On the set of normality and iteration of $e^z - 1$

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1°. Introduction

The theory of iteration of a rational or entire function $f(z)$ of the complex variable z treats the sequence of iterates $\{f_n(z)\}$ defined by

$$f_0(z) = z, \quad f_1(z) = f(z), \quad f_{n+1}(z) = f(f_n(z)), \qquad n = 1, 2, \ldots,$$

and are respectively rational or entire according as $f(z)$ is.

In the theory developed by Fatou [4, 5] and Julia [7] a fundamental role is played by the set $\mathfrak{F}(f)$ of those points of the complex plane where $f_n(z)$ is not normal, in the sense of Montel. $\mathfrak{F}(f)$ is a *perfect* set [4, 5] whose complement $C(f)$ consists of an atmost countably infinite set of components G_i each of which is a maximal domain where $\{f_n(z)\}$ is normal.

If $f(z)$ is rational then it is known [4] that the number of components G_i of $C(f)$ is 0, 1, 2, or ∞. For entire $f(z)$ we shall prove

Theorem 1. *If $f(z)$ is entire which is not a polynomial and if the set $C(f)$ has a finite number of disjoint components, then it is atmost one.*

The determination of $\mathfrak{F}(f)$ and G_i's corresponding to a given $f(z)$ is a problem of considerable difficulty, particularly when $f(z)$ is entire. So it is that despite the need urged by Fatou ([5]) to establish by numerous examples the various ways in which $\mathfrak{F}(f)$ can divide the plane only a few examples have been worked out so far [5, 6, 8]. In this note we shall consider the iteration of $e^z - 1$.

All the cases of theorem can indeed occur. For example the function kze^z, where k is suitable constant considered by BAKER [2] has no component of normality, the function sin z considered by TÖPFER [8] has an infinite number of components of normality. For $e^z - 1$ we shall show that $C(f)$ has a single component.

2°. Preliminaries

A set D is said to be *invariant* under the iteration of $f(z)$ if $f(D) \subset D$, and *completely invariant* if in addition $f_{-1}(D) \subset D$ for all the branches of the inverse function.

The following lemma is well known [4, 5, 7].

Lemma 1. *The set $\mathfrak{F}(f)$ and its complement $C(f)$ are completely invariant.*

Among the components of $C(f)$ there may occur completely invariant ones. We have ([1])

Lemma 2. [Baker] *If $f(z)$ is entire transcendental then $C(f)$ has atmost one completely invariant component.*

Lemma 3. *Let G be a component of $C(f)$ such that some sequence $\{f_{n_k}\}$, n_k strictly increasing, $k=1, 2, \ldots$, has a nonconstant limit function $\varphi(z)$ in G. Then $C(f)$ has a component G^*, which contains $\varphi(G)$ and which is mapped one to one onto itself by some iterate $f_p(z)$ and $\psi(z) \equiv z$ is a limit function of sequence $\{f_{m_k}\}$, m_k increasing in G^*. Further $f(z)$ is univalent in G^*.*

If such a component G^* exists then it is called a *singular domain*.

The proof of this lemma was given by Fatou [4] for rational functions. A proof under more general circumstances is given by H. Cremer [3].

If $w=f_n(z)$ we say that w is *successor* of z and z is a *predecessor* of w in both cases of order n.

A value z_0 is said to be *Fatou exceptional*, if it has atmost a finite set of predecessors. It is easy to see that an entire function can have atmost one such exceptional value.

Lemma 4. [Fatou, 5.] *If α is any finite value other than a Fatou exceptional one and if $\beta \in \mathfrak{F}$ then there exists a sequence of integers $n_k \to \infty$ and values $\beta_k \to \beta$ such that $f_{n_k}(\beta_k)=\alpha$.*

Finally we have [4, 5]

Lemma 5. *The set $\mathfrak{F}(f)$ is identically equal to the extended plane if it has an interior point.*

3°. Proof of Theorem 1.

Suppose that G_i, $i=1, 2, \ldots, N$ are the disjoint components of $C(f)$ where $N<\infty$. For any G_i consider $f_{-1}(G_i)$ for any branch $f_{-1}(z)$ of the inverse function f_{-1} of f. Then $f_{-1}(G_i)$ will be in $C(f)$ by lemma 1, i.e. will be in a number of G_j. If G_j is a component meeting any $f_{-1}(G_i)$, then since $f(G_j)$ belongs to a single component of $C(f)$, we must have $f(G_j) \subset G_i$. Clearly then, $f_{-1}(G_i)$ and $f_{-1}(G_k)$, $i \neq k$ will constitute different sets of domains and so f_{-1} must induce a permutation π among the G_i, such that $f_{-1}(G_i) \subset G_{\pi(i)}$. There is an integer n for which $\pi^n=1$ and for this n we have $f_{-n}(G_i) \subset G_i$ for each $i=1, 2, \ldots, N$ where $f_{-n}(G_i)$ means the predecessor of order n of G_i. Thus each G_i is completely invariant for the function f_n, which is also entire transcendental. By Lemma 2. N cannot be greater than 1 and the theorem is proved.

