Publ. Math. Debrecen
49 / 1-2 (1996), 157–165

Free subgroups of the group of units in group algebras

By A. A. BOVDI (Debrecen)

Abstract. We give necessary conditions under which the group of units of a group algebra over a field does not contain a free subgroup of rank 2 and these conditions with some restriction are sufficient.

Introduction

Let K be a commutive ring and $t(G)$ the set of torsion elements of G. The following problem of Hartley is a very interesting one:

When does the group of units $U(KG)$ of a group ring KG not contain a free group of rank 2?

The first result has been obtained by B. Hartley and P. Pickel [8]:

Let G be a solvable-by-finite group and suppose that $U(\mathbb{Z}G)$ does not contain a free group of rank two. Then $t(G)$ is an abelian group or a hamiltonian 2-group and every subgroup of $t(G)$ is normal in G.

We would like to deal with this problem for the group of units $U(KG)$ of a group algebra KG. J. Z. Gonsalves [5] gave necessary and sufficient conditions for this problem in case G is finite or some infinite solvable group [6, 7]. We extend this result and generalize Gonsalves' theorems.

We now define for an arbitary group G the normal subgroup

$$\Lambda(G) = \big\{ g \in G \mid [H : C_H(g)] < \infty \quad \text{for every finitely generated subgroup } H \text{ of } G \big\}.$$

Mathematics Subject Classification: Primary 16S34; Secondary 16U60, 20C07.
Key words and phrases: group ring, unit group, free group.
Research was supported by Hungarian National Fund for Scientific Research grant No. T014279.

Of course, the torsion part $\Lambda^+(G)$ of $\Lambda(G)$ is a normal subgroup and $\Lambda(G)/\Lambda^+(G)$ is a torsion free abelian group [10].

Theorem 1. *Let K be a field of characteristic 0 or p and suppose that $U(KG)$ does not contain a free subgroup of rank two. Then one of following conditions holds:*

1. *G is abelian;*
2. *G is a torsion group and K is algebraic over its prime field $\mathbb{F}_p$;*
3. *K is a field of characteristic 0 and*
   a. *$\Lambda^+(G)$ is an abelian subgroup and each of its subgroups is normal in G;*
   b. *the centralizer $C_G(\Lambda^+(G))$ contains all elements of finite order of G;*
   c. *for every $a \in \Lambda^+(G)$, which is not central in G, K contains no root of unity of order equal to the order of a;*
4. *K is a field of characteristic p and K is not algebraic over its prime field $\mathbb{F}_p$ and*
   a. *the p-Sylow subgroup P of $\Lambda^+(G)$ is normal in G and $A = \Lambda^+(G)/P$ is abelian group;*
   b. *the centralizer $C_{G/P}(A)$ contains all elements of finite order of G/P;*
   c. *if A is noncentral in G/P and G/P is non-torsion, then the algebraic closure L of $\mathbb{F}_p$ in K is finite and for all $g \in G/P$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r satisfies that $[L : \mathbb{F}_p]$ divides r.*
5. *G is not a torsion group, K is algebraic over its prime field $\mathbb{F}_p$ and*
   a. *the p-Sylow subgroup P of $\Lambda^+(G)$ is normal in G and $A = \Lambda^+(G)/P$ is an abelian group;*
   b. *if A is noncentral in G/P then the algebraic closure L of $\mathbb{F}_p$ in K is finite and for all elements g of infinite order in G/P and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r satisfies that $[L : \mathbb{F}_p]$ divides r.*

Corollary 1. *Let K be a field of characteristic 0 or p and G a group such that $t(G) = \Lambda^+(G)$ and $G/t(G)$ is a unique product group. Then $U(KG)$ does not contain a free group of rank two if and only if G does not contain a free group of rank two and one of the following statements holds:*

1. *G is abelian;*
2. *G is a torsion group and K is algebraic over its prime field $\mathbb{F}_p$;*

3. K is a field of characteristic 0 and
  - *a. $t(G)$ is an abelian subgroup and each of its subgroups is normal in G;*
  - *b. for every $a \in t(G)$, which is not central in G, K contains no root of unity of order equal to the order of a;*

