Symmetric generalized topological structures I

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PREFACE

This paper is based on M. W. LODATO's Ph. D. thesis: *On topologically induced generalized proximity relations*, the author's Ph. D. thesis: *Symmetric generalized uniform and proximity spaces*, the paper: *Test for correctness of a uniform space* by A. G. MORDKOVIČ, and the paper: *Correct spaces* by V. A. EFREMOVIČ, A. G. MORDKOVIČ, and V. JU. SANDBERG.

In Lodato's thesis the notion of an ordinary proximity space is generalized in such a way as to obtain a partial generalization of the classical theorem which states that every separated proximity space (X, δ) is a dense subspace of a unique compact T_2 space $(Y, \mathfrak{I})$ such that $(A\delta B)$ in X iff $\bar{A} \cap \bar{B} \neq \emptyset$ where the closure is taken in Y.

In the author's thesis the notion of an ordinary uniform space is generalized in such a way that every such uniformity generates a Lodato generalized proximity in a natural way. It is then shown that the classical theorem which states that every proximity class of ordinary uniformities contains one and only one totally bounded uniformity can be generalized to these generalized uniform and proximity spaces in such a way that the classical theorem follows as an immediate corollary. Generalizations and partial generalizations are also obtained for many other classical theorems concerning uniform continuity, p-continuity, uniform convergence, convergence in proximity, completeness, and compactness.

The correct spaces of Efremovič, Mordkovič, and Sandberg are a special subclass of the author's generalized uniform spaces. The axioms for a correct space are almost as strong as those for an ordinary uniform space. It can be shown that every correct space has a completion.

In the last chapter we introduce the concept of a symmetric generalized topological group and investigate the relationship between these groups and symmetric generalized uniform spaces.

Part II of this paper will appear in a subsequent issue of this journal.

NOTATION

Let X be a set.
$P(X)$ denotes the power set of X.
$P(X \times X)$ denotes the power set of $(X \times X)$.
Δ denotes the set: $\{(x, x) | x \in X\}$.

Let $\mathcal{J}$ be a topology on X.
A° denotes the interior of A with respect to $\mathcal{J}$.
$\bar{A}$ denotes the closure of A with respect to $\mathcal{J}$.
Let $U \subset (X \times X)$. Let $A \subset X$ and let $x \in X$.
U^{-1} denotes the set: $\{(x, y)|(y, x) \in U\}$.
$U[x]$ denotes the set: $\{y|(x, y) \in U\}$.
$U[A]$ denotes the set: $\cup \{U[x]|x \in A\}$.
$\{x\}$ will be denoted simply as x.
Let δ be a relation on $P(X)$.
$(A, B) \in \delta$ is denoted $A\delta B$.
$(A, B) \notin \delta$ is denoted $A\bar{\delta}B$.
The expression "if and only if" will be abbreviated "iff".

I

Symmetric generalized proximity spaces

Let X be a non-void set. Let δ be a relation on $P(X)$ We write $A\delta B$ for $(A, B) \in \delta$ and $A\bar{\delta}B$ for $(A, B) \notin \delta$. Consider the following axioms:

(P. 1) $A\delta B$ implies $B\delta A$.

(P. 2) $C\delta(A \cup B)$ iff either $C\delta A$ or $C\delta B$.

(P. 3) $\varnothing\bar{\delta}A$ for every $A \subset X$.

(P. 4) $x\delta x$ for all $x \in X$.

(P. 4)′ $x\delta y$ implies $x = y$ for all x, y in X.

(P. 5) $A\delta B$ and $b\delta C$ for all $b \in B$ implies that $A\delta C$.

(P. 5)′ $A\bar{\delta}B$ implies the existence of C and D such that $C \cap D = \varnothing$, and $A\bar{\delta}(X - C)$ $B\bar{\delta}(X - D)$.

(1. 1) *Definition.* δ is a [separated] symmetric generalized proximity on X iff δ satisfies (P. 1), (P. 2), (P. 3), (P. 4), [(P. 4)′], and (P. 5).

(1. 2) *Definition.* δ is a [separated] proximity on X iff δ satisfies (P. 1), (P. 2), (P. 3), (P. 4), [(P. 4)′], and (P. 5)′.

(1. 3) *Remark.* If δ is a (symmetric generalized) proximity on X, then we call (X, δ) a (symmetric generalized) proximity space.

(1. 4) *Remark.* Our definition of a symmetric generalized proximity is the same as that given by S. Leader in [21].

(1. 5) *Lemma. Let (X, δ) be either a proximity space or a symmetric generalized proximity space. If $A \subset A_1 \subset X$ and $B \subset B_1 \subset X$ and $A\delta B$, then $A_1\delta B_1$.*

Proof. $A_1 = A \cup (A_1 - A)$; but by (P. 1) $B\delta A$. Hence by (P. 2) $B\delta A_1$; so that by (P. 1) $A_1\delta B$. By a similar argument we have $A_1\delta B_1$.

(1. 6) **Theorem.** *If (X, δ) is a proximity space, then (X, δ) is a symmetric generalized proximity space.*

Proof. The following is essentially the proof given by M. W. Lodato in [23]. It is sufficient to show that (P. 5)′ implies (P. 5). Suppose $A\delta B$ and $b\delta C$ for all $b \in B$, but $A\bar{\delta}C$. Then by (P. 5)′ there exist E and F such that $A\bar{\delta}(X-F)$, $C\bar{\delta}(X-E)$ and $E\cap F=\varnothing$. Suppose $A\delta E$. Then since $(X-F)\supset E$, we have by Lemma (1. 5) that $A\delta(X-F)$ which is a contradiction. Hence $A\bar{\delta}E$. Now $B\cap(X-E)=\varnothing$; for if there exists some $b\in(X-E)$, then by Lemma (1. 5) and (P. 1) $b\delta C$ would imply $C\delta(X-E)$. Hence $B\subset E$. But then since $A\delta B$, we have by Lemma (1. 5) that $A\delta E$, and this is a contradiction.

(1. 7) **Theorem.** *Let X be a non-void set. Let δ be a relation on $P(X)$. δ is a symmetric generalized proximity on X iff δ satisfies:*

(L. 1) $A\delta B$ implies $B\delta A$.

(L. 2) *$C\delta(A\cup B)$ implies either $C\delta A$ or $C\delta B$.*

(L. 3) *$A\delta B$ implies $A\neq\varnothing$ and $B\neq\varnothing$.*

(L. 4) *$A\cap B\neq\varnothing$ implies $A\delta B$.*

(L. 5) *$A\delta B$ and $b\delta C$ for all $b\in B$ implies that $A\delta C$.*

PROOF. Suppose δ satisfies (L. 1), (L. 2), (L. 3), (L. 4), and (L. 5). (L. 1)=(P. 1) and (L. 5)=(P. 5). It is clear that (L. 3) implies (P. 3) and (L. 4) implies (P. 4). We show that if $C\delta A$ or $C\delta B$ then $C\delta(A\cup B)$. Suppose $C\delta A$. Then since $(A\cup B)\supset A$, we have by (L. 4) that $a\delta(A\cup B)$ for every a in A; consequently, by (L. 5) we have that $C\delta(A\cup B)$. Similarly, if $C\delta B$, then $C\delta(A\cup B)$.

Conversely, suppose δ satisfies (P. 1), (P. 2), (P. 3), (P. 4), and (P. 5). It is clear that (P. 2) implies (L. 2) and (P. 3) implies (L. 3). We show that (P. 4) implies (L. 4). Let $x_0\in(A\cap B)$. By (P. 4)$x_0\delta x_0$. Hence by Lemma (1. 5) we have $A\delta B$. (L. 1)= =(P. 1) and (L. 5)=(P. 5).

(1. 8) *Remark.* M. W. Lodato in [23] defines a symmetric generalized proximity on a non-void set X to be a relation δ on X that satisfies (L. 1), (L. 2), (L. 3), (L. 4), and (L. 5). In a manner similar to our proof of Theorem (1. 7) he shows that his definition is equivalent to Definition (1. 1).

(1. 9) **Theorem.** *Let (X,δ) be a symmetric generalized proximity space. The function $f:P(X)\to P(X)$ defined by $x\in f(A)$ iff $x\delta A$ is a Kuratowski closure function.*

PROOF. The following is essentially the proof given by S. Leader in [21]. We derive the four Kuratowski closure axioms.

$f(\varnothing)=\varnothing$: Suppose there exists a point $x\in f(\varnothing)$. Then $x\delta\varnothing$ which contradicts (P. 3).

$A\subset f(A)$: Let $x\in A$. By (L. 4) $x\delta A$, and hence $x\in f(A)$.

$f(A\cup B)=f(A)\cup f(B)$: Suppose $x\in f(A\cup B)$. Then $x\delta(A\cup B)$ which by (P. 2) implies that $x\delta A$ or $x\delta B$ and hence that $x\in f(A)\cup f(B)$.

For the reverse inclusion suppose that $x\in f(A)\cup f(B)$. Then either $x\delta A$ or $x\delta B$. Suppose $x\delta A$. Then by Lemma (1. 5) we have $x\delta(A\cup B)$.

$f(f(A))\subset f(A)$: Suppose $x\in f(f(A))$. Then $x\delta f(A)$. But $a\delta A$ for all $a\in f(A)$. Hence by (L. 5) $x\delta A$; so that $x\in f(A)$.

(1. 10) *Definition.* The topology (which we denote $\mathscr{I}(\delta)$) induced on X by the Kuratowski closure function f in Theorem (1. 9) is called the p-topology (or proximity topology) on X.

(1. 11) *Definition.* A topological space $(X,\mathscr{I})$ is symmetric iff for every x, y in X, $x\in\bar{y}$ implies $y\in\bar{x}$.

(1. 12) *Remark*. Note that any T_1 space is symmetric. Also, for equivalent formulations of Definition (1. 11) see Appendix *I*.

(1. 13) **Theorem.** *Let $(X, \mathscr{I})$ be a symmetric topological space. The relation δ_0 on $P(X)$ defined by $A\delta_0 B$ iff $\bar{A}\cap\bar{B}\neq\varnothing$ is a symmetric generalized proximity on X such that $\mathscr{I}(\delta_0)=\mathscr{I}$. Furthermore, if (X, δ) is a symmetric generalized proximity space such that $\mathscr{I}(\delta)=\mathscr{I}$, then $\delta_0\subset\delta$.*

PROOF. The following is essentially the proof given by M. W. Lodato in [23]. It is easily shown that δ_0 satisfies (L. 1), (L. 2), (L. 3) and (L. 4). We must show that δ_0 satisfies (L. 5). Suppose for some point b and a set C we have $\bar{b}\cap\bar{C}\neq\varnothing$. Then there exists *a* point $c\in\bar{C}$ such that $c\in\bar{b}$. Since $\mathscr{I}$ is symmetric we have that $b\in\bar{c}\subset\bar{C}$. Hence, if $\bar{A}\cap\bar{B}\neq\varnothing$ and $\bar{b}\cap\bar{C}\neq\varnothing$ for every $b\in B$, then $\bar{B}\subset\bar{C}$; so that $\bar{A}\cap\bar{C}\neq\varnothing$. Consequently, δ_0 satisfies (L. 5).

