - y(N)$, then $y(k) \neq k$. But, by $x = y \circ x$, $x(k) = x(y(k))$. This is clearly a contradiction.

Lemma 1b. *S is not idempotent.*

Lemma 1c. *S satisfies property* (L).

The PROOFS follow from Lemma 1a and Theorem 1.

Lemma 1d. *S is right regular.*

PROOF. Let $x \in S$. We put $x^2 = x \circ x$. Since card $(N - x(N)) = \aleph_0$, hence there exist two sets A, B such that $N - x(N) = A \cup B$, card $A = \aleph_0 =$ card B and $A \cap B = \emptyset$. Evidently card $(N - x^2(N)) = \aleph_0$. Let φ be a one-to-one mapping of $N - x^2(N)$ onto A. Put

$$y(n) = \begin{cases} x(k), & n \in x^2(N) \quad \text{and} \quad n = x^2(k); \\ \varphi(n), & n \in N - x^2(N). \end{cases}$$

It is easily checked that $y \in S$ and $x = x^2 \circ y$.

Remark. It follows from the dual of Theorem 4 that S does not comply with property (R).

II

Theorem 5. *A semigroup S has property* (M) *if and only if S has properties* (L), (R) *and if the following implication*

$$(4) \qquad x = axb \Rightarrow x = ax \quad \text{or} \quad x = xb$$

holds for every $x, a, b \in S$.

PROOF. 1. Let S have property (M). If $L(x)=L(y)$ for $x, y\in S$, then $x\in L(x)=L(y)\subset M(y)$ and $y\in L(y)=L(x)\subset M(x)$. This implies that $M(x)=M(y)$ and thus we have $x=y$. Therefore S has property (L). Similarly we obtain that S has property (R). Now, we shall prove (4). If $x=axb$ for $x, a, b\in S$, then $x\in M(ax)$ and $ax\in M(x)$. This implies that $M(x)=M(ax)$ and thus we have $x=ax$.

2. Let S have properties (L), (R) and let the implication (4) hold. Clearly the following implication

$$(4') \qquad x=axb \Rightarrow x=ax \quad \text{and} \quad x=xb$$

holds for every $x, a, b\in S$. Suppose that S does not satisfy property (M). Then there exist $x, y\in S$ such that $M(x)=M(y)$ and $x\neq y$. From this it follows that $L(x)\neq L(y)$ and $R(x)\neq R(y)$. Since $x\in M(y)$, hence either $x=ay$ or $x=yb$ or $x=ayb$ for some $a, b\in S$.

(i) Let $x=ay$. Since $y\in M(x)$, hence either $y=xd$ or $y=cxd$ for some $c, d\in S$. If $y=xd$, then $x=axd$. By (4′) we have $x=xd=y$ which is a contradiction. If $y=cxd$, then $x=acxd$. It follows from (4′) that $x=acx$ and $x=xd$. According to Theorem 1, we have $x=cx=cxd=y$ which is a contradiction.

(ii) Let $x=yb$. Similarly we get a contradiction.

(iii) Let $x=ayb$. Since $y\in M(x)$, hence either $y=cx$ or $y=xd$ or $y=cxd$ for some $c, d\in S$. If $y=cx$, then $x=acxb$. By (4′) we have $x=acx$. It follows from Theorem 1 that $x=cx=y$ which is a contradiction. If $y=xd$, then $x=axdb$. By (4′) we have $x=xdb$. According to the dual of Theorem 1, we obtain $x=xd=y$ which is a contradiction. If $y=cxd$, then $x=acxdb$. By (4′) we have $x=acx$ and $x=xdb$. It follows from Theorem 1 and from its dual that $x=cx$ and $x=xd$. This implies that $x=cx=cdx=y$ which is a contradiction.

This proves that S has property (M).

Corollary. *A periodic semigroup S has property* (M) *if and only if S has properties* (L) *and* (R).