4. K is a field of characteristic p and
  - *a. the p-Sylow subgroup P of $t(G)$ is normal in G and $A = t(G)/P$ is an abelian group;*
  - *b. if A is noncentral in G/P and G/P is non-torsion, then the algebraic closure L of $\mathbb{F}_p$ in K is finite and for all $g \in G/P$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r satisfies that $[L : \mathbb{F}_p]$ divides r.*

Corollary 2. *Let K be a field of characteristic 0 or p and G a solvable group such that $t(G) = \Lambda^+(G)$ and $G/t(G)$ is a unique product group. Then $U(KG)$ either contains a free group of rank two or $U(KG)$ has a normal p-subgroup N such that the factorgroup $U(KG)/N$ is a solvable group.*

Clearly, if G is a locally nilpotent group, then $t(G) = \Lambda^+(G)$. If $U(KG)$ does not contain a free group of rank two and $U(K)$ has an element of infinite order, we propose that $t(G) = \Lambda^+(G)$. The last question is very difficult and was answered affirmatively if

1. (Hartley and Pickel [8]) *K is a field of characteristic 0 and G is a solvable-by-finite group;*

2. (Gonsalves [7]) *G is a solvable-by-finite group without p-elements, K is a field of characteristic p not algebraic over its prime subfield $\mathbb{F}_p$, and if $p = 2$ then the degree of transcendence of K over $\mathbb{F}_2$ is at least 2.*

Bist Vikas [2] obtained a necessary and sufficient condition for the commutator subgroup of the group of units $U(KG)$ of group algebras to be torsion if G is a locally finite or a locally FC-group. As a consequence of Theorem 1 we have also the following result.

Corollary 3. *Let K be a field of characteristic 0 or p and G a group such that $t(G) = \Lambda^+(G)$. Then the derived subgroup of $U(KG)$ is torsion if and only if the derived subgroup of G is torsion and one of the following conditions hold:*

1. G is abelian;

2. G is a torsion group and K is algebraic over its prime subfield $\mathbb{F}_p$;

3. K is a field of characteristic 0 and $t(G)$ is a central subgroup of G;

4. K is a field of characteristic p, the p-Sylow subgroup P of $t(G)$ is normal in G and $A = t(G)/P$ is abelian group;
   a. A is a central subgroup of G/P;
   b. if A is noncentral in G/P and G/P is non-torsion, then K is finite and for all $g \in G/P$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r satisfies that $[K : \mathbb{F}_p]$ divides r.

We wish to prove Theorem 1, Corollary 1 and 2. For this we need the following statments, which are well-known.

Lemma 1. *1.1. Suppose that the characteristic of the field K does not divide the order of the finite abelian subgroup A of G and the element $g \in N_G(A)$ does not commute with a primitive idempotent e of the group algebra KA. Then the elements $e_{11} = e, e_{12} = eg, e_{21} = g^{-1}e, e_{22} = g^{-1}eg$ are matrix units. Let $f = e_{11} + e_{22}$ and*

$$W = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL(2, K).$$

Then

$$w = 1 - f + a_1e_{11} + a_2e_{12} + a_3e_{21} + a_4e_{22} \in U(KG)$$

and the map $W \to w$ is a monomorphism of $GL(2, K)$ into $U(KG)$.

1.2. [12]. *If the characteristic of the field K is zero and $n > 1$ is an integer then the matrices*

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

generate in GL(2,K) a free subgroup of rank 2;

1.3. [1]. *Let K be any commutative ring and $G = \langle u \rangle$ an infinite cyclic group. Then the matrices*

$$A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}; \quad P = \begin{pmatrix} 1+u & u \\ -u & 1-u \end{pmatrix}$$

are invertible over the group ring KG. The matrices A and $B = PAP^{-1}$ are free generators of a noncyclic free group;

1.4. If the characteristic of the field K is p, u is transcendental over the prime field and

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}; \quad P = \begin{pmatrix} 1+u & u \\ -u & 1-u \end{pmatrix}$$

then $\langle A, PBP^{-1} \rangle$ is a non-cyclic free subgroup of $GL(2, K)$.