To show that $\mathscr{I}(\delta_0)=\mathscr{I}$ it is sufficient to show that $x\delta_0 B$ iff $x\in\bar{B}$. Clearly, $x\in\bar{B}$ implies $\bar{x}\cap\bar{B}\neq\varnothing$. Hence $x\delta_0 B$.

Conversely, suppose $x\delta_0 B$. Then for some y we have $y\in(\bar{x}\cap\bar{B})$. Hence $\bar{y}\subset\bar{B}$ and $y\in\bar{x}$. But since $\mathscr{I}$ is symmetric, $y\in\bar{x}$ implies $x\in\bar{y}$. Hence $x\in\bar{B}$.

Suppose $A\delta_0 B$. Then $\bar{A}\cap\bar{B}\neq\varnothing$, and if $x\in\bar{A}\cap\bar{B}$, then since (L. 1) we have $x\delta A$, $x\delta B$ as a consequence of $A\delta x$, $x\delta B$. Hence $A\delta B$ from (L. 5).

(1. 14) *Corollary*. A topology $\mathscr{I}$ on X is the proximity topology for some symmetric generalized proximity on X iff $\mathscr{I}$ is symmetric.

PROOF. Suppose there exists a symmetric generalized proximity δ on X such that $\mathscr{I}(\delta)=\mathscr{I}$. Let $x\in\bar{y}$. Then $x\delta y$; so that $y\delta x$ and $y\in\bar{x}$.

The converse is an immediate consequence of Theorem (1. 13).

(1. 15) *Remark*. Contrast Corollary (1. 14) with the classical theorem which states that a topology $\mathscr{I}$ on X is the proximity topology for some proximity on X iff $\mathscr{I}$ is completely regular.

(1. 16) *Definition*. Let (X, δ) be a symmetric generalized proximity space. The set $B\in P(X)$ is a *p-neighborhood* of a set $A\in P(X)$ (notation: $A\ll B$) iff $A\bar{\delta}(X-B)$.

(1. 17) **Theorem.** *Let (X, δ) be a symmetric generalized proximity space. Let $\mathscr{I}(\delta)$ be the topology on X. Then for all A, B in $P(X)$ we have:*

(a) *$A\delta B$ iff $\bar{A}\delta\bar{B}$.*
(b) *$A\ll B$ implies $\bar{A}\ll B$.*
(c) *$A\ll B$ implies $A\ll B^{\circ}$.*

PROOF OF (a). Suppose $A\delta B$. Since $\bar{A}\supset A$ and $\bar{B}\supset B$ we have by Lemma (1. 5) that $\bar{A}\delta\bar{B}$.

Conversely, suppose $\bar{A}\delta\bar{B}$. By definition for all $b\in\bar{B}$ we have that $b\delta B$; hence by (P. 5) $\bar{A}\delta B$ so $B\delta\bar{A}$. But for all $a\in\bar{A}$ we have that $a\delta A$; so that by (P. 5) $B\delta A$. Hence $A\delta B$.

PROOF OF (b). $A\ll B$ implies $A\bar{\delta}(X-B)$. By Theorem (1. 17a) we have that $\bar{A}\bar{\delta}(\overline{X-B})$; so that by Lemma (1. 5) $\bar{A}\bar{\delta}(X-B)$. Consequently, $\bar{A}\ll B$.

PROOF OF (c). Since $A \ll B$ it is easily shown by (P. 1) that $(X-B) \ll (X-A)$. By Theorem (1. 17b) $(\overline{X-B}) \ll (X-A)$. Again by (P. 1) it is easily shown that $(X-(X-A)) \ll (X-(\overline{X-B}))$. Hence $A \ll B^\circ$.

We next obtain a p-neighborhood characterization of symmetric generalized proximity spaces.

(1. 18) **Theorem.** *The relation $\ll$ in a symmetric generalized proximity space (X, δ) satisfies the following conditions:*

(Q. 1) $X \ll X$.

(Q. 2) $A \ll B$ *implies* $A \subset B$.

(Q. 3) $A \subset B \ll C \subset D$ *implies* $A \ll D$.

(Q. 4) $A \ll B_k$, $k=1, 2$ *iff* $A \ll (B_1 \cap B_2)$.

(Q. 5) $A \ll B$ *implies* $(X-B) \ll (X-A)$.

(Q. 6) $A \ll B$ *implies that, for all* C, $A \ll C$ *or there exists* $x \in (X-C)$ *with* $x \ll B$.

If δ *is separated, then*

(Q. 7) $x \ll (X-y)$ *iff* $x \neq y$.

Conversely, let a relation $\ll$ *satisfying* (Q. 1) *through* (Q. 6) *be defined on* $P(X)$. *Then* δ, *defined by* $A\bar{\delta}B$ *iff* $A \ll (X-B)$, *is a symmetric generalized proximity on* X. *Furthermore,* B *is a p-neighborhood of* A *with respect to* δ *iff* $A \ll B$; *and if* (Q. 7) *is satisfied, then* δ *is a separated symmetric generalized proximity on* X.

PROOF. The proof of (Q. 1), (Q. 2), (Q. 3), (Q. 4), and (Q. 5) is straightforward and is left to the reader.

(Q. 6): Suppose $A \ll B$, and it is not the case that $A \ll C$. Furthermore, suppose for every $x \in (X-C)$ it is not the case that $x \ll B$. Then $A\delta(X-C)$ and for all $x \in (X-C)$ we have that $x\delta(X-B)$; so that by (P. 5) $A\delta(X-B)$ which is a contradiction.

(Q. 7): Suppose $x \ll (X-y)$. Then $x\bar{\delta}y$; so that by (P. 4) $x \neq y$.

Conversely, suppose $x \neq y$. Then by (P. 4)$'$ $x\bar{\delta}y$. Consequently, $x\bar{\delta}(X-(X-y))$.

PROOF OF THE CONVERSE of Theorem (1. 18).

(L. 1): Suppose $A\bar{\delta}B$. Then $A \ll (X-B)$; so that by (Q. 5) $B \ll (X-A)$. Consequently, $B\bar{\delta}A$.

(L. 2): Suppose $A\bar{\delta}B$ and $A\bar{\delta}C$. Then $A \ll (X-B)$ and $A \ll (X-C)$. Hence by (Q. 4) $A \ll ((X-B) \cap (X-C))$; so that $A \ll (X-(B \cup C))$. Consequently, $A\bar{\delta}(B \cup C)$.

(L. 3): Let $A \subset X$. Then by (Q. 1) $A \subset X \ll X$. Hence by (Q. 3) $A \ll X$; so that $A\bar{\delta}\varnothing$. Now suppose $A\delta B$. Then $B \neq \varnothing$. But by (L. 1) $B\delta A$. Hence $A \neq \varnothing$.

(L. 4): $A\bar{\delta}B$ implies $A \ll (X-B)$; consequently, by (Q. 2) we have that $A \subset$ $\subset (X-B)$; so that $A \cap B = \varnothing$.

(L. 5): Clearly, by taking the contrapositive of (L. 5) it is sufficient to show that $A\bar{\delta}B_1$ implies $A\bar{\delta}C_1$ or there exists $x \in C_1$ such that $x\bar{\delta}B_1$. Let $B_1 = (X-B)$ and $C_1 = (X-C)$. $A\bar{\delta}(X-B)$ implies by definition that $A \ll B$; consequently, by (Q. 6) $A \ll C$ or there exists an $x \in (X-C)$ such that $x \ll B$. Hence $A\bar{\delta}(X-C)$ or there exists an $x \in (X-C)$ such that $x\bar{\delta}(X-B)$.

Hence δ is a symmetric generalized proximity on X.

Suppose B is a p-neighborhood of A with respect to δ. Then $A\bar{\delta}(X-B)$; so that $A \ll B$.

Conversely, suppose $A \ll B$. Then $A\bar{\delta}(X-B)$. Consequently, B is a p-neighborhood of A with respect to δ.

Suppose $\ll$ satisfied (Q. 7), and $x\bar{\delta}y$. Then $x \ll (X-y)$; so that $x \neq y$.

Conversely, suppose $x \neq y$. Then $x \ll (X-y)$; so that $x\bar{\delta}y$.

This completes the proof of Theorem (1. 18).

The following theorem which may appear unrelated at this time will be referred to in the sequel.

(1. 19) **Theorem.** *Let δ be a symmetric generalized proximity on X. Let $A \subset X$. Then $A^\circ = \{x | x \ll A\}$.*

Proof. It is easily shown that $C \in \mathscr{I}(\delta)$ iff for every $x \in C$ $x \ll C$. Let $B = \{x | x \ll A\}$. It is clear that $A^\circ \subset B \subset A$. Consequently, it is sufficient to show that if $x \in B$, then $x \ll B$. Let $x \in B$. Then $x \ll A$; hence by (Q. 6) $x \ll B$ or there exists $x_1 \in (X-B)$ such that $x_1 \ll A$. But if $x_1 \ll A$, then $x_1 \in B$; hence $x \ll B$.

We next obtain a characterization of separated symmetric generalized proximity spaces which is analogous to the classical result of Smirnov which states that every separated proximity space (X, δ) is a dense subspace of a unique compact T_2 space $(Y, \mathscr{I})$ such that $A\delta B$ in X iff $\bar{A} \cap \bar{B} \neq \varnothing$ where the closure is taken in Y. For a proof of this theorem based on the concept of a "cluster" see [17].

(1. 20) *Definition* (S. Leader [17]). A class of subsets of X is a cluster (denoted π) from a symmetric generalized proximity space (X, δ) iff π satisfies:

(C. 1) $A\delta B$ for all $A, B \in \pi$.

(C. 2) $A \cup B \in \pi$ implies that either $A \in \pi$ or $B \in \pi$.

(C. 3) If $B\delta A$ for every $A \in \pi$, then $B \in \pi$.

The proofs of the rest of the theorems in this chapter are essentially the same as those given by Lodato in [23].

(1. 21) **Theorem.** *Let (X, δ) by a symmetric generalized proximity space. The class π_x of all subsets of X which are close to x is a cluster from (X, δ).*

Proof. We verify (C. 1), (C. 2), and (C. 3).

(C. 1): Suppose $A, B \in \pi_x$. Then $x\delta A$ and $x\delta B$; so that by (L. 1) and (L. 5) we have that $A\delta B$.

(C. 2): Suppose $A \cup B \in \pi_x$. Then $x\delta(A \cup B)$; so that by (L. 2) $x\delta A$ or $x\delta B$. Hence $A \in \pi_x$ or $B \in \pi_x$.

(C. 3): By (L. 4) $x \in \pi_x$. Suppose $B\delta A$ for every $A \in \pi_x$. Then $B\delta x$; so that $B \in \pi_x$.

(1. 22) *Remark.* The following facts about clusters are easily established. Let (X, δ) be a symmetric generalized proximity space. Let π and π' be clusters from (X, δ). (A) If $A \in \pi$, and $A \subset B \subset X$, then $B \in \pi$. (B) If $A \in \pi$ and $a\delta B$ for every $a \in A$, then $B \in \pi$. (C) If $\pi \subset \pi'$, then $\pi = \pi'$. (D) If $x \in \pi$, then $\pi = \pi_x$, the class of all subsets A of X such that $A\delta x$. (E) Let $A \subset X$. Then $A \in \pi$ or $(X-A) \in \pi$. (F) Let $A \subset X$. If A has a non-void intersection with every element of π, then $A \in \pi$. (G) $\varnothing \notin \pi$, $X \in \pi$ for every π.

