A characterization of biperfect topogenous orders

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As is known, biperfect topogenous orders play an important role in the syntopogenous build-up of general topology (see [1], in particular Chapter 5). Their importance is largely due to the fact that a natural one-to-one correspondence exists between the biperfect topogenous orders and the reflexive relations defined on a given set ([1], theorem (5. 39)—(5. 41)).

In this short note we are going to establish a result, running parallel as it were, to the theorem just quoted. Biperfect topogenous orders over a given set will be shown to be equivalent to a class of entities, more complicated than reflexive relations, but endowed with a simple meaning that may appeal to our intuition.

We start with two definitions, one of them well-known:

Definition 1. A biperfect topogenous order on a set E is a relation $<$ defined on the set of all subsets of E, satisfying the following axioms:

$(O1)$ $\emptyset < \emptyset$, $E < E$;
$(O2)$ $A < B$ implies $A \subseteq B$;
$(O3)$ $A \subseteq A' < B' \subseteq B$ implies $A < B$;
$(B_\cap)$ $A < B_i$ for $i \in I$ implies $A < \bigcap \{B_i | i \in I\}$;
$(B_\cup)$ $A_i < B$ for $i \in I$ implies $\bigcup \{A_i | i \in I\} < B$. ■

Definition 2. A kernelled covering of a set E is a triple (γ, π, f), with γ a covering of E, π a partition of E and f a one-to-one mapping of π onto γ, satisfying $P \subseteq f(P)$ for any $P \in \pi$. ■

The intuitive background of the second definition is quite simple. If γ is a covering of E, then for each $x \in E$ let us choose a set $G \in \gamma$ containing x. (In other words: for each $x \in E$, choose a member of St (x, γ).) Now, two points of E will be considered equivalent, if the same set has been chosen for them. If we still make correspond to each equivalence class its chosen covering set, then we obtain a kernelled covering (γ_1, π, f), γ_1 being the class of chosen sets[1]), π the partition formed by the equivalence classes just defined, with the correspondence f clearly one-to-one.

Thus we see that each covering of a set gives raise in a natural way to kernelled coverings, these being essentially the same as the selection functions operating on the stars of the space's points.

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[1]) γ_1 is a subset of γ. There are cases when this subset is necessarily proper. (E.g. cover a set by all its subsets.)

Let now $<$ be a biperfect topogenous order. Put

$$U_{<}(x) = \{y | x \not< E - y\} = \bigcap \{V | x < V\}.$$

Then $\gamma_{<} = \{U_{<}(x) | x \in E\}$ is a cover of E. — Write now $x \sim y$ if and only if $U_{<}(x) = U_{<}(y)$.

This is clearly an equivalence relation, and if we still write $\bar{x} = \{y | x \sim y\}$ and $f_{<}(\bar{x}) = U_{<}(x)$, then the function $f_{<}$ will be a one-to-one mapping of the partition $\pi_{<} = \{\bar{x} | x \in E\}$ onto the cover $\gamma_{<}$, satisfying $\bar{x} \subseteq f_{<}(\bar{x})$ for any $\bar{x} \in \pi_{<}$. Thus we have established the following

Proposition 1. *If $<$ is a biperfect topogenous order on the set E, then $\mathfrak{A}_{<} = (\gamma_{<}, \pi_{<}, f_{<})$ is a kernelled covering of E.* ■[2])

Let now a kernelled covering $\mathfrak{A} = (\gamma, \pi, f)$ of E be given. We write $A < B$ iff $x < B$ for any $x \in A$, and $x < B$ iff $x \in P \subseteq f(P) \subseteq B$ for some $P \in \pi$. The relation $< = <_{\mathfrak{A}}$ so defined is a biperfect topogenous order on E; the verification is straightforward.[3]) We thus have

Proposition 2. *If $\mathfrak{A}$ is a kernelled covering of the set E, then $<_{\mathfrak{A}}$ is a biperfect topogenous order on E.* ■

The following proposition will show that the mappings $< \to \mathfrak{A}_{<}$ and $\mathfrak{A} \to <_{\mathfrak{A}}$ are one-to-one correspondences, inverse to each other, between the sets of all biperfect topogenous orders and all kernelled coverings on E.