Lemma 2 (Passman [10]). *Let the Sylow p-subgroup P of $\Lambda^+(G)$ be normal in G. Then $\Lambda^+(G)/P = \Lambda^+(G/P)$ and the ideal $\mathcal{I}(P)$ generated by all elements of form $h-1$ $(h \in P)$ in KG is a nilideal.*

Proof of Theorem 1. Let $U(KG)$ not contain noncyclic free subgroups. It is known [9] that $\Lambda^+(G)$ is a locally finite group. Let us further suppose that if K is algebraic over its prime field $\mathbb{F}_p$ then G is a non-torsion group.

We shall prove that

1. *$\Lambda^+(G)$ is either abelian or, if p is the characteristic of K, the derived subgroup of $\Lambda^+(G)$ is p-group and*

2. *every idempotent of $K\Lambda^+(G)$ commute with elements of infinite order of $G \mod \mathcal{I}(P)$, where P is the Sylow p-subgroup P of $\Lambda^+(G)$ and $P = 1$, if K is a field of characreristic 0.*

First we consider the case, when the group of units $U(K)$ of K contains an element of infinite order.

Let H be a finite subgroup of $\Lambda^+(G)$ and I a maximal nilpotent ideal of KH. By the Artin-Wedderburn theorem

$$KH/I \cong D^{(1)}_{n_1} \oplus D^{(2)}_{n_2} \oplus \cdots \oplus D^{(s)}_{n_s},$$

where $D^{(i)}_{n_i}$ is the ring of all $n_i \times n_i$ matrices over the division ring $D^{(i)}$. Thus

$$U(KH)/1+I \cong GL(n_1, D^{(1)}) \times GL(n_2, D^{(2)}) \times \cdots \times GL(n_s, D^{(s)}).$$

Clearly, $D^{(i)}$ is finite dimensional over the centre, which contains a subfield K with an element of infinite order. If $D^{(i)}$ is a noncommutative division ring, then by Tits' theorem [11] easy to see [5] that the group of units of $D^{(i)}$ has a non-cyclic free subgroup. Since $1+I$ is a p-group and the group $U(KH)/1+I$ does not contain noncyclic free subgroups, we conclude that $D^{(i)}$ is a field with an element of infinite order and by Lemma 1.2 and 1.4 $n_i = 1$ for all $i = 1, 2, \dots, s$. Thus KH/I is a commutative ring, which contains no nilpotent elements.

Consequently, if the characteristic of K does not divide the order of the group H then H is an abelian group, and in the opposite case if K is of characteristic p and H contains p elements then $H \cap (1+I)$ is a p-group and the group $H/H \cap (1+I)$ is abelian and has no p-elements. Indeed, if $g \in H \setminus (H \cap (1+I))$ is a p-element then $g - 1 + I$ is a nilpotent element of KH/I, which is a contradiction. Thus we proved that $\Lambda^+(G)$ is either abelian or the characteristic of K is p and the derived subgroup of $\Lambda^+(G)$ is a p-group. It follows that the Sylow p-subgroup P of $\Lambda^+(G)$ is a normal subgroup of G. By Lemma 2 the ideal $\mathcal{I}(P)$ of KG generated by elements of form $h-1$ with $h \in P$ is a nilideal and, clearly,

$$U(KG)/1+\mathcal{I}(P) \cong U(KG/P).$$

By Lemma 2 $\Lambda^+(G)/P = \Lambda^+(G/P)$ and we conclude that $\Lambda^+(G/P)$ is abelian and $U(KG/P)$ does not contain a non-cyclic free subgroup.