(1. 23) *Definition.* Let $(Y, \mathcal{T})$ be a topological space. Let $X \subset Y$. X is *regularly dense* in Y iff given M open in Y and $p \in M$ there exists $E \subset X$ such that $p \in \overline{E} \subset M$, the closure being taken in Y.

(1. 24) **Theorem.** *Let $(Y, \mathcal{T})$ be a topological space. If X is regularly dense in Y, then X is dense in Y. If $(Y, \mathcal{T})$ is regular and X is dense in Y, then X is regularly dense in Y.*

PROOF. Let $p \in Y$. Since Y is open there exists $E \subset X$ such that $p \in \overline{E} \subset \overline{X} \subset Y$. Hence $Y \subset \overline{X} \subset Y$; so that $\overline{X} = Y$.

Suppose $(Y, \mathcal{T})$ is regular and X is dense in Y. Let M be an open subset of Y and let $p \in M$. Then there exists an open subset N such that $p \in N$ and $\overline{N} \subset M$. Let $E = N \cap X$. Then since X is dense in Y we have that $\overline{E} = \overline{N}$; so that $p \in \overline{E} \subset M$.

(1. 25) **Theorem.** *Let X be a non-void set. Let δ be a relation on $P(X)$. The following are equivalent:*

(I) *There exists a T_1 topological space $(Y, \mathcal{T})$ and a mapping $f: X \to Y$ such that $f(X)$ is a regularly dense in Y and $(A\delta B)$ in X iff $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$ in Y.*

(II) *(X, δ) is a symmetric generalized proximity space that has the property that given $(A\delta B)$ in X there exists a cluster π to which both A and B belong.*

PROOF. We first show (I) implies (II). To show that (X, δ) is a symmetric generalized proximity space we verify (L. 1), (L. 2), (L. 3), (L. 4), and (L. 5). Clearly, (L. 1), (L. 2), (L. 3), and (L. 4) are immediate by the properties of closure.

(L. 5): Suppose $A\delta B$ and $b\delta C$ for all $b \in B$. Then $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$ and $\overline{f(b)} \cap \cap \overline{f(C)} \neq \emptyset$ for all $b \in B$; so that since $(Y, \mathcal{T})$ is T_1 $f(b) \in \overline{f(C)}$ for all $b \in B$. Thus $f(B) \subset \overline{f(C)}$ or $\overline{f(B)} \subset \overline{f(C)}$ so that $\overline{f(A)} \cap \overline{f(C)} \neq \emptyset$. Consequently, $A\delta C$.

Suppose $A\delta B$. Then there exists $t \in \overline{f(A)} \cap \overline{f(B)}$. Let $\pi = \{S \subset X \mid t \in \overline{f(S)}\}$. Clearly $A \in \pi$ and $B \in \pi$. We now show that π is a cluster from (X, δ). Clearly, π satisfies (C. 1) and (C. 2).

(C. 3): Suppose that $\overline{f(D)} \cap \overline{f(C)} \neq \emptyset$ for every $C \in \pi$ but that $D \notin \pi$, i.e., $t \notin \overline{f(D)}$. Then $t \in Y - \overline{f(D)}$ and since $f(X)$ is regularly dense in Y there exists a subset E of X such that $t \in \overline{f(E)} \subset Y - \overline{f(D)}$. Thus there exists an $E \in \pi$ such that $\overline{f(D)} \cap \overline{f(E)} = \emptyset$ which is a contradiction. Hence $D \in \pi$.

We now show that (II) implies (I). Let $x \in X$. Let $\pi_x = \{E \subset X \mid x\delta E\}$. By Theorem (1. 21) π_x is a cluster from (X, δ). Let $A \subset X$. Let $\mathcal{A} = \{\pi_a \mid a \in A\}$. Let $\mathcal{A}^* = \{\pi \mid \pi$ is a cluster from (X, δ) and $A \in \pi\}$. Let $Y = \{\pi \mid \pi$ is a cluster from $(X, \delta)\}$. Clearly, for every $A \subset X$ we have that $\mathcal{A} \subset \mathcal{A}^* \subset Y$. We say that a subset $A \subset X$ *absorbs* a subset $\mathcal{B} \subset Y$ iff $\mathcal{B} \mid \mathcal{A}^*$. For every $\mathcal{B} \subset Y$ let $\overline{\mathcal{B}} = \{\pi \mid E \in \pi$ if E absorbs $\mathcal{B}\}$.

We show that for every $A \subset X \overline{\mathcal{A}} = \mathcal{A}^*$. For suppose $\pi \in \overline{\mathcal{A}}$. Then since A absorbs $\mathcal{A}$, $A \in \pi$; so that $\pi \in \mathcal{A}^*$. Conversely, if $\pi \in \mathcal{A}^*$ then $A \in \pi$. Let $E \in \pi_a$ for every $\pi_a \in \mathcal{A}$, i.e., $E\delta a$ for every $a \in A$ and hence $A \subset \overline{E}$. Then by Remark (1. 22B) $E \in \pi$; so that $\pi \in \overline{\mathcal{A}}$.

We now show that the operator defined above is a Kuratowski closure operator.

$\mathcal{B} \subset \overline{\mathcal{B}}$: Immediate since if E absorbs $\mathcal{B}$, then $E \in \pi$ for every $\pi \in \mathcal{B}$.

$\overline{\emptyset} = \emptyset$: Suppose $\pi \in \overline{\emptyset}$. Clearly, every subset of X absorbs $\overline{\emptyset}$; so that every subset of X is in π. In particular, $\emptyset$ and X are in π. Hence by (C. 1) $\emptyset\delta X$ which is a contradiction.

$\overline{\overline{\mathscr{B}}} \subset \overline{\mathscr{B}}$: Suppose $\pi \in \overline{\overline{\mathscr{B}}}$ and that E absorbs $\mathscr{B}$. Then E absorbs $\overline{\mathscr{B}}$. Hence $E \in \pi$; so that $\pi \in \overline{\mathscr{B}}$.

$\overline{(\mathscr{B} \cup \mathscr{B}')} = \overline{\mathscr{B}} \cup \overline{\mathscr{B}'}$: Suppose that $\pi \in \overline{(\mathscr{B} \cup \mathscr{B}')}$ and that A absorbs $\mathscr{B}$ and A' absorbs $\mathscr{B}'$. Then, by Remark (1. 22A), $A \cup A'$ absorbs $\mathscr{B} \cup \mathscr{B}'$ so that $A \cup A' \in \pi$. But, by (C. 2), this means that either $A \in \pi$ or $A' \in \pi$, that is $\pi \in \overline{\mathscr{B}}$ or $\pi \in \overline{\mathscr{B}'}$. Thus $\overline{\mathscr{B} \cup \mathscr{B}'} \subset \overline{\mathscr{B}} \cup \overline{\mathscr{B}'}$. To show the reverse inclusion let $\pi \in \overline{\mathscr{B}} \cup \overline{\mathscr{B}'}$. Then $\pi \in \overline{\mathscr{B}}$ or $\pi \in \overline{\mathscr{B}'}$. Now if E absorbs $\mathscr{B} \cup \mathscr{B}'$, then E absorbs $\mathscr{B}$ and also absorbs $\mathscr{B}'$. Hence $E \in \pi$; so that $\pi \in \overline{\mathscr{B} \cup \mathscr{B}'}$.

We now show that the topology induced by the Kuratowski closure operator above is T_1. Suppose $\pi' \in \bar{\pi}$ where π and π' are in Y. Then every set in π is also in π'. Thus, $\pi \subset \pi'$, and by Remark (1. 22C) $\pi = \pi'$. Hence $\bar{\pi} = \pi$ for every point $\pi \in Y$.

Define a map $f: X \to Y$ by $f(x) = \pi_x$. Clearly $f(A) = \mathscr{A}$ for every subset A of X.

We now show that $A\delta B$ in X iff $\overline{f(A)} \cap \overline{f(B)} \neq \varnothing$ in Y. Clearly it is sufficient to show that $A\delta B$ in X iff $\mathscr{A}^* \cap \mathscr{B}^* \neq \varnothing$ in Y. Suppose $A\delta B$. Then by hypothesis there exists a cluster π to which both A and B belong. Hence $\pi \in \mathscr{A}^* \cap \mathscr{B}^*$. Conversely, if $\pi \in \mathscr{A}^* \cap \mathscr{B}^*$ then $A \in \pi$ and $B \in \pi$; so that by (C. 1) $A\delta B$.

We now show that $f(X)$ is regularly dense in Y. Suppose α is open in Y and that $\pi \in \alpha$. Then $\pi \notin Y - \alpha = \overline{(Y - \alpha)}$. Hence there exists some subset E of X such that E is in every cluster of $Y - \alpha$ but that E is not in π. Consequently, by (C. 3), there is a $C \in \pi$ such that $E\bar{\delta}C$. But since $\mathscr{C}^*$ is the set of all clusters to which C belongs, we have $\pi \in \mathscr{C}^*$. And since E belongs to every cluster in $Y - \alpha$ and $E\bar{\delta}C$, then C cannot belong to any cluster in $Y - \alpha$, by (C. 1). Hence $\mathscr{C}^* \subset \alpha$ and $f(X)$ is regularly dense in Y.

(1. 26) **Theorem.** *Let X be a non-void set. Let δ be a relation on $P(X)$. The following are equivalent:*

(I) *There exists a T_1 topological space $(Y, \mathscr{I})$ in which X can be topologically imbedded as a regularly dense subset; so that $(A\delta B)$ in X iff $\bar{A} \cap \bar{B} \neq \varnothing$ in Y.*

(II) *(X, δ) is a separated, symmetric generalized proximity space that has the property that given $(A\delta B)$ in X there exists a cluster π to which both A and B belong.*

Proof. We first note that if (X, δ) is separated, then every cluster π from (X, δ) possesses at most one point.

We show that (I) implies (II). By Theorem (1. 25) it is sufficient to show that (X, δ) is separated. But this is clear since $x^* \cap y^* \neq \varnothing$ implies that $x = y$.

We next show that (II) implies (I). Again by Theorem (1. 25) it is sufficient to show that the mapping f is a topological imbedding. It is clear that f is 1—1, onto $f(x)$. To show that f is bicontinuous we must show that if $A \subset X$, $x \in \bar{A}$ iff $\pi_x \in \mathrm{cl}\,(\mathscr{A})$ where $\mathrm{cl}\,(\mathscr{A})$ is the closure of $\mathscr{A}$ in $f(X)$ relative to $(Y, \mathscr{I})$.

Suppose $x \in \bar{A}$ and E absorbs $\mathscr{A}$. Then E is a member of every π_a in $\mathscr{A}$ and it follows that $a\delta E$ for every $a \in A$. Thus $A \subset \bar{E}$ and since $A \in \pi_x$ we have by Remark (1. 22B) that $E \in \pi_x$; so that $\pi_x \in \mathrm{cl}\,(\mathscr{A})$.

Conversely, suppose $\pi_x \in \mathrm{cl}\,(\mathscr{A})$. Then since A absorbs $\mathscr{A}$ we have $A \in \pi_x$, i.e., $A\delta x$ and hence $x \in \bar{A}$.