Proposition 3.

(I) *Let $< \to \mathfrak{A}_{<}$ and $\mathfrak{A} \to <_{\mathfrak{A}}$. If $\mathfrak{A} = \mathfrak{A}_{<}$, then $<_{\mathfrak{A}} \; = \; <$.*

(II) *Let $\mathfrak{A} \to <_{\mathfrak{A}}$ and $< \to \mathfrak{A}_{<}$. If $< = <_{\mathfrak{A}}$, then $\mathfrak{A}_{<} = \mathfrak{A}$.*

PROOF. (I) If $\mathfrak{A} = \mathfrak{A}_{<}$, then each of the following statements is equivalent to the next one:

$$x <_{\mathfrak{A}} B;$$

$$x \in \bar{y} \subseteq U_{<}(y) \subseteq B \quad \text{for some} \quad y \in E;$$

$$x \in \bar{x} \subseteq U_{<}(x) \subseteq B;$$

$$U_{<}(x) \subseteq B;$$

$$x < B.$$

Thus $<_{\mathfrak{A}} = <$ results proved.

(II) It is clear that if $< = <_{\mathfrak{A}}$ for some kernelled covering $\mathfrak{A}$, then in view of "$x < B$ iff $x \in P \subseteq f(P) \subseteq B$" we have $U_{<}(x) = f(P)$. Clearly, if x runs through E, $f(P)$ will run through γ. This proves $\gamma_{<} = \gamma$. Also $\pi_{<} = \pi$, because $x \sim y$, i.e. $U_{<}(x) = U_{<}(y)$ iff x and y belong to the same member of the partition π. By what has

[2]) If the relation $<$ is a semi-topogenous order (i.e. if it is supposed to satisfy only the first three of the five conditions listed in Definition 1), then the two formulae used to define $U_{<}(x)$ fail to be equivalent. This yields two possibilities for the definition of $U_{<}(x)$, the validity of Proposition 1. being preserved by each of the two.

[3]) Of course, π being a partition, the set $P \in \pi$ is uniquely determined by x, a fact needed in establishing property $(B_{\cap})$.

already been said, we clearly have also $f_{<}=f$, since to a given member of $\pi_{<}=\pi$ there belongs the same $f(P)=U_{<}(x)$ according to $\mathfrak{A}$ and to $\mathfrak{A}_{<}=\mathfrak{A}(<_{\mathfrak{A}})$. ∎

Thus we have proved the two notions of biperfect topogenous order and kernelled covering to be equivalent.

Definition 3. A biperfect topogenous order $<$ on a set E is subordinated to a covering γ of E, if $<=<_{\mathfrak{A}}$ for some kernelled covering $\mathfrak{A}=(\alpha, \pi, f)$ such that $\alpha \leqq \gamma$[4]).

A covering γ is superposed to the biperfect topogenous order $<$, if $<$ is subordinated to γ. ∎

One easily sees that the symmetrical perfect topogenous order defined by

$$A <_{\gamma} B \quad \text{if and only if} \quad \mathrm{St}(A, \gamma) \subseteqq B$$

is the intersection of all the biperfect topogenous orders subordinated to γ.