4°. Iteration of e^z—1.

Let $f(z) = e^z - 1$, $z = x + iy$.

(a) Since $|f(z)+1| = |e^z| = e^x < 1$ if $x<0$, we see that $x<0$ implies $\operatorname{Re} f(z)<0$, i.e., the left half plane H: $\operatorname{Re} z<0$ is invariant and hence $\{f_n(z)\}$ is normal in H. Also for $x<0$, $x<f(x)<0$ i.e. $\{f_n(x)\}$, $x<0$, is a monotone increasing sequence which must converge some limit $t(<0)$ for which $f(t)=t$. Then t must be 0. Thus $\{f_n(x)\}$ converges to 0 for all $x<0$, and hence since $\{f_n(z)\}$ is normal in H, $\{f_n(z)\}$ converges to 0 for all z in H. Thus H belongs to an invariant component of $C(f)$ which is a maximal domain of normality, say G^*.

(b) We now notice that G^* extends across the imaginary axis except at the countable set of points $z=2n\pi i$, n integer. Because for any point $z=iy$ $(y\neq 2n\pi i)$ of the imaginary axis $f(z) = e^{iy}-1 \in H \subset C(f)$. Then by complete invariance of $C(f)$ [Lemma 1.] $z=iy\in H$. Since $f(z) = z+z^2/2+\cdots$, it is clear that $\{f_n(z)\}$ cannot be normal at $z=0$, i.e. $0\in\mathfrak{F}$. Since $f(2n\pi i)=0$ we have [Lemma 1.] $2n\pi i\in\mathfrak{F}$. Hence not only does H belong to G^* but so do all points of the imaginary axis, except the points $z=2n\pi i$, n integer.

(c) We now show that G^* is *completely invariant.*

We prove

Lemma 6. *Let G be a component of $C(g)$ where g is an entire or rational function. Let $\alpha\in G$ be such that* (i) *α is not a singularity of any branch of g_{-1}.* (ii) *$g(\alpha)\in G$ and $g_{-1}(\beta)\in G$ for all the branches of the inverse function. Then G is completely invariant.*

Proof. Let $z\in G$ be any point. We need to show that $g(z)\in G$ and $g_{-1}(z)\in G$, for all the branches of the inverse function.

Since G is a domain we can join α to z by a curve γ lying wholly in G. We note that $\gamma\subset C(g)$ and $\partial G\subset\mathfrak{F}$, where ∂G is the boundary of G.

First suppose $g(z)\notin G$. Now $g(z)\in C(g)$ and $g(\gamma)$ is a continuous curve joining $g(\alpha)\in G$ to $g(z)\notin G$. This implies that $g(\gamma)$ must cross ∂G i.e. there is a point δ which belongs to $g(\gamma)$ and ∂G at the same time. This is impossible since $g(\gamma)$ belongs to $C(g)$ and ∂G belongs to $\mathfrak{F}$.

Next we show that $g_{-1}(z)$ belongs to G for all the branches of the inverse function. Suppose this is not true.

Now any $z\in G$ can be joined to $\alpha\in G$ by a polygonal path. By slight variations in the sides of this path, so small that they leave it (the path) in G, we can ensure (Gross' Star Theorem) that a given branch p of $g_{-1}(z)$ can be continued from z along the path to a regular branch over α, lying arbitrarily near α and so along a path right upto α. Since α is not a singularity of $g_{-1}(z)$ the continuation extends further over to α itself, i.e. p may be obtained from a branch q of $g_{-1}(\alpha)$ by continuation along a polygonal path γ in G. But every branch q of $g_{-1}(\alpha)$ is in G by assumption. Also by Lemma 1. we know that $g_{-1}(z)$ maps $\gamma\subset C(g)$ into $C(g)$. This fact gives us a contradiction as in the first case. *This completes the proof of the lemma.*

Proof of (c). Consider the point $z=-2$ which belongs to G^*. Clearly $f(-2)$ belongs to G^* and $f_{-1}(-2) = \log(-1) = (2n+1)\pi i$ belongs to G^*, by (b) above. Also -2 is not a singularity of $f_{-1}(z)$. Hence by lemma 6, it follows that G^* is completely invariant.