(1. 27) *Remark.* In [25] Lodato obtains by means of the concept of a "bunch" an improvement of Theorem (1. 26).

II.

Symmetric generalized uniform spaces

Let X be a non-void set. Let $\mathcal{U}$ be a non-void subset of $P(X\times X)$. Consider the following axioms:

(M. 1) For every $U\in\mathcal{U}\, U\supset\Delta$.

(M. 2) $\cap\{U|U\in\mathcal{U}\}=\Delta$.

(M. 3) For every $U\in\mathcal{U}\, U=U^{-1}$.

(M. 4) For every $A\in P(X)$ and U, V in $\mathcal{U}$ there is a $W\in\mathcal{U}$ such that $W[A]\subset\subset U[A]\cap V[A]$.

(M. 5) For every U, V in $\mathcal{U}$ $(U\cap V)\in\mathcal{U}$.

(M. 6) For every A, B in $P(X)$ and $U\in\mathcal{U}$, if $V[A]\cap B\neq\varnothing$ for all $V\in\mathcal{U}$, then there exists $x\in B$ and there exists a $W\in\mathcal{U}$ such that $W[x]\subset U[A]$.

(M. 7) For every $U\in\mathcal{U}$ there exists a $V\in\mathcal{U}$ such that $(V\circ V)\subset U$.

(M. 8) If $U\in\mathcal{U}$ and $V\subset(X\times X)$ and $U\subset V$ and $V=V^{-1}$, then $V\in\mathcal{U}$.

(2. 1) *Remark*. It is clear that (M. 5) implies (M. 4). The following simple argument shows that (M. 7) implies (M. 6): Let A, B be in $P(X)$ and let $U\in\mathcal{U}$. By (M. 7) there exists a $V\in\mathcal{U}$ such that $V\circ V\subset U$. But by hypothesis there exists an $x\in V[A]\cap B$; hence there exists $z\in A$ such that $(z,x)\in V$. Let $p\in V[x]$. Then $(x,p)\in V$; hence $(z,p)\in U$; so that $p\in U[A]$, and $V[x]\subset U[A]$.

(2. 2) **Theorem.** *Let $\mathcal{U}$ be a subset of $P(X\times X)$ with the property that for all $U\in\mathcal{U}$, U^{-1} contains a member of $\mathcal{U}$. Define a relation $\delta(\mathcal{U})$ on $P(X)$ by $A\delta(\mathcal{U})B$ iff $U[A]\cap B\neq\varnothing$ for all $U\in\mathcal{U}$. Then $\delta(\mathcal{U})$ satisfies* (L. 1), (L. 2), (L. 3), (L. 4), *and* (L. 5) *iff $\mathcal{U}$ satisfies* (M. 1), (M. 4), *and* (M. 6).

PROOF. Suppose $\mathcal{U}$ satisfies (M. 1), (M. 4), and (M. 6). We show that $\delta(\mathcal{U})$ satisfies (L. 1), (L. 2), (L. 3), (L. 4), and (L. 5). To simplify the notation we will write δ in place of $\delta(\mathcal{U})$.

(L. 1): Suppose $A\bar{\delta}B$. There exists by hypothesis a $U\in\mathcal{U}$ such that $U[A]\cap\cap B=\varnothing$. Suppose $U^{-1}[B]\cap A\neq\varnothing$. Let $x_0\in U^{-1}[B]\cap A$. Then $x_0\in U^{-1}[B]$; so that there exists $y_0\in B$ such that $(y_0,x_0)\in U^{-1}$, and consequently $(x_0,y_0)\in U$. But this means that $y_0\in U[A]\cap B$ which is a contradiction. Hence $U^{-1}[B]\cap A=\varnothing$. But by hypothesis $U^{-1}\supset V$ where $V\in\mathcal{U}$; so that $V[B]\cap A=\varnothing$. Hence $B\bar{\delta}A$.

(L. 2): Suppose $C\bar{\delta}A$ and $C\bar{\delta}B$. Then there exists $U\in\mathcal{U}$ and $V\in\mathcal{U}$ such that $U[C]\cap A=\varnothing$ and $V[C]\cap B=\varnothing$. But by (M. 4) there exists a $W\in\mathcal{U}$ such that $W[C]\subset U[C]\cap V[C]$. Consequently, $W[C]\cap(A\cup B)=\varnothing$; so that $C\bar{\delta}(A\cup B)$.

(L. 3): Immediate from the definition of δ.

(L. 4): Suppose $A\cap B\neq\varnothing$. By (M. 1) for all $U\in\mathcal{U}$ $U[A]\cap B\neq\varnothing$. Consequently, $A\delta B$.

(L. 5): Suppose $A\delta B$ and $b\delta C$ for all $b\in B$, but $A\bar{\delta}C$. Then there exists a $U\in\mathcal{U}$ such that $U[A]\cap C=\varnothing$. But since $A\delta B$, we have that $V[A]\cap B\neq\varnothing$ for all $V\in\mathcal{U}$; so that by (M. 6) there exists $x_0\in B$ and there exists $W\in\mathcal{U}$ such that $W[x_0]\subset\subset U[A]\subset(X-C)$. But this implies that $W[x_0]\cap C=\varnothing$; so that $x_0\bar{\delta}C$ which is a contradiction since $x_0\in B$.

Conversely, suppose δ satisfies (L. 1), (L. 2), (L. 3), (L. 4), and (L. 5). We show that $\mathcal{U}$ satisfies (M. 1), (M. 4), and (M. 6).

(M. 1): Let $x \in X$. Let $U \in \mathscr{U}$. $x \cap x \neq \emptyset$ implies by (L. 4) that $x\delta x$. Consequently, $U[x] \cap x \neq \emptyset$; so that $(x, x) \in U$. Hence $U \supset \Delta$.

(M. 4): Suppose not true. Then there exists $A \in P(X)$ and U, V in $\mathscr{U}$ such that for every $W \in \mathscr{U}$ there exists $x \in W[A]$ such that $x \notin U[A] \cap V[A]$. For each $W \in \mathscr{U}$ let $B(W) = \{x | x \in W[A]$ and $x \notin U[A] \cap V[A]\}$. Let $B = \bigcup \{B(W) | W \in \mathscr{U}\}$. Suppose there exists $U_a \in \mathscr{U}$ such that $U_a[A] \cap B = \emptyset$. Then since $B(U_a) \subset B$, we have that $U_a[A] \subset U[A] \cap V[A]$, but by assumption this is not possible. Hence $M[A] \cap B \neq \emptyset$ for all $M \in \mathscr{U}$; so that $A\delta B$. Let $B_1 = (B - U[A])$ and $B_2 = (B - V[A])$. Clearly, $U[A] \cap B_1 = \emptyset$ and $V[A] \cap B_2 = \emptyset$; so that $A\bar{\delta}B_1$ and $A\bar{\delta}B_2$. Consequently, by (L. 2) $A\bar{\delta}(B_1 \cup B_2)$. But by the definition of B we have that $B = B_1 \cup B_2$. Hence $A\bar{\delta}B$ which is a contradiction.

(M. 6): Suppose not true. Then there exists A, B in $P(X)$ and $U \in \mathscr{U}$ such that $V[A] \cap B \neq \emptyset$ for all $V \in \mathscr{U}$ and for every $b \in B$ and for every $W \in \mathscr{U}$ we have that $W[b] \cap (X - U[A]) \neq \emptyset$. Consequently, $A\delta B$ and $b\delta(X - U[A])$ for every $b \in B$; so that by (L. 5) $A\delta(X - U[A])$. But $U[A] \cap (X - U[A]) = \emptyset$; so that $A\bar{\delta}(X - U[A])$. Hence our assumption leads to a contradiction.

(2. 3) *Definition.* Let $\mathscr{U}$ be a non-void subset of $P(X \times X)$. $\mathscr{U}$ is *separated* iff $\mathscr{U}$ satisfies (M. 2). $\mathscr{U}$ is a *symmetric generalized uniformity* on X iff $\mathscr{U}$ satisfies (M. 1), (M. 3), (M. 4), (M. 6), and (M. 8). $\mathscr{U}$ is a *correct uniformity* on X iff $\mathscr{U}$ satisfies (M. 1), (M. 3), (M. 4), (M. 7), and (M. 8). $\mathscr{U}$ is a *symmetric uniformity* on X iff $\mathscr{U}$ satisfies (M. 1), (M. 3), (M. 5), (M. 7), and (M. 8).

(2. 4) *Remark.* If $\mathscr{U}$ is a symmetric generalized uniformity on X, then $(X, \mathscr{U})$ is called a symmetrix generalized uniform space. Similarly, we define a correct uniform space and a symmetric uniform space.

(2. 5) *Remark.* Note that if $\mathscr{V}$ is any classical uniformity, then $\{V \in \mathscr{V} | V = V^{-1}\}$ is a symmetric uniformity.

(2. 6) **Theorem.** *Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. The function $g: P(X)$ into $P(X)$ defined by $x \in g(A)$ iff $U[x] \cap A \neq \emptyset$ for all $U \in \mathscr{U}$ is a Kuratowski closure function.*

PROOF. We derive the four Kuratowski closure axioms.

$g(\emptyset) = \emptyset$: Suppose there exists a point $x \in g(\emptyset)$. Then $U[x] \cap \emptyset \neq \emptyset$ for every $U \in \mathscr{U}$ which of course is not possible.

$A \subset g(A)$: Let $x \in A$. Then $U[x] \cap A \neq \emptyset$ for all $U \in \mathscr{U}$; so that $x \in g(A)$.

$g(A \cup B) = g(A) \cup g(B)$: It is clear that $g(A) \cup g(B) \subset g(A \cup B)$. Let $x \in g(A \cup B)$. Suppose $x \notin (g(A) \cup g(B))$. Then there exists U_1, U_2 in $\mathscr{U}$ such that $U_1[x] \cap A = \emptyset$ and $U_2[x] \cap B = \emptyset$. Then by (M. 4) there exists a $W \in \mathscr{U}$ such that $W[x] \subset U_1[x] \cap \cap U_2[x]$. But $W[x] \cap (A \cup B) \neq \emptyset$; so that $x \in (g(A) \cup g(B))$.

$g(g(A)) = g(A)$: Let $x \in g(A)$. Then $U[x] \cap A \neq \emptyset$ for every $U \in \mathscr{U}$; so that $U[x] \cap g(A) \neq \emptyset$ for every $U \in \mathscr{U}$. Hence $x \in g(g(A))$.

Conversely, suppose $x \in g(g(A))$. Let $U \in \mathscr{U}$. Then $V[x] \cap g(A) \neq \emptyset$ for every $V \in \mathscr{U}$. But by (M. 6) there exists $x_0 \in g(A)$, and there exists $W \in \mathscr{U}$ such that $W[x_0] \subset \subset U[x]$. But $W[x_0] \cap A \neq \emptyset$; so that $U[x] \cap A \neq \emptyset$. Consequently $x \in g(A)$.

(2. 7) *Remark.* Note that since $g(A) = \{x | x\ \delta(\mathscr{U}) A\}$ it is possible to derive Theorem (2. 6) directly from Theorem (1. 9) and Theorem (2. 2).