Let K be a field of characteristic 0 or p. We put $P=1$ if the characteristic of K is 0 and in this case $\mathcal{I}(P)=0$. We shall prove below that every idempotent of $K\Lambda^+(G)$ commutes with elements of infinite order of G modulo $\mathcal{I}(P)$. Suppose that for the idempotent $e = a_1g_1 + a_2g_2 + \cdots + a_sg_s$ of $K\Lambda^+(G/P)$ we have $eg \neq ge$ for some $g \in G$. Clearly, $C = \langle g^m g_i g^{-m} \mid m \in \mathbb{Z},\ (i = 1,2,\ldots,s)\rangle$ is a finite abelian subgroup in $K\Lambda^+(G/P)$ and $g \in N_{G/P}(C)$. Since $\Lambda^+(G/P)$ has no p-elements in case of characteristic p, there exists a primitive idempotent which does not commute with g, and by Lemma 1.1 $GL(2,K)$ is isomorphic to a subgroup of $U(KG)$. This is impossible by Lemma 1.2 and 1.4, because K has an element of infinite order. Therefore all idempotents of $K\Lambda^+(G/P)$ are central in KG/P.

Let the field K be algebraic over its prime subfield $\mathbb{F}_p$ of characteristic p, and assume that G is not a torsion group. Suppose further that H is a finite subgroup of $\Lambda^+(G)$ such that $\mathbb{F}_pH/I$ is a non-commutative ring, where I is a maximal nilpotent ideal of KH. Since a finite division ring is a field, there exists a two-sided ideal J of LH such that $J+I/I \cong L_n$, the ring of all $n \times n$ matrices over the finite field L, and $n > 1$. It follows that there exist matrix units $e_{11}, e_{12}, e_{21}, e_{22}$ in KH [[9], 3.8.1 Theorem]. Clearly, $\operatorname{Supp}(e_{ij}) \subseteq \Lambda^+(G)$ and by definition of $\Lambda^+(G)$ we get an element $u \in G$ of infinite order such that $ue_{ij} = e_{ij}u$ for all i,j. Let $f = e_{11} + e_{22}$ and

$$w = 1 - f + (1+u)e_{11} + ue_{12} - ue_{21} + (1-u)e_{22} \in U(KG)$$
$$v = 1 - f + ue_{11} + u^{-1}e_{22} \in U(KG)$$

By Lemma 1.3 $\langle v, w\rangle$ is a non-cyclic free group and this forces a contradiction.

Now put $P=1$ if the characteristic of K is 0 and assume that K contains an element of infinite order. Let q be a prime number and suppose that the q-element $c \in G/P$ does not belong to $C_{G/P}(A)$. We choose some element $h \in A$ such that $(c,h) \neq 1$. Since $h \in \Lambda^+(G/P)$, thus the subgroup $H = \langle h, c\rangle$ is a finite and nonabelian. Clearly, the subgroup $H \cap \Lambda^+(G/P)$ is normal in H. Since K contains an element of infinite order and the subgroup $U(KH)$ has no non-cyclic free subgroups, by the facts proved above this leads to $q = p$ and the Sylow p-subgroup of H is normal in H. Thus H is abelian, which is impossible. Therefore, the centralizer $C_{G/P}(A)$ contains all elements of finite order of G/P. As we have seen above, the Sylow p-subgroup P of $\Lambda^+(G)$ is normal in G, the factor group $A = \Lambda^+(G)/P$ is abelian and every idempotent of $K\Lambda^+(G)$ commutes with elements of infinite order of G modulo $\mathcal{I}(P)$. It implies that all idempotents of KA are central in KG/P, we can construct for every element $a \in A$ of order n the idempotent $e = \frac{1}{n}\sum_{i=1}^{n} a^i$, which is central. Then $ge = eg$ for all $g \in G/P$ and this follows that $\langle a \rangle$ is normal in G/P.

Suppose that $a \in A$ is not central in G/P and K contains the root of unity ζ of order equal to the order of a. Then $g^{-1}ag = a^k \neq a$ and the idempotent $e = \frac{1}{n}\sum_{i=1}^{n} \zeta^i a^i$ satisfies the condition $ge \neq eg$, which is a contradiction.