(d) *The positive real axis belongs to the set* $\mathfrak{F}$

We have already shown [in (b)] that

$$0 \in \mathfrak{F}. \tag{1}$$

Take $x_0 > 0$ and suppose $x_0 \notin \mathfrak{F}$. Then $\{f_n(z)\}$ is normal in some neighbourhood $N: |z - x_0| < 2R$ of x_0. We observe that for $x_0 > 0$

$$\lim_{n\to\infty} f_n(x_0) = \infty \tag{2}$$

and since $\{f_n(z)\}$ is normal in N we have $\lim_{n\to\infty} f_n(z) = \infty$ locally uniformly in N. Thus

$$|f_n(z)| > 2 \quad \text{for some} \quad n > n_0 \text{ in } M: |z - x_0| < R \tag{3}$$

This implies

$$\mathrm{Re}\,[f_{n-1}(z)] > 0 \quad \text{for} \quad z \in M \tag{4}$$

Now

$$f_n'(x_0) = \prod_{k=0}^{n-1} f'(f_{k-1}(x_0)) > \exp(f_{n-1}(x_0)) \to \infty \tag{5}$$

by (2).

Thus by *Bloch's Theorem*, the disc M contains a subdomain which is mapped by $f_{n-1}(z)$ on to a domain U_1 containing a disc, say, U_2 of radius $R \cdot B \cdot f_{n-1}'(x_0)$ where B is the Bloch constant. By (4) U_1 must lie on the right half plane $\mathrm{Re}\, z > 0$, and by (5) the radius of U_2 can be made arbitrarily large for large enough n.

Let the disc U_2 be of radius $> 2\pi$ and let d denote the vertical diameter of the disc U_2. The equation of d is, say, $\mathrm{Re}\, z = \lambda$ where $\lambda > 2$ [by (3)] for $n > n_0$.

We notice that $d \subset C(f)$, since $N \subset C(f)$. Now $f(z)$ maps the vertical diameter d of U_2 to the circle of radius λ with centre at -1. We call this circle $f(d)$.

Since $f(d)$ meets G^*, we must have $f(d) \subset G^*$. Also from the maximum modulus principle, it is clear that G^* is simply connected. Thus the interior of $f(d)$ must belongs to G^*, i.e. to $C(f)$. But this is absurd since the interior $f(d)$ contains the point 0 which belongs to $\mathfrak{F}$ [by (1)]. Hence $x_0 > 0$ belongs to $\mathfrak{F}$.

Summerising these results we have

Theorem 2. *For* $f(z) = e^z - 1$, *the half plane* $\mathrm{Re}\, z < 0$ *is an invariant domain and is contained in a completely invariant component* G^* *of* $C(f)$. *In* G^* *we have* $\lim_{n\to\infty} f_n(z) = 0$. *The domain* G^* *includes all points of the imaginary axis except points of the form* $z = 2n\pi i$, *n integer, which belong to* $\mathfrak{F}$. *Further, the positive real axis* R^+ *belongs to* $\mathfrak{F}$. *The set* $\mathfrak{F}$ *may therefore be defined as* [by lemma 4 and since $\mathfrak{F}$ is perfect] *as consisting of* R^+ *together with all its predecessors and points of accumulations. By the periodicity of* $f(z)$, $\mathfrak{F}$ *also contains the reys*

$$d^k: y = 2\pi k i, \ k \text{ integer.}$$

TÖPFFER [8] made some statements about the iteration of $e^z - 1$, without proof. Theorem 1 contains essentially these statements.

Next we prove

Theorem 3. *G^* is the only component of $C(f)$.*

PROOF. Suppose there exists another component G_1, which must necessarily belong to the right half plane Re $z>0$. This is because the left half plane is contained in G^* and since G^* is completely invariant we must also have $f_n(G_1)\cap G^* = \emptyset$. Let $\beta\in G_1$ and $M: |z-\beta| < R$ be a neighbourhood of β, whose closure lies in G_1. Since $\{f_n(z)\}$ is normal in M every convergent subsequence tends either to a nonconstant or to a constant limit.