(2. 8) *Definition.* The topology induced on X by the Kuratowski closure function g in the above theorem is called the uniform topology on X induced by $\mathcal{U}$ (notation $\mathcal{I}(\mathcal{U})$).

(2. 9) **Theorem.** *Let $(X, \mathcal{U})$ be a symmetric generalized uniform space. Then $A \in \mathcal{I}(\mathcal{U})$ iff for every $x \in A$ there exists a $U \in \mathcal{U}$ such that $U[x] \subset A$.*

PROOF. Suppose $A \in \mathcal{I}(\mathcal{U})$. Then $(X-A)$ is closed. Let $x \in A$. Since $x \notin \overline{(X-A)}$, there exists a $U \in \mathcal{U}$ such that $U[x] \cap (X-A) = \emptyset$; so that $U[x] \subset A$.

Conversely, suppose $x \in A$ and there exists $U \in \mathcal{U}$ such that $U[x] \subset A$. Then $x \notin \overline{(X-A)}$; so that $(X-A)$ contains all its accumulation points. Hence $(X-A)$ is closed; so that A is open.

(2. 10) *Remark.* Note that since $U[x] \cap A \neq \emptyset$ for every $U \in \mathcal{U}$ iff $x\,\delta(\mathcal{U})A$ we have that if $(X, \mathcal{U})$ is a symmetric generalized uniform space, then $\mathcal{I}(\mathcal{U}) = \mathcal{I}(\delta(\mathcal{U}))$.

The following theorem and corollary are very important for the development of the theory of symmetric generalized uniform spaces.

(2. 11) **Theorem.** *Let $(X, \mathcal{U})$ be a symmetric generalized uniform space. Then for every $A \in P(X)$ we have that $A^\circ = \{x | U[x] \subset A$ for some $U \in \mathcal{U}\}$.*

PROOF. Let $B = \{x | U[x] \subset A$ for some $U \in \mathcal{U}\}$. It is clear that $A \supset B \supset A^\circ$. Consequently, it is sufficient to show that $(X-B)$ is closed. Suppose $y \in \overline{(X-B)}$. Then $V[y] \cap (X-B) \neq \emptyset$ for every $V \in \mathcal{U}$. Suppose $y \in B$. Then there exists $U_1 \in \mathcal{U}$ such that $U_1[y] \subset A$. But then by (M. 6) there exists $x \in (X-B)$ and there exists $W \in \mathcal{U}$ such that $W[x] \subset U_1[y] \subset A$. So that $x \in B$ which is a contradiction. Consequently, $y \in (X-B)$ and $(X-B)$ is closed.

(2. 12) *Corollary.* For every $x \in X \{U[x] | U \in \mathcal{U}\}$ is a base for the neighborhood system of x.

PROOF. Let $x \in X$. Let M be an open set that contains x. There exists by Theorem (2. 9) a $U \in \mathcal{U}$ such that $U[x] \subset M$. But by Theorem (2. 11) we have that $x \in (U[x])^\circ$. Hence $U[x]$ is a neighborhood of x.

(2. 13) *Remark.* Note that it is possible to derive Theorem (2. 11) from Theorem (1. 19) if in the latter theorem we let $\delta = \delta(\mathcal{U})$.

(2. 14) **Theorem.** *Let $(X, \mathcal{U})$ be a symmetric generalized uniform space. Then for every $A \subset X$ we have that $\bar{A} = \cap \{U[A] | U \in \mathcal{U}\}$.*

PROOF. Let $x \in \bar{A}$. Then $U[x] \cap A \neq \emptyset$ for all $U \in \mathcal{U}$; so that $x \in U^{-1}[A]$ for all $U \in \mathcal{U}$. But since $U = U^{-1}$, this implies $x \in U[A]$ for all $U \in \mathcal{U}$.

Conversely, suppose $x \in U[A]$ for all $U \in \mathcal{U}$. Then $x \in U^{-1}[A]$ for all $U \in \mathcal{U}$; so that $U[x] \cap A \neq \emptyset$ for all $U \in \mathcal{U}$. Hence $x \in \bar{A}$.

(2. 15) **Theorem.** *The following are equivalent for any symmetric generalized uniform space $(X, \mathcal{U})$:*

(a) *$\mathcal{I}(\mathcal{U})$ is a T_0 topology.*

(b) $\cap \{U | U \in \mathcal{U}\} = \Delta$

(c) *$\mathcal{I}(\mathcal{U})$ is a T_1 topology.*

PROOF. We first show (a) implies (b). Assume $\mathscr{I}(\mathscr{U})$ is T_0, and $x \neq y$. Suppose there exists an open set M such that $y \in M$ and $x \notin M$. Then by Corollary (2. 12) there exists a $U \in \mathscr{U}$ such that $U[y] \subset M$. Consequently, $x \notin U[y]$; so that $(x, y) \notin U$. Hence $\bigcap \{U | U \in \mathscr{U}\} = \Delta$.

We now show (b) implies (c). Assume $\bigcap \{U | U \in \mathscr{U}\} = \Delta$, and suppose $x \neq y$. Then $(x, y) \notin U_1$ and $(y, x) \notin U_1$ for some $U_1 \in \mathscr{U}$. Hence $y \notin U_1[x]$ and $x \notin U_1[y]$; so that by Corollary (2. 12) we have that $\mathscr{I}(\mathscr{U})$ is T_1.

(2. 16) *Definition.* $\mathscr{B}$ is a *base for a symmetric generalized uniformity* $\mathscr{U}$ on X iff (1) $\mathscr{B} \subset \mathscr{U}$ and (2) for every $U \in \mathscr{U}$ there exists a $V \in \mathscr{B}$ such that $V \subset U$.

(2. 17) *Definition.* $\mathscr{S}$ is a *subbase for a symmetric generalized uniformity* $\mathscr{U}$ on X iff $\mathscr{B}$, the set of all finite intersections of elements of $\mathscr{S}$, is a base fo $\mathscr{U}$.

(2. 18) *Remark.* In the sequel we will show that a base for $\mathscr{U}$ need not be a subbase for $\mathscr{U}$.

(2. 19) *Remark.* If $\mathscr{B}$ is a base for $\mathscr{U}$ on X, and if each element of $\mathscr{B}$ is open with respect to the product topology on $(X \times X)$, then $\mathscr{B}$ is called an *open base*. Similarly we define a *closed base*.

(2. 20) **Theorem.** *Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. If $\mathscr{U}$ has a closed base then $\mathscr{I}(\mathscr{U})$ is regular.*

(2. 21) **Lemma.** *Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. If V is closed in $(X \times X)$ where the topology on $(X \times X)$ is the product topology of $\mathscr{I}(\mathscr{U})$, then for each $x \in X V[x]$ is closed with respect to $\mathscr{I}(\mathscr{U})$.*

PROOF *of Lemma* (2. 21). Let $x_0 \in X$. Let $\{y_n | n \in D\}$ be a net in $V[x_0]$. Then $\{(x_0, y_n) | n \in D\}$ is a net in V. Suppose $\{y_n\}$ converges to b. We know the constant net $\{x_0\}$ converges to x_0. Hence $\{(x_0, y_n) | n \in D\}$ converges to $(x_0, b) \in V$; so that $b \in V[x_0]$ and $V[x_0]$ is closed.

PROOF *of Theorem* (2. 20). This is an immediate consequence of Lemma (2. 21) and Corollary (2. 12).

(2. 22) **Theorem.** *$\mathscr{B}$, a subset of $P(X \times X)$, is a base for some symmetric generalized uniformity on X iff $\mathscr{B}$ satisfies* (M. 1), (M. 3), (M. 4), *and* (M. 6).

PROOF. Clearly, if $\mathscr{B}$ is a base for some symmetric generalized uniformity on X, then $\mathscr{B}$ satisfies (M. 1) (M. 3), (M. 4), and (M. 6).

Conversely, let $\mathscr{U} = \{U | U = U^{-1}$ and $U \supset V$ for some $V \in \mathscr{B}\}$. Clearly, $\mathscr{U}$ satisfies (M. 1), (M. 3), and (M. 8). We now show $\mathscr{U}$ satisfies (M. 4). Let $A \in P(X)$ and let U, V be in $\mathscr{U}$. There exist U_1, V_1 in $\mathscr{B}$ such that $U \supset U_1$ and $V \supset V_1$. But since $\mathscr{B}$ satisfies (M. 4), we have that there exists $W \in \mathscr{B}$ such that $W[A] \subset U_1[A] \cap V_1[A]$. But $U_1[A] \cap V_1[A] \subset U[A] \cap V[A]$. Consequently, $\mathscr{U}$ satisfies (M. 4). We now show $\mathscr{U}$ satisfies (M. 6). Let A, B be in $P(X)$ and $U \in \mathscr{U}$, and suppose $V[A] \cap B \neq \varnothing$ for all $V \in \mathscr{U}$. Then $V[A] \cap B \neq \varnothing$ for all $V \in \mathscr{B}$. But there exists $U_1 \in \mathscr{B}$ such that $U \supset U_1$. But since $\mathscr{B}$ satisfies (M. 6), we have that there exists an $x \in B$ and there exists a $W \in \mathscr{B}$ such that $W[x] \subset U_1[A]$. But $U_1[A] \subset U[A]$. Consequently, $\mathscr{U}$ satisfies (M. 6).

(2. 23) **Theorem.** *Let (X, δ) be a symmetric generalized proximity space. Then there exists a symmetric generalized uniformity $\mathscr{U}_1(\delta)$ on X such that $\delta(\mathscr{U}_1(\delta)) = \delta$.*

PROOF. For every A, B in $P(X)$ let $U_{A,B} = ((X\times X)-(A\times B)\cup(B\times A))$. Let $\mathscr{V}=\{U_{A,B}|A\bar{\delta}B\}$. It is clear that $\mathscr{V}$ satisfies (M. 1) and (M. 3). We now show that $A\bar{\delta}B$ iff for some C, $D\, C\bar{\delta}D$ and $U_{C,D}[A]\cap B=\emptyset$. Suppose $A\bar{\delta}B$ and there exists $t\in U_{A,B}[A]\cap B$: Then there exists $s\in A$ such that $(s,t)\in U_{A,B}$. But this is a contradiction since $(s,t)\in(A\times B)$. Hence $U_{A,B}[A]\cap B=\emptyset$. Conversely, suppose there exists C, D such that $C\bar{\delta}D$ and $U_{C,D}[A]\cap B=\emptyset$. We first assert that $A\subset C\cup D$; for if $t\in A-(C\cup D)$, then $U_{C,D}[t]=X$ and so also $U_{C,D}[A]=X$, a contradiction. Next we show that $A\subset C$ or $A\subset D$. Suppose there exist $t_1\in A$, $t_2\in A$ such that $t_1\in C$ and $t_2\in D$. Then $U_{C,D}[t_1]=(X-D)$ and $U_{C,D}[t_2]=(X-C)$. But since $C\bar{\delta}D$, we know by (L. 4) that $(X-C)\cup(X-D)=X$. Hence $U_{C,D}[t_1]\cup U_{C,D}[t_2]=X$; so that $U_{C,D}[A]=X$ which is a contradicton. Consequently, $A\subset C$ or $A\subset D$. Suppose the first case is true. Then $U_{C,D}[A]=(X-D)$; so that $B\subset D$, and by Lemma (1. 5) $A\bar{\delta}B$. The proof in the second case is similar.