PROOF. It suffices to prove (L) and (R)$\Rightarrow$(4). Let $x=axb$ for $x, a, b\in S$. It follows from the Corollary of Theorem 1 that every subgroup of S consists of a single element. There exists a positive integer n such that $a^n=e\in E$ and $b^n=f\in E$. Thus we have $x=axb=a^2xb^2=\dots=a^nxb^n=exf$. This implies that $x=exf=aexf=ax$.

Example 2. The following example shows that a nonperiodic semigroup S having properties (L) and (R) need not have necessarily property (M).

Let C be the set of all integers. Denote by H the set of all one-to-one mappings of $C\times C$ onto itself and define a multiplication in H as follows: let $x, y\in H$, then $z=x\circ y$, where $z(n, m)=y(x(n, m))$ for all $n, m\in C$. Evidently, H is a group.

Put

(i) $u(n, m)=(n+1, m)$ for $n, m\in C$;

(ii) $v(n, m)=(n, m+(-1)^n)$ for $n, m\in C$.

It is clear that $u, v\in H$. Let S be a subsemigroup of H generated by u, v, i.e.

$$x\in S \Leftrightarrow x=x_1\circ x_2\circ\dots\circ x_k,$$

where either $x_i=u$ or $x_i=v$. Since H is a group, evidently we have

Lemma 2a. *S is cancellative.*

Lemma 2b. *Let e denote the identical mapping on* $C \times C$. *Then* $e \in H - S$.

Proof. This is straightforward.

Lemma 2c. *S possesses the properties* (L) *and* (R).

Proof. If $x = a \circ b \circ x$ for some $x, a, b \in S$, then by Lemma 2a $a \circ b = e$ which contradicts Lemma 2b. According to Theorem 1, S has property (L). Similarly, we prove that S has property (R).

Lemma 2d. *S does not satisfy property* (M).

Proof. It is easily checked that $u = v \circ u \circ v$, $u \neq v \circ u$ and $u \neq u \circ v$. According to Theorem 5, S does not fulfill property (M).

Theorem 6. *The following conditions on a semigroup S are equivalent:*

1. *S is periodic with property* (M);
2. *S is periodic, every subgroup of S consists of a single element and for each* $a, b \in S$ *the implication*

$$ab \in K_e, \quad a \in K_g, \quad b \in K_f \ (e, f, g \in E) \Rightarrow fe = e = eg$$

holds;

3. *for each* $a, b \in S$ *there exists a positive integer n such that*

$$(ab)^n = b(ab)^n = (ab)^n a.$$

The proof follows from Theorem 2, its dual and from the Corollary of Theorem 5.

Corollary 1. (Cf. Satz 5 [1].) *If S is a nilpotent semigroup, then S has property* (M).

A semigroup S is called *weakly commutative* if for every $a, b \in S$ there exist $x, y \in S$ and a positive integer k such that $(ab)^k = bx = ya$. (See [4].)

Corollary 2. *If S is a periodic semigroup having property* (M), *then S is weakly commutative.*

Theorem 7. *An idempotent semigroup S has property* (M) *if and only if S is commutative.*

Proof. An immediate consequence of Theorem 3 and Corollary to Theorem 5.

Theorem 8. (Cf. Satz 3 [1].) *Let S be a semigroup having property* (M). *Then the following conditions on S are equivalent:*

1. *S is left regular;*
2. *S is right regular;*
3. *S is regular;*
4. *S is intraregular;*
5. *S is idempotent;*
6. *S is a semilattice.*

Proof. This follows from Theorem 4 and its dual, Theorem 5, the Corollary of Theorem 5 and Theorem 7.

Theorem 9. *A periodic commutative semigroup S has property* (M) *if and only if every subgroup of S consists of a single element.*

PROOF. 1. If S has property (M), then by Theorem 6 every subgroup of S consists of a single element.

2. Let every subgroup of S consist of a single element. If $a \in K_g$, $b \in K_f$ ($f, g \in E$), then $ab \in K_e$, where $e = fg$. Thus we have $fe = ffg = fg = e = fg = fgg = eg$. It follows from Theorem 6 that S has property (M).

References

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