As a matter of fact, let $<$ be the intersection of all the biperfect topogenous orders subordinated to γ. Then

$$A<B \quad \text{iff} \quad A<_{\mathfrak{A}} B \quad \text{for each} \quad \mathfrak{A}=(\alpha, \pi, f) \quad \text{with} \quad \alpha \leqq \gamma$$

Equivalently, $A<B$ iff for each $x \in A$ and any $G \in \gamma$ satisfying $x \in G$, one has $x \in G \subseteqq B$.[5]) Thus

$$A<B \quad \text{iff} \quad \bigcup_{x \in A} St(x, \gamma) = St(A, \gamma) \subseteqq B,$$

i.e. $<=<_{\gamma}$. ∎

The one-to-one correspondence existing between biperfect topogenous orders and kernelled coverings over a given set E makes it natural to adopt the following

Definition 4. If $\mathfrak{A}_1=(\alpha_1, \pi_1, f_1)$ and $\mathfrak{A}_2=(\alpha_2, \pi_2, f_2)$ are kernelled covers and $<_{\mathfrak{A}_1}$ and $<_{\mathfrak{A}_2}$ are the corresponding biperfect topogenous orders over a set E, then

$$\mathfrak{A}_1 \leqq \mathfrak{A}_2 \quad \text{if and only if} \quad <_{\mathfrak{A}_1} \subseteqq <_{\mathfrak{A}_2}. \text{ ∎}^{6)}$$

If we wish to obtain an "inner" characterization of the partial order thus introduced, we arrive at the following

Proposition 4. $\mathfrak{A}_1 \leqq \mathfrak{A}_2$ *iff for any* $P \in \pi_1$ *and* $Q \in \pi_2$, $P \cap Q \neq \emptyset$ *implies* $f_2(Q) \subseteqq f_1(P)$.

Corollary. *If* $\mathfrak{A}_1 \leqq \mathfrak{A}_2$, *then* $\alpha_2 \leqq \alpha_1$ *for the first components.*

PROOF. The condition is necessary:

Let $\mathfrak{A}_1 \leqq \mathfrak{A}_2$. If $P \cap Q \neq \emptyset$ $(P \in \pi_1, Q \in \pi_2)$, then $x \in P \cap Q$ for some $x \in E$, and $x \in P \subseteqq f_1(P)$ implies $x <_1 f_1(P)$.

Then, however, $x <_2 f_1(P)$, i.e. $x \in Q' \subseteqq f_2(Q') \subseteqq f_1(P)$ for some $Q' \in \pi_2$. But π_2 is a partition of E, so $Q'=Q$ and $f_2(Q) \subseteqq f_1(P)$.

The condition is sufficient:

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[4]) I.e. for $A \in \alpha$ there is $G \in \gamma$ such that $A \subseteqq G$.

[5]) This in view of the "selection-function" aspect of the notion of kernelled covering.

[6]) Sometimes, for simplicity' sake we write $<_1$ instead of $<_{\mathfrak{A}_1}$.

Suppose it is satisfied, and let $x <_1 B$, i.e. $x \in P \subseteqq f_1(P) \subseteqq B$ for some $P \in \pi_1$. Now for the uniquely determined $Q \in \pi_2$ satisfying $x \in Q$ we have $x \in Q \subseteqq f_2(Q) \subseteqq \subseteqq f_1(P) \subseteqq B$, hence $x <_2 B$. By the biperfectness of $<_2$ this yields $<_1 \subseteqq <_2$, i.e. $\mathfrak{A}_1 \leqq \mathfrak{A}_2$.

This completes the proof of the proposition.

Now let $\mathfrak{A}_1 \leqq \mathfrak{A}_2$. Since f_2 is an onto mapping, each member of α_2 can be written in the form $f_2(Q)$, $Q \in \pi_2$, and if $x \in Q$ then for $x \in P \in \pi_1$ we have $f_2(Q) \subseteqq f_1(P) \in \alpha_1$. Hence $\alpha_2 \leqq \alpha_1$. $\blacksquare$

In the same manner as one partially orders the kernelled covers of a given set by referring to the corresponding biperfect topogenous orders, one can also define unions and intersections of kernelled covers with the help of their biperfect topogenous counterparts.

References

[1] Á. Császár, Foundations of general topology, *Oxford*, 1963.

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