By the above argument and Theorem (2. 2) we have that $\mathscr{V}$ also satisfies (M. 4) and (M. 6). Consequently, by Theorem (2. 22) $\mathscr{U}_1(\delta)=\{U|U=U^{-1}$ and $U\supset V$ for some $V\in\mathscr{V}\}$ is a symmetric generalized uniformity on X. It is clear that $\delta(\mathscr{U}_1(\delta))=\delta$.

(2. 24) *Corollary.* A topology $\mathscr{I}$ on X is the uniform topology for some symmetric generalized uniformity on X iff $\mathscr{I}$ is symmetric.

PROOF. This is an immediate consequence of Corollary (1. 14), Theorem (2. 23), and Remark (2. 10).

(2. 25) *Remark.* Contrast Corollary (2. 24) with the classical theorem which states that a topology $\mathscr{I}$ on X is the uniform topology for some uniformity on X iff $\mathscr{I}$ is completely regular.

(2. 26) *Definition.* If we are given δ, a symmetric generalized proximity on X, then the class of symmetric generalized uniformities $\mathscr{U}$ on X such that $\delta(\mathscr{U})=\delta$ is called a *proximity class of symmetric generalized uniformities on X* and is denoted by $\pi(\delta)$. Similarly, if δ is a proximity on X, then the class of symmetric uniformities $\mathscr{U}$ on X such that $\delta(\mathscr{U})=\delta$ is called a *proximity class of symmetric uniformities on X* and is denoted $\pi^*(\delta)$.

(2. 27) **Theorem.** *Let (X,δ) be a symmetric generalized proximity space. Let $\mathscr{U}\in\pi(\delta)$. Then*

(a) *$A\delta B$ iff for every $U\in\mathscr{U}(A\times B)\cap U\neq\emptyset$.*

(b) *$A\ll B$ iff there exists $U\in\mathscr{U}$ such that $B\supset U[A]$.*

PROOF (a). Let $\mathscr{U}\in\pi(\delta)$. Suppose $(A\times B)\cap U\neq\emptyset$ for every $U\in\mathscr{U}$. Then $U[A]\cap$ $\cap B\neq\emptyset$ for every $U\in\mathscr{U}$; so that $A\delta B$.

Conversely, suppose $A\delta B$ and $U\in\mathscr{U}$. Then since $\mathscr{U}\in\pi(\delta)$, there exists $b\in U[A]\cap$ $\cap B$. Hence there exists $a\in A$ with $(a,b)\in U$; so that $(A\times B)\cap U\neq\emptyset$.

PROOF (b). Let $\mathscr{U}\in\pi(\delta)$. Suppose $A\ll B$. Then $a\bar{\delta}(X-B)$; so that there exists $U\in\mathscr{U}$ such that $U[A]\cap(X-B)=\emptyset$. Hence $B\supset U[A]$.

Conversely, suppose there exists $U\in\mathscr{U}$ such that $B\supset U[A]$; then $U[A]\cap$ $\cap(X-B)=\emptyset$. Hence $A\bar{\delta}(X-B)$; so that $A\ll B$.

(2. 28) **Theorem.** *Let (X,δ) be a symmetric generalized proximity space. Then $\mathscr{U}_1(\delta)$ (as constructed in Theorem (2. 23)) is the least element of $\pi(\delta)$ (where the partial order on $\pi(\delta)$ is set inclusion).*

PROOF. Let $\mathscr{U} \in \pi(\delta)$. Let $U_{A,B} \in \mathscr{U}_1(\delta)$, $A\bar{\delta}B$. Then by Theorem (2. 27) part (a) there exists $V \in \mathscr{U}$ such that $(A \times B) \cap V = \varnothing$. But since $V = V^{-1}$ we have that $(B \times A) \cap V = \varnothing$. Hence $U_{A,B} \supset V$; so that $U_{A,B} \in \mathscr{U}$.

(2. 29) **Theorem.** *Let (X, δ) be a symmetric generalized proximity space. The union, $\mathscr{B}$, of an arbitrary family of members of $\pi(\delta)$ is a base for a symmetric generalized uniformity in $\pi(\delta)$.*

PROOF. It is clear that $\mathscr{B}$ satisfies (M. 1) and (M. 3). By the definition of $\pi(\delta)$, we have that $A\delta B$ iff, for every $U \in \mathscr{B}$, $U[A] \cap B \neq \varnothing$. Hence by Theorem (2. 2) we have that $\mathscr{B}$ satisfies (M. 4) and (M. 6). Consequently, by Theorem (2. 22) $\mathscr{B}$ is a base for a uniformity on X which clearly is in $\pi(\delta)$.

(2. 30) *Corollary.* Let (X, δ) be a symmetric generalized proximity space. Then $\pi(\delta)$ has a greatest element.

PROOF. This is an immediate consequence of Theorem (2. 29).

(2. 31) *Remark.* C. DOWKER in [7] has shown that a proximity class of symmetric uniformities may fail to have a greatest element.

(2. 32) **Lemma.** *Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. If $(x, y) \in V$ for every $V \in \mathscr{U}$ and $(y, z) \in V$ for every $V \in \mathscr{U}$, then $(x, z) \in V$ for every $V \in \mathscr{U}$.*

PROOF. Let $U \in \mathscr{U}$. By hypothesis $V[x] \cap y \neq \varnothing$ for every $V \in \mathscr{U}$. Hence by (M. 6) there exists $W_1 \in \mathscr{U}$ such that $W_1[y] \subset U[x]$. But $V[y] \cap z \neq \varnothing$ for every $V \in \mathscr{U}$; consequently, there exists $W_2 \in \mathscr{U}$ such that $W_2[z] \subset W_1[y]$. Hence $z \in W_1[y] \subset U[x]$; so that $(x, z) \in U$.

(2. 33) **Theorem.** *Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. If $\mathscr{U}$ has a least element (with respect to set inclusion), then $(X, \mathscr{U})$ is a symmetric uniform space.*

PROOF. It is immediate by the hypothesis that $\mathscr{U}$ satisfies (M. 5). We now show that $\mathscr{U}$ satisfies (M. 7). Let $U \in \mathscr{U}$ and let V be the least element in $\mathscr{U}$. Suppose $(x, y) \in V$ and $(y, z) \in V$. Then by Lemma (2. 32) we have that $(x, z) \in V$; so that $(V \circ V) \subset V \subset U$.

(2. 34) *Definition.* A *decomposition* of a set X is a disjoint family $\mathscr{D}$ of subsets of X whose union is X. A decomposition $\mathscr{D}$ of a topological space $(X, \mathscr{I})$ is *upper-semicontinuous* iff for each $D \in \mathscr{D}$ and each open set A containing D there exists an open set B such that $D \subset B \subset A$, and B is the union of members of $\mathscr{D}$.

(2. 35) **Theorem.** *Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. Let $R = \bigcap \{U \mid U \in \mathscr{U}\}$. Then R is an equivalence relation on X, and X/R is an upper semi-continuous decomposition of $(X, \mathscr{I}(\mathscr{U}))$.*

PROOF. Clearly, R is reflexive and symmetric, and by Lemma (2. 32) R is transitive. Let $A \in \mathscr{I}(\mathscr{U})$. Let $x \in A$. Then there exists $U \in \mathscr{U}$ such that $U[x] \subset A$. But $R \subset U$ for every $U \in \mathscr{U}$. Hence $R[x] \subset A$ for every $x \in A$. Hence $A = \bigcup \{R[x] \mid x \in A\}$. But $R[x] \in X/R$ for every $x \in A$. Consequently, X/R is an upper semi-continuous decomposition of $(X, \mathscr{I}(\mathscr{U}))$.

(2. 36) *Example.* There exists a symmetric generalized uniform space that does not satisfy (M. 5).

PROOF. Let δ be the usual proximity for the reals X. Let $\mathscr{U}_1(\delta)$ be the symmetric generalized uniformity on X as constructed in Theorem (2. 23). Let $A=[1, 2]$; $B=[2, 3]$; $A_1=[3, 4]$; $B_1=[4, 5]$. Clearly $A\bar{\delta}A_1$ and $B\bar{\delta}B_1$. We will show that there does not exist P, Q such that $P\bar{\delta}Q$ and $U_{P,Q} \subset U_{A,A_1} \cap_{B,B_1}$. For suppose there does exist such a P and Q. Then

$$(P\times Q)\cup(Q\times P)\supset(A\times A_1)\cup(A_1\times A)\cup(B\times B_1)\cup(B_1\times B);$$

so that $(P\cup Q) \supset [1, 5]$. This may hold only in the case of $P\supset[1, 5]$ or $Q\supset[1, 5]$ by the connectedness of the interval. In the first case $Q\cap[1, 5] =\varnothing$, i.e. $Q\times P$ contains no point of the set on the right hand side, from which $Q\supset[1, 5]$ follows, which is a contradiction.

(2. 37) *Remark.* Example (2. 36) shows that a base for a symmetric generalized uniformity $\mathscr{U}$ on X may not necessarily be a subbase for $\mathscr{U}$ because a base need not be closed with respect to finite intersections.

III.

P-correct and totally bounded spaces

In this chapter we obtain a generalization of a theorem of Alfsen-Fenstad, Gál, and Smirnov which states that every proximity class of symmetric uniformities contains one and only one totally bounded uniformity.

Let X be a non-void set. For every A, B in $P(X)$ let $U_{A,B} = (X\times X)-((A\times B)\cup \cup(B\times A))$.

(3. 1) *Definition.* Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. $(X, \mathscr{U})$ is *p-correct* iff there exists a symmetric generalized proximity δ on X such that the family $\mathscr{S}=\{U_{A,B}|A\bar{\delta}B\}$ is a subbase for $\mathscr{U}$. δ is called the *generator proximity* for $\mathscr{U}$.

(3. 2) *Remark.* On page 194 in [38] W. J. PERVIN states without proof that if (X, δ) is a proximity space with a proximity class $\pi^*(\delta)$ of symmetric uniformities, then $\mathscr{S}=\{U_{A,B}|A\bar{\delta}B\}$ is a subbase for a uniformity, $\mathscr{U}$, which is in $\pi^*(\delta)$. This construction is the dual of that in Lemma 3. 4 in [11]. Note that it omits the topology parameter from consideration.

(3. 3) *Definition.* Let $(X, \mathscr{U})$ be a symmetric generalized uniform space. $(X, \mathscr{U})$ is *totally bounded* iff for every $U\in\mathscr{U}$ there exists $x_1, \dots, x_n$ in X such that $X = \cup\{U(x_k)|k=1, \dots, n\}$.

(3. 4) *Remark.* Note that if $(X, \mathscr{U})$ is a symmetric uniform space then it can be shown that $(X, \mathscr{U})$ is totally bounded iff for every $U\in\mathscr{U}$, there exists a finite family of sets $\{A_1, \dots, A_k\}$ such that $\cup\{A_k|\ k=1, \dots, n\}=X$ and such that $\cup\{(A_k\times A_k)|\ k=1, \dots, n\} \subset U$.