In the first case there is a component [by Lemma 3.] G_2 of $C(f)$ mapped one to one onto itself by some iterate $f_p(z)$ of $f(z)$. Clearly $G_2\neq G^*$ and G_2 together will all its images $f(G_2), f_2(G_2), \dots, f_{p-1}(G_2)$ lies in the right half plane Re $z>0$. Moreover z is a limit function of some sequence $\{f_{n_k}\}$ in G_2. Take a point z_1 in G_2 and its images $f_n(z_1)$. Then $f_{n_k}(z_1)\to z_1$ (as $k\to\infty$) and $\left|f\left(f_{n_k}(z_1)\right)\right| \to |e^{z_1}|$ (as $k\to\infty$) and so $> 1+\delta > 1$ for large k. Hence for large n,

$$|f_n'(z_1)| = \prod_{i=0}^{n-1} \left|f'\left(f_i(z_1)\right)\right| = \prod_{i=0}^{n-1} \left|\exp\left(f_i(z_1)\right)\right| \to \infty.$$

Thus by *Bloch's theorem* $f_n(G_2)$ contains a disc of radius $>2\pi$ if n is large enough. This disc lies in $C(f)$ and also in the right half plane Re $z>0$. This implies that it must meet one of the rays d^k of Theorem 1. This is a contradiction since the rays d^k belong to $\mathfrak{F}$. Hence there is no subsequence of $\{f_n(z)\}$ with nonconstant limit function.

In the disc M defined above any convergent subsequence of $\{f_n(z)\}$ thus has a constant limit. Since Re $[f_n(\beta)] > 0$ for all $\beta\in G_1$, we see that

$$|f_m'(\beta)| = \prod_{i=0}^{m-1} \left|f'\left(f_i(\beta)\right)\right| = \prod_{i=0}^{m-1} \left|\exp\left(f_k(\beta)\right)\right| > 1$$

and by Bloch's theorem the image of $f_m(G_2)\supset f_m(M)$ contains a disc of some fixed radius for all m. Thus since no nonconstant limits exist, the only possible limit of any subsequence is ∞. But then $f_n(z)\to\infty$ uniformly in M. Thus $|f_n(\beta)|>2\pi$ for $n>N_0$. It follows that there is $\eta>0$ such that Re $[f_n(\beta)]>\eta$ for every $n>N_0$. For if this were not true then

$$|f_{n+1}(\beta)-1| < e^\eta \quad \text{and} \quad |f_{n+1}(\beta)| < 1+e^\eta < 2\pi, \text{ if } \eta \text{ is small.}$$

Thus for $n>N_0$, we have

$$|f_m'(\beta)| > \prod_{i=N_0}^{m-1} \left|\exp\left(f_i(\beta)\right)\right| = \exp\left[\eta(m-1-N_0)\right] \to \infty \qquad (m\to\infty).$$

Then Bloch's theorem gives us a contradiction just as in the case of nonconstant limit function.

Thus G^* is the only component of $C(f)$.

Hence G^* together with its boundary points [by Lemma 5.] covers the whole plane.

Theorem 4. *Every point interior to a line d^k of theorem 2 is inaccessible from G^**

PROOF. Since $\mathfrak{F}$ is invariant under $z \to z+2\pi i$, we need only to show that every interior point of d^0 (i.e. the positive real axis) is inaccessible from G^*. The same argument holds for any d^k.

Let P be any point to d^0 which is suppose, accessible from G^*. Then there is a simple Jordan arc, say mP lying entirely in G^* except for the end point P. We may for example assume m on the imaginary axis and mP in the right half plane except for m. Also the origin 0 is clearly accessible from G^* by any path in the left half plane H. Join m to 0 by a Jordan arc $mn0$ lying in $G^* \cap H$ except for m and 0. Then $L=0Pmn$ is a simple closed Jordan curve dividing the plane into two parts. Let D be the set of predecessors od every order of the rays d^{+1} and d^{-1}. Then D is symmetric with respect to the real axis since $f(z)$ is real for real z. Every point of $0P$ (=segment of the real axis joining 0 to P) is a limit point of points belonging to the set D (say predecessors of d^{+1}) [by Lemma 4.] and since these are symmetric with respect to the real axis, it follows that there are points of D inside L. Furthermore the curves in D go to infinity since d^{+1} and d^{-1} do. Hence they (curves in D) must meet either $Pmn0$ or $0P$ at some point. But this is is impossible since all f_n are real on $0P$ and $D \subset \mathfrak{F}$ cannot meet $Pmn0 \subset C(f)$. Thus every point of d^0 (save for 0 and ∞) is inaccessible from G^*. The same property applies to antecedent curves.

The proof of Theorem 4. is now complete.

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