A theorem on bounded analytic functions

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Introduction. Let $f(z) = a_1 z + a_2 z^2 + \cdots$, $0 < a_1 \leqq 1$, be analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfy $|f(z)| \leqq 1$ in E. Since $\operatorname{Re} f'(0) = a_1 > 0$, it follows that there exists a positive number r_0 such that $\operatorname{Re} f'(z) > 0$ whenever $|z| < r_0$. The purpose of this note is to determine the number r_0.

Theorem. *Let* $f(z) = a_1 z + a_2 z^2 + \ldots$, $0 < a_1 \leqq 1$, *be analytic in* E *and satisfy* $|f(z)| \leqq 1$ *for all* z *in* E. *Then* $\operatorname{Re} f'(z) > 0$ *in* $|z| < r_0$, *where*

$$r_0 = \frac{a_1}{1 + \sqrt{1 - a_1^2}}.$$

The number r_0 *cannot be replaced by a larger one.*

PROOF. Define

$$F(z) = \frac{\dfrac{f(z)}{z} - a_1}{1 - a_1 \dfrac{f(z)}{z}}.$$

Then $F(z)$ is analytic in E, $F(0) = 0$ and $|F(z)| \leqq 1$ in E. Therefore, by Schwarz' Lemma it follows that

$$\left| \frac{\dfrac{f(z)}{z} - a_1}{1 - a_1 \dfrac{f(z)}{z}} \right| \leqq |z| \tag{1}$$

for all $z \in E$.

Squaring both the sides of (1) we obtain

$$\left| \frac{f(z)}{z} - a_1 \right|^2 = \left| \frac{f(z)}{z} \right|^2 - 2a_1 \operatorname{Re} \frac{f(z)}{z} + a_1^2 \leqq$$

$$\leqq r^2 \left[1 - 2a_1 \operatorname{Re} \frac{f(z)}{z} + a_1^2 \left| \frac{f(z)}{z} \right|^2 \right], \qquad |z| = r.$$

Or

$$(1-a_1^2r^2)\left|\frac{f(z)}{z}\right|^2-(1-r^2)2a_1\,\mathrm{Re}\,\frac{f(z)}{z}\leqq r^2-a_1^2.$$

Or

$$\left|\frac{f(z)}{z}\right|^2-\frac{(1-r^2)}{1-a_1^2r^2}\,2a_1\,Re\left\{\frac{f(z)}{z}\right\}+\frac{(1-r^2)^2a_1^2}{(1-a_1^2r^2)}\leqq\frac{r^2-a_1^2}{1-a_1^2r^2}+\frac{(1-r^2)^2a_1^2}{(1-a_1^2r^2)^2}.$$

I.e.

$$\left|\frac{f(z)}{z}-\frac{(1-r^2)}{1-a_1^2r^2}\,a_1\right|^2\leqq\frac{r^2(1-a_1^2)^2}{(1-a_1^2r^2)^2},$$

or

(2) $$\left|\frac{f(z)}{z}-\frac{(1-r^2)a_1}{1-a_1^2r^2}\right|\leqq\frac{r(1-a_1^2)}{(1-a_1^2r^2)}.$$

Put

(3) $$G(z)=\frac{f(z)}{z}.$$

Then $G(z)$ is analytic in E. Also, since $|f(z)|\leqq|z|$ ($f(z)$ satisfies the hypotheses of Schwarz's Lemma) in E, it follows that $|G(z)|\leqq 1$ in E. Also we have

(4) $$|G'(z)|\leqq\frac{1-|G(z)|^2}{1-r^2}$$

([1], p. 168, equ. (28)) for all z in E.

Differentiating (3) and making use of (4) we obtain the following inequality:

(5) $$\left|f'(z)-\frac{f(z)}{z}\right|\leqq\frac{r\left[1-\left|\frac{f(z)}{z}\right|^2\right]}{1-r^2}.$$

From (5) we see that $f'(z)$ lies in the circle whose centre is at $f(z)/z$ and whose radius is equal to $r\{1-|f(z)/z|^2\}/(1-r^2)$. Therefore, we conclude that

$$\mathrm{Re}\,f'(z)\geqq\mathrm{Re}\,\frac{f(z)}{z}-r\,\frac{\left[1-\left|\frac{f(z)}{z}\right|^2\right]}{1-r^2}=\frac{r}{(1-r^2)}\left[\left|\frac{f(z)}{z}\right|^2+\frac{1-r^2}{r}\,\mathrm{Re}\,\frac{f(z)}{z}-1\right]=$$

(6) $$=\frac{r}{(1-r^2)}\left[\left|\frac{f(z)}{z}\right|^2+\frac{1-r^2}{r}\,\mathrm{Re}\,\frac{f(z)}{z}+\frac{(1-r^2)^2}{4r^2}-1-\frac{(1-r^2)^2}{4r^2}\right]=$$

$$=\frac{r}{(1-r^2)}\left[\left|\frac{f(z)}{z}+\frac{1-r^2}{2r}\right|^2-\frac{(1+r^2)}{4r^2}\right].$$

Now, from (2) we have

$$\left|\left\{\frac{f(z)}{z}+\frac{1-r^2}{2r}\right\}-\left\{\frac{a_1(1-r^2)}{1-a_1^2r^2}+\frac{1-r^2}{2r}\right\}\right|\leqq\frac{r(1-a_1^2)^2}{1-a_1^2r^2},$$

from which it follows that

(7)

$$\left|\frac{f(z)}{z}+\frac{1-r^2}{2r}\right|^2 \geqq \left[\frac{a_1(1-r^2)}{1-a_1^2r^2}+\frac{1-r^2}{2r}-\frac{r(1-a_1^2)}{1-a_1^2r^2}\right]^2=\frac{(1+a_1r-3r^2+a_1r^3)^2}{4r^2(1-a_1r)^2}.$$

Making use of (7) in (6) we conclude that $\mathrm{Re}\, f'(z)>0$ provided

$$\frac{r}{(1-r^2)}\left[\frac{(1+a_1r-3r^2+a_1r^3)^2}{4r^2(1-a_1r)^2}-\frac{(1+r^2)^2}{4r^2}\right]>0,$$

or

$$\frac{a_1r^2-2r+a_1}{(1-a_1r)^2}>0,$$

i.e.

$$r<\frac{a_1}{1+\sqrt{1-a_1^2}}=r_0.$$

To show that our result is sharp, we consider the function

$$f_0(z)=\frac{z(a_1-z)}{1-a_1z}, \qquad 0<a\leqq 1,$$

which satisfies the hypotheses of our theorem. A little compulation revelas that $f_0'(z)=0$ when $z=r_0$. This shows that the number r_0 cannot be replaced by any larger one. The theorem is therefore proved.

Now it is known ([2], [3]) that if $F(z)$ is regular in a convex domain D and if $\mathrm{Re}\, F'(z)>0$ in D then $F(z)$ is univalent in D. We, therefore, have the following corollary:

Corollary: *If $f(z)$ satisfies the hypotheses of our theorem then $f(z)$ is univalent in $|z|<r_0$.*

References

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[2] K. Noshiro, On the theory of schlicht functions, *J. Fac. of Sci., Hokkaido Imperial Univ., Sapparo*, **2** (1934—35), 129—155.
[3] S. E. Warschawski, On the higher derivatives at the boundary in conformal mapping *Trans. Amer. Math. Soc.* **38** (1935), 31—340.

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