(3. 5) *Remark.* It is clear that $\mathscr{U}_1(\delta)$, as constructed in Theorem (2. 23), is totally bounded. For if $U_{A,B}\in\mathscr{U}_1(\delta)$, let x_a be any element in A and let x_b be any element

in B. Then $U_{A,B}[x_a] = (X-B)$ and $U_{A,B}[x_b] = (X-A)$; so that since $A \cap B = \emptyset$, $U_{A,B}[x_a] \cup U_{A,B}[x_b] = X$.

The following lemma is crucial for the development of the theory of p-correct symmetric generalized uniform spaces.

(3. 6) **Lemma.** *Let* $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ *be n-tuples of non-void subsets of a set X. Let* $U = U_{A_1, B_1} \cap \ldots \cap U_{A_n, B_n}$. *Let* $I_1 = \{k_1, \ldots, k_p\}$ and $I_2 = \{j_1, \ldots, j_q\}$ *be subsets of* $\{1, \ldots, n\}$. *Suppose* $x_0 \in (A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q})$ *and* $x_0 \notin A_i$ *if* $i \notin I_1$ *and* $x_0 \notin B_i$ *if* $i \notin I_2$. *Then* $U[x_0] = E$, *where E is equal to*

$$(X - B_{k_1}) \cap \ldots \cap (X - B_{k_p}) \cap (X - A_{j_1}) \cap \ldots \cap (X - A_{j_q}).$$

(3. 7) *Remark.* In the sequel to simplify the language we will abbreviate the hypothesis of Lemma (3. 6) as follows: "Suppose $x_0 \in (A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q})$ and x_0 is in no other A_i or B_i."

Proof *of Lemma* (3. 6). By DeMorgan's law.

$$U = (X \times X) - \left(\bigcup_{i=1}^{n} [(A_i \times B_i) \cup (B_i \times A_i)] \right).$$

Suppose $t \in U[x_0]$. Then $(x_0, t) \in U$; so that since $x_0 \in (A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q}$ we have that $t \notin B_{k_i}$ $i = 1, \ldots, p$ and $t \notin A_{j_i}$ $i = 1, \ldots, q$. Consequently, $t \in E$ and $E \supset U[x_0]$. To show the reverse inclusion, suppose there exists $t_1 \in (E - U[x_0])$. Then $(x_0, t_1) \notin U$; so that (x_0, t_1) is an element of $\bigcup_{i=1}^{n} [A_i \times B_i) \cup (B_i \times {}_iA)]$. Suppose $(x_0, t_1) \in (A_m \times B_m)$ where $1 \leqq m \leqq n$. Then since $t_1 \in E$, we have that $m \neq k_i$ for $i = 1, \ldots, p$; so that $x_0 \in A_m$ and $m \notin I_1$ which is a contradiction. Suppose $(x_0, t_1) \in \in (B_m \times A_m)$ where $1 \leqq m \leqq n$. Then since $t_1 \in E$, we have that $m \neq j_i$ for $i = 1, \ldots, q$; so that $x_0 \in B_m$ and $m \notin I_2$ which is a contradiction. Hence $E = U[x_0]$.

(3. 8) *Remark.* Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ be n-tuples of non-void subsets of a set X. Let $I_1 = \{k_1, \ldots, k_p\}$ and $I_2 = \{j_i, \ldots, j_q\}$ be any two subsets of $\{1, \ldots, n\}$ and let

$E = \{x | x \in A_i \text{ iff } i \in I_1 \text{ and } x \in B_i \text{ iff } i \in I_2\}$. If $E \neq \emptyset$, we call E a *residual intersection of the A_i and B_i*.

It is clear that residual intersections are mutually disjoint; so that $\mathscr{R}$, the family of all residual intersections of the A_i and B_i, provides a decomposition of $\bigcup \{(A_i \cup B_i) | i = 1, \ldots, n\}$ into mutually disjoint sets.

(3. 9) **Theorem.** *Let* $(X, \mathscr{U})$ *be a p-correct symmetric generalized uniform space. Then* $(X, \mathscr{U})$ *is totally bounded.*

Proof. Let $U \in \mathscr{U}$, and let δ be a generator proximity for $\mathscr{U}$. Then there exists a finite family of sets $A_1, \ldots, A_n$; $B_1, \ldots, B_n$ such that $A \bar{\delta} B_i$ for $i = 1, \ldots, n$ and $U_{A_1, B_1} \cap \ldots \cap U_{A_n, B_n} = V \subset U$. Now if $\bigcup \{(A_i \cup B_i) | i = 1, \ldots, n\} \neq X$, then for any $x_0 \in X - \bigcup \{(A_i \cup B_i) | i = 1, \ldots, n\}$ we have that $V[x_0] = X$, and the theorem follows; so we assume that $\bigcup \{(A_i \cup B_i) | i = 1, \ldots, n\} = X$. Let $\mathscr{R}$ be the family of all residual intersections of the A_i and B_i. From each $R \in \mathscr{R}$ choose one and only one point and denote that point x_R. Let $S = \{x_R | R \in \mathscr{R}\}$. Clearly, since $\mathscr{R}$ is finite, S is also finite. We now show that $V[S]=X$. Let $z\in X$. Since we assume that $\cup\{(A_i\cup B_i)|i=1,\dots,n\}=X$, we have that $z\in R$ for some $R\in\mathcal{R}$. Consequently, for some $k_1,\dots,k_p$; $j_1,\dots,j_q$, $z\in(A_{k_1}\cap\dots\cap A_{k_p}\cap B_{j_1}\cap\dots\cap B_{j_q})$ and z is in no other A_i or B_i. But by the definition of S there exists x_R in S such that $x_R\in(A_{k_1}\cap\dots\cap A_{k_p}\cap B_{j_1}\cap\dots\cap B_{j_q})$ and x_R is in no other A_i or B_i. By Lemma (3. 6) we have that $V[x_R]$ is equal to $(X-B_{k_1})\cap\dots\cap(X-B_{k_p})\cap(X-A_{j_1})\cap\dots\cap(X-A_{j_q})$. But since $A_i\bar{\delta}B_i$ for all i we have that $z\notin B_{k_i}$ for $i=1,\dots,p$ and $z\notin A_{j_i}$ for $i=1,\dots,q$. Consequently, $z\in V[x_R]$. But z is an arbitrary point in X. Hence $V[S]=X$; so that $U[S]=X$.

(3. 10) **Theorem.** *Let $(X,\mathcal{U})$ be a p-correct symmetric generalized uniform space. Then $(X,\mathcal{U})$ has an open base.*

Proof. Let $U\in\mathcal{U}$. Then there exists a finite family of sets $A_1,\dots,A_n$; $B_1,\dots,B_n$ such that $A_i\bar{\delta}B_i$ for $i=1,\dots,n$ and $(U_{A_1,B_1}\cap\dots\cap U_{A_n,B_n})=V$ is contained in U. But for each i $1\leqq i\leqq n$, $\bar{A}_i\supset A_i$ and $\bar{B}_i\supset B_i$; so that $U_{A_i,B_i}\supset U_{\bar{A}_i,B_i}$. But by Theorem (1. 17a) $\bar{A}_i\bar{\delta}\bar{B}_i$ for $i=1,\dots,n$; so that $U_{\bar{A}_i,B_i}\in\mathcal{U}$ for $1\leqq i\leqq n$. But it is easily shown that $U_{\bar{A}_i,B_i}$ is open for $i=1,\dots,n$. Hence V is open.

Note that we assume that $\mathcal{I}(U)=\mathcal{I}(\delta)$ and $\delta(U)=\delta$. Thisis established in (3. 19).

(3. 11) *Remark*. It is clear that $\mathcal{U}_1(\delta)$ as constructed in Theorem (2. 23) has an open base; for if $U_{A,B}$ is an element of $\mathcal{U}_1(\delta)$, then by the same argument that is given above we have that $U_{A,B}\supset U_{\bar{A},B}$; $U_{\bar{A},B}\in\mathcal{U}_1(\delta)$; and $U_{\bar{A},B}$ is open.

(3. 12) *Remark*. It will be shown (6. 20) that there exists a symmetric generalized uniform space that does not have an open base.

(3. 13) **Theorem.** *A symmetric uniform space $(X,\mathcal{U})$ is totally bounded iff for some proximity δ on X the family $\mathcal{S}=\{U_{A,B}|A\bar{\delta}B\}$ is a subbase for $(X,\mathcal{U})$.*

(3. 14) **Lemma.** *Suppose $\{A_i\}$ and $\{B_i\}$ $i=1,\dots,n$ are finite sequences of non-void subsets of a set X such that for all i $A_i\supset B_i$ and $\cup\{B_i|i=1,\dots,n\}=X$. Then we have that*

$$F=(X\times X)-\bigcup_{i=1}^{n}[[(X-A_i)\times B_i]\cup[B_i\times(X-A_i)]]\subset\bigcup_{i=1}^{n}[A_i\times A_i].$$

Proof *of Lemma* (3. 14). Let $(x,y)\in F$. Then since $\cup\{B_i|i=1,\dots,n\}=X$ we have that $(x,y)\in(B_{k_1}\times B_{k_2})$ where $1\leqq k_1\leqq n$ and $1\leqq k_2\leqq n$. But is it clear that $(x,y)\notin[(X-A_{k_2})\times B_{k_2}]$; so that since $y\in B_{k_2}$, $x\in A_{k_2}$. But $A_{k_2}\supset B_{k_2}$. Hence $(x,y)\in(A_{k_2}\times A_{k_2})$.

(3. 15) **Lemma.** *Let (X,δ) be a proximity space. Let $\mathcal{U}$ be a totally bounded symmetric uniformity on X that is in $\pi^*(\delta)$, a proximity class of symmetric uniformities on X. Then for every $U\in\mathcal{U}$ there exist sets $A_1,\dots,A_n$; $B_1,\dots,B_n$ such that $U\supset U_{A_1,B_1}\cap\dots\cap U_{A_n,B_n}$ and $A_i\bar{\delta}B_i$ for $i=1,\dots,n$.*

Proof *of Lemma* (3. 15). Let $U\in\mathcal{U}$. We know there exists $V\in\mathcal{U}$ such that $V=V^{-1}$ and $(V\circ V\circ V)\subset U$. Then since $(X,\mathcal{U})$ is totally bounded, there exists sets $B_1,\dots,B_n$ such that $\bigcup_{i=1}^{n}[B_i]=X$ and $\bigcup_{i=1}^{n}[B_i\times B_i]\subset V$. Let $A_i=V[B_i]$. Since $V[B_i]\cap(X-V[B_i])=\varnothing$, $A_i\gg B_i$, $i=1,\dots,n$. Also, by a straightforward calculation we can show for $i=1, \ldots, n$ that $(A_i \times A_i) \subset V \circ V \circ V$. Hence we have that $\bigcup_{i=1}^{n} [A_i \times A_i] \subset U$. But by Lemma (3. 14)

$$(X \times X) - \bigcup_{i=1}^{n} [[(X-A_i) \times B_i] \cup [B_i \times (X-A_i)]] \subset \bigcup_{i=1}^{n} [A_i \times A_i];$$

so that

$$U_{B_1, X-A_1} \cap \ldots \cap U_{B_n, X-A_n} \subset U.$$

and

$$B_i \bar{\delta} (X-A_i) \quad \text{for} \quad i = 1, \ldots, n.$$

PROOF *of Theorem* (3. 13). Suppose for some proximity δ on X $\mathscr{S} = \{U_{A,B} | A \bar{\delta} B\}$ is a subbase for $\mathscr{U}$. Then $\mathscr{U}$ is a p-correct symmetric generalized uniformity on X, and hence by Theorem (3. 9) $\mathscr{U}$ is totally bounded.