Let A be noncentral in the non-torsion group G/P and K algebraic over its prime subfield $\mathbb{F}_p$ of characteristic p. Then every idempotent of KA commutes with elements of infinite order of G. If g is an element of infinite order of G/P, then we apply Coelho's theorem [3] for the group algebra $K\langle g, \Lambda^+(G/P)\rangle$ and the Conditions 4.c and 5.b of Theorem 1 holds. □

Proof of Corollary 1. It is easy to see that there remained to prove sufficiency of these conditions. Let us first assume that K is a field of characteristic p. Let $\mathcal{I}(P)$ be the ideal of KG generated by elements of form $h - 1$ with $h \in P$ and let $\bar{G} = G/P$. By Lemma 3 $\mathcal{I}(P)$ is a nilideal,

$$U(KG)/1 + \mathcal{I}(P) \cong U(K\bar{G}) \tag{2}$$

and $\Lambda^+(G)/P = \Lambda^+(\bar{G})$. It implies that $\Lambda^+(\bar{G})$ is an abelian group, $t(\bar{G}) = \Lambda^+(\bar{G})$ and $\bar{G}/\Lambda^+(\bar{G})$ are unique product groups. Since $1 + \mathcal{I}(P)$ is a p-group, it is enough to prove that $U(K\bar{G})$ does not contain a free group of rank two.

We shall suppose below that K is a field of characteristic p or 0 and $t(G) = \Lambda^+(G)$ is an abelian group such that if K is a field of characteristic p then $\Lambda^+(G)$ has no p-elements. Clearly, KG is isomorphic to the crossed product S of $G/t(G)$ and $Kt(G)$.

Let $\{u_h \mid h \in G/t(G)\}$ be a $Kt(G)$-basis of S and $u_{h_1}u_{h_2} = u_{h_1h_2}\lambda_{h_1,h_2}$, where $\lambda_{h_1,h_2} \in U(Kt(G)$. Then the units x_i $(i = 1, 2, \dots, n)$ are in S and the elements x_i, x_i^{-1} can be expressed as

$$x_i = \sum_{h\in G/t(G)} t_h\alpha_h^{(i)}, \quad x_i^{-1} = \sum_{h\in G/t(G)} t_h\beta_h^{(i)},$$

where $\alpha_h^{(i)}, \beta_h^{(j)} \in Kt(G)$. Clearly, the support subgroup L of the elements $\{\alpha_h^{(i)}, \beta_h^{(j)} \mid i = 1, 2, \dots, n, h \in G/t(G)\}$ is a finite abelian subgroup of G. By the theorem of Coelho and Polcino Milies [4] all idempotens of $Kt(G)$ are central in KG. Since the idempotent $\frac{1}{|L|}\sum_{h\in L} h$ is central, the subgoup L is a normal subgroup in G and KL is a semisimple algebra. Thus KL contains the orthogonal primitive idempotens $e_1, e_2, \dots, e_m$ such that $e_1 + e_2 + \cdots + e_m = 1$ and KLe_i is a field. It is easy to see that KLe_i is invariant under transformation with the elements $u_g\,(g \in G)$ and $\alpha_h^{(j)}e_i, \beta_h^{(j)}e_i \in KLe_i$. Since by assumption $G/t(G)$ is a unique product group, the equality $(x_je_i)(x_j^{-1}e_i) = e_i$ gives $x_je_i = g_{ij}a_{ij}e_i$ and $x_i^{-1}e_i = g_{ij}^{-1}a_{ij}^{-1}e_i$, where $g_{ij} \in G$ and $a_{ij} \in U(KLe_i)$. It follows

$$x_j = \sum_{i=1}^{m} g_{ij}a_{ij}e_i, \quad x_j^{-1} = \sum_{i=1}^{m} g_{ij}^{-1}a_{ij}^{-1}e_i. \tag{3}$$

Clearly, $U(KLe_i)$ is a normal subgroup of $M_i = G\cdot U(KLe_i)$ and M_i does not contain a free group of rank two. There exists a monomorphism of $\langle x_1, x_2\rangle$ into the direct product $M = M_1 \times M_2 \times \cdots \times M_m$. It is easy to see that the direct product of groups, which does not contain free subgroups of rank two, contains also no noncyclic free subgroups. Therefore, the subgroup $\langle x_1, x_2\rangle$ and also the group of units $U(KG)$ does not contain a noncyclic free subgroup. □

Proof of Corollary 2. Suppose that $U(KG)$ does not contain a free group of rank two. As in the proof of Corollary 1, if K is a field of characteristic p then $1 + \mathcal{I}(P)$ is a p-group. By (2) we may only consider the case, when K is a field of characteristic p or 0 and $t(G) = \Lambda^+(G)$ is an abelian group such that if K is a field of characteristic p then $\Lambda^+(G)$ has no p-elements. We shall prove that $U(KG)$ is a solvable group.