Conversely, suppose $\mathscr{U}$ is totally bounded. It is known (cf. [40] Theorem (21. 14) and Theorem (21. 15)) that for some proximity δ on X $\mathscr{U} \in \pi^*(\delta)$, a proximity class of symmetric uniformities on X. Suppose $A_i \bar{\delta} B_i$ for $i=1, \ldots, n$. For each i, $i=1, \ldots, n$ there exists a symmetric $V_i \in \mathscr{U}$ such that $(A_i \times B_i) \cap V_i = \varnothing$, and hence such that $U_{A_i, B_i} \supset V_i$. Consequently, we have that $U = (U_{A_1, B_1} \cap \ldots \cap U_{A_n, B_n}) \supset (V_1 \cap \ldots \cap V_n)$; so that $U \in \mathscr{U}$. By this fact and Lemma (3. 15) we have that the family $\mathscr{S} = \{U_{A,B} | A \bar{\delta} B\}$ is a subbase for $\mathscr{U}$.

The following theorem is a generalization of the theorem of Alfsen-Fenstad, Gál, and Smirnov which is mentioned a, the beginning of this chapter.

(3. 16) **Theorem.** *Let (X, δ) be a symmetric generalized proximity space.* There *exists in $\pi(\delta)$ one and only one p-correct symmetric generalized uniformity, $\mathscr{U}_2(\delta)$, on X.*

(3. 17) **Lemma.** *Let (X, δ) be a symmetric generalized proximity space. Let $(C_1, \ldots, C_n)$ and $(D_1, \ldots, D_n)$ be n-tuples of non-void subsets of X such that $C_i \bar{\delta} D_i$ for $i=1, \ldots, n$. Then $(C_1 \cap \ldots \cap C_n) \bar{\delta} (D_1 \cup \ldots \cup D_n)$.*

PROOF *of Lemma* (3. 17). Suppose that $(C_1 \cap \ldots \cap C_n)\ \delta (D_1 \cup \ldots \cup D_n)$. Then by (P. 2) $(C_1 \cap \ldots \cap C_n) \delta D_k$ where $1 \leqq k \leqq n$. But $C_k \supset (C_1 \cap \ldots \cap C_n)$; so that by Lemma (1. 5) $C_k \delta D_k$ which is a contradiction.

(3. 18) **Lemma.** *Let (X, δ) be a symmetric generalized proximity space. Then $P \bar{\delta} Q$ iff there exists n-tuples $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of subsets of X such that $(U_{A_1, B_1} \cap \ldots \cap U_{A_n, B_n})[P] \cap Q = \varnothing$, and $A_i \bar{\delta} B_i$ for $i=1, \ldots, n$.*

PROOF *of Lemma* (3. 18). If $P \bar{\delta} Q$, then by the same argument that is given in the beginning of the proof of Theorem (2. 23) $U_{P,Q}[P] \cap Q = \varnothing$.

Conversely, let $V = U_{A_1, B_1} \cap \ldots \cap U_{A_n, B_n}$. Since $V[P] \cap Q = \varnothing$ we have $P \subset \cup$ $\cup \{(A_i \cup B_i) | i=1, \ldots, n\}$. Let $\mathscr{A} = \{E_1, \ldots, E_m\}$ be the pairwise disjoint family of all residual intersections of the A_i and B_i that have a non-void intersection with P. Clearly, $P \subset M = \cup \{E_c | c=1, \ldots, m\}$. By Lemma (3. 6) since $\mathscr{A}$ is a pairwise disjoint family, if $t_1 \in (P \cap E_c)$ and $t_2 \in (P \cap E_c)$ where $1 \leqq c \leqq m$, then $V[t_1] = V[t_2]$. Let $F_c = V[t_c]$ for $c=1, \ldots, m$ where t_c is a fixed point in E_c. Then we have that $V[P] = \cup \{F_c | c=1, \ldots, m\}$. But since $V[P] \cap Q = \varnothing$ we have that $Q \subset (X - V[P])$; so that by DeMorgan's law $Q \subset N$ where $N = \cap \{(X - F_c) | c=1, \ldots, m\}$. Let $E_c \in \mathscr{A}$

where $1 \leqq c \leqq m$. We may assume that $E_c \subset E_c^* = A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q}$ for some $k_1, \ldots, k_p$; $j_1, \ldots, j_q$ and E_c intersects no other A_i or B_i. Consequently, by Lemma (3. 6) and DeMorgan's law $(X - F_c) = (B_{k_1} \cup \ldots \cup B_{k_p} \cup A_{j_1} \cup \ldots \cup A_{j_q})$. Hence by Lemma (3.17) $E_c^* \bar{\delta} (X - F_c)$ where $1 \leqq c \leqq m$; so that by Lemma (1. 5) $E_c \bar{\delta} (X - F_c)$ where $1 \leqq c \leqq m$. Hence again by Lemma (3. 17) $M \bar{\delta} N$; so that by Lemma (1. 5) $P \bar{\delta} Q$.

(3. 19) **Lemma.** *Let $(X, \mathscr{U})$ be a p-correct symmetric generalized uniform space with generator proximity δ. Then $\delta(\mathscr{U}) = \delta$.*

PROOF *of Lemma* (3. 19). Suppose $P \bar{\delta} Q$. Then by Lemma (3. 18) there exists $U \in \mathscr{U}$ such that $U[P] \cap Q = \emptyset$; so that $P \overline{\delta(\mathscr{U})} Q$.

Conversely, suppose $P \overline{\delta(\mathscr{U})} Q$. Then there exists $V \in \mathscr{U}$ such that $V[P] \cap Q = \emptyset$; so that by Lemma (3. 18) $P \bar{\delta} Q$.

PROOF *of Theorem* (3. 16). Let $\mathscr{S} = \{U_{A,B} | A \bar{\delta} B\}$. Let $\mathscr{B} =$ {all finite intersections of members of $\mathscr{S}$}. It is clear that $\mathscr{B}$ satisfies (M. 1) and (M. 3). By Lemma (3. 18) and Theorem (2. 2) we have that $\mathscr{B}$ also satisfies (M. 4) and (M. 6). Consequently, by Theorem (2. 22) we have that $\mathscr{U}_2(\delta) = \{U | U = U^{-1}$ and $V \supset U$ for some $V \in \mathscr{B}\}$ is a symmetric generalized uniformity on X. It is clear that $\mathscr{U}_2(\delta)$ is p-correct, and by Lemma (3. 19) that $\mathscr{U}_2(\delta) \in \pi(\delta)$. We now show that $\mathscr{U}_2(\delta)$ is the only p-correct symmetric generalized uniformity on X that is in $\pi(\delta)$. For suppose $\mathscr{V} \in \pi(\delta)$ and $(X, \mathscr{V})$ is p-correct with generator proximity δ_1. Clearly, $\delta_1 \neq \delta$ if $\mathscr{U}_2(\delta) \neq \mathscr{V}$. But by Lemma (3. 19) we have that $\delta(\mathscr{V}) = \delta_1$ which is a contradiction, since we assume $\mathscr{V} \in \pi(\delta)$. Hence $\mathscr{V} = \mathscr{U}_2(\delta)$.

(3. 20) *Remark*. We note that if U, V are in $\mathscr{U}_2(\delta)$ (as constructed in Theorem (3. 16)), then $(U \cap V) \in \mathscr{U}_2(\delta)$. Hence if δ is the usual proximity on the reals X, then $\mathscr{U}_1(\delta)$ (as constructed in Theorem (2. 23)) is properly contained in $\mathscr{U}_2(\delta)$ (cf. Theorem (2. 28) and Example (2. 36)). Hence we see that a proximity class of symmetric generalized uniformities may contain two distinct totally bounded uniformities. It is easily shown that a proximity class may contain more than two distinct totally bounded uniformities (cf. (6. 20)).

(3. 21) *Corollary*. (Alfsen-Fenstad, Gál, Smirnov). Let (X, δ) be a proximity space. There exists in $\pi(\delta)$ one and only one totally bounded symmetric uniformity on X.

PROOF. By Theorem (3. 13), Theorem (3. 16), and Remark (3. 20), it is sufficient to show that $\mathscr{U}_2(\delta)$ satisfies (M. 7). We note that if $V_i \circ V_i \subset U_i$ for $i = 1, \ldots, n$, then $(V_1 \cap \ldots \cap V_n) \circ (V_1 \cap \ldots \cap V_n) \subset U_1 \cap \ldots \cap U_n$ where V_i and U_i for $i = 1, \ldots, n$ are subsets of $(X \times X)$. Consequently, it is sufficient to show that for each $U_{A,B} \in \mathscr{U}_2(\delta)$ there exists a $V \in \mathscr{U}_2(\delta)$ such that $V \circ V \subset U_{A,B}$. We now show the existence of such a V. By (P. 5) there exist sets C and D such that $C \cap D = \emptyset$ and $C \gg A$ and $D \gg B$. Let $V = (U_{A, X-C}) \cap (U_{B, X-D})$. We show $V \circ V \subset U_{A,B}$. Suppose $(x, y) \in V$ and $(y, z) \in V$. We must show that $(x, z) \in U_{A,B}$ or equivalently that $(x, z) \notin (A \times B) \cup (B \times A)$. Clearly, if $x \notin (A \cup B)$, then for every $t \in X$ we have that $(x, t) \in U_{A,B}$. Hence we may assume that $x \in (A \cup B)$. Two cases now occur. *Case 1* $x \in A$ and *Case 2* $x \in B$. These are the only possibilities for x since $A \cap B = \emptyset$.

Claim 1. If $x \in A$, then $z \notin B$. For suppose $z \in B$. Then $(y, z) \in (C \times B)$. But since $C \cap D = \emptyset$, $(X-D) \supset C$; so that $((X-D) \times B) \supset (C \times B)$. Hence $(y, z) \notin V$ which is a contradiction. By a similar argument we get

Claim 2. If $x \in B$, then $z \notin A$.

By Claim 1 if $x \in A$, then $(x, z) \notin (A \times B)$; so that $(x, z) \in U_{A,B}$. By Claim 2 if $x \in B$, then $(x, z) \notin (B \times A)$; so that $(x, z) \in U_{A,B}$.

(3. 22) *Remark.* The symmetric generalized uniformity $\mathscr{U}_2(\delta)$ as constructed in Theorem (3. 16) satisfies (M. 5), but might fail to satisfy (M. 7). For let $(X, \mathscr{I})$ be any symmetric topological space which is not completely regular. Define the relation δ_0 on $P(X)$ by $(A\delta_0 B)$ iff $\bar{A} \cap \bar{B} \neq \emptyset$, so that $\mathscr{I}(\delta_0) = \mathscr{I}$. Then $\mathscr{U}_2(\delta_0)$ cannot satisfy (M. 7): for if so, then $\mathscr{U}_2(\delta_0)$ would be a symmetric uniformity and hence $\mathscr{I}$ would be a completely regular topology.

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