Let $n = 2^{t+1}$ and $x_1, x_2, \dots, x_n \in U(KG)$. As in the proof of Corollary 1, there exists a normal abelian subgroup L and every element x_i can be represented in the form (3). Let us define inductively

$$(x_1, x_2)^\circ = (x_1, x_2) = x_1^{-1}x_2^{-1}x_1x_2$$
$$(x_1, x_2, x_3, x_4)^\circ = ((x_1, x_2)^\circ, (x_3, x_4)^\circ$$

and if $s = 2^t$, then

$$(x_1, x_2, \dots, x_n)^\circ = ((x_1, x_2, \dots, x_s)^\circ, (x_1, x_2, \dots, x_s)^\circ).$$

Since L is a normal abelian subgroup, it is easy to show that if $a, b \in U(KLe_i)$ and $g, h \in G$ then $(ag, bh) = c(g, h)$ for some $c \in U(KLe_i)$. If $t+1$ is derived length of G, then

$$\begin{aligned}(x_1, x_2, \dots, x_n)^\circ &= \sum_{i=1}^{m} (g_{i1}a_{i1}, g_{i2}a_{i2}, \dots, g_{in}a_{in})^\circ e_i \\ &= \sum_{i=1}^{m} c_i (g_{i1}, g_{i2}, \dots, g_{in})^\circ e_i\end{aligned}$$

for some $c_i \in U(KLe_i)$ and $(g_{i1}, g_{i2}, \dots, g_{in})^\circ = 1$.

We proved that $(x_1, x_2, \dots, x_n)^\circ \in U(KL)$ and this implies the solvability of $U(KG)$. □

References

[1] A. A. Bovdi, On group algebras with solvable unit groups, *Contemporary Math.* **131** (1992 (Part 1)), 81–90.
[2] Vikas Bist, Groups of units of group algebras, *Commun. in Algebra* **20 (6)** (1992), 1747–1761.
[3] S. P. Coelho, A note on central idempotents in group rings, *Proc. Edinburgh Math. Soc.* **30** (1987), 69–72.
[4] S. P. Coelho and C. Polcino Milies, A note on central idempotents in group rings, II, *Proc. Edinburgh Math. Soc.* **31** (1988), 211–215.
[5] J. Z. Gonsalves, Free subgroups of units in group rings, *Bull. Can. Math. Soc.* **27 (3)** (1984), 309–312.
[6] J. Z. Gonsalves, Free subgroups in the group of units of group rings, II, *J. Number Theory* **21 (2)** (1985), 121–127.
[7] J. Z. Gonsalves, Free subgroups and the residual nilpotence of the group of units of modular and p-adic group rings, *Bull. Can. Math. Soc.* **29 (3)** (1986), 321–328.
[8] B. Hartly and P. F. Pickel, Free subgroups in the unit groups of integral group rings, *Can. J. Math.* **32 (6)** (1980), 1342–1352.
[9] N. Jacobson, Structure of rings, Vol. 36, *Amer. Math. Soc. Colloquium*, *Providence, R.I.*, 1964.
[10] D. S. Passman, A new radical for group rings?, *J. Algebra* **28 (3)** (1974), 556–572.
[11] J. Tits, Free subgroups in linear groups, *J. Algebra* **20** (1972), 250–270.
[12] B. A. F. Wehrfritz, Infinite linear groups, *Springer Verlag*, *Berlin*, 1973.

ADALBERT BOVDI
INSTITUTE OF MATHEMATICS,
KOSSUTH UNIVERSITY,
H–4010 DEBRECEN, PF. 12,
HUNGARY
E-mail: bodibela@tigris.klte.hu

(Received October 31, 1995)