Radius of convexity of convex sum of univalent functions

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Introduction

Let S denote the class of functions $f(z)$, normalized so that $f(0)=0$, $f'(0)=1$, that are regular univalent in E, the open unit disc. For a fixed real number λ, $0<\lambda<1$, consider the function $h_\lambda(z)$ given by

$$h_\lambda(z) = (1-\lambda)z+\lambda f(z). \tag{1}$$

TRIMBLE [4] showed that if f is convex then $h_\lambda(z)$ is close-to-convex (and hence univalent) in E for all admissible values of λ and if $\lambda \geqq \frac{2}{3}$, then $h_\lambda(z)$ is starlike univalent in E.

Let J denote the class of functions $f(z)$ normalized as above, for which $\operatorname{Re} f'(z)>0$ for z in E. Then it is well known [3] that J is a proper subclass of S. In a recent paper [1], the author considered the class of function $h_\lambda(z)$ defined by (1) when $f\in J$ and obtained its radius of convexity. In this paper we propose to consider a similar problem. Let J_B denote the subclass of J whose functions $f(z)$ are characterized by the property that

$$|f'(z)-1| < 1 \qquad z\in E. \tag{2}$$

Let $h_\lambda(z)$ be defined by (1) and $f\in J_B$. Then we obtain the radius of convexity of the class of functions $h_\lambda(z)$ so defined. It may be remarked that while in [1], ZMOROVIC's technique [5] was found to work well, here a different method had to be adopted.

Radius of convexity

Theorem *For a fixed λ, $0<\lambda<1$ let $h_\lambda(z)$ be defined by* (1) *and $f\in J_B$. Then $h_\lambda(z)$ is convex in*

$$|z| < r_\lambda = \begin{cases} \left(\dfrac{\sqrt{(1-\lambda)(1+3\lambda)}-(1-\lambda)}{2\lambda}\right)^{1/2} & \text{for } 0<\lambda\leqq \dfrac{\sqrt{5}+1}{4}, \\ \dfrac{1}{2\lambda} & \text{for } \dfrac{\sqrt{5}+1}{4}\leqq\lambda<1. \end{cases} \tag{3}$$

These results are sharp in the sense that for each λ, there exists a function $h_\lambda(z)$ which is not convex in a larger circle.

PROOF. Since the function $\psi(z)=f'(z)-1$ is bounded by one, $\psi(0)=0$, we have by Schwarz lemma

$$|f'(z)-1| = |\psi(z)| \leqq |z|.$$

Thus we may write

$$f'(z)-1 = z\Phi(z) \tag{4}$$

where $\Phi(z)$ is regular and $|\Phi(z)|\leqq 1$ in E. Differentiating (1) and substituting the value of $f'(z)$ from (4), we obtain

$$h_\lambda'(z) = 1+\lambda z\Phi(z) \tag{5}$$

so that

$$\operatorname{Re}\left\{1+\frac{zh_\lambda''(z)}{h_\lambda'(z)}\right\} = \operatorname{Re}\left\{1+\frac{\lambda z(\Phi(z)+z\Phi'(z))}{1+\lambda z\Phi(z)}\right\} \geqq \tag{6}$$

$$\geqq 1-\max_{|z|=r}\max_{\Phi}\lambda r\left\{\frac{|\Phi|+r|\Phi'|}{1-\lambda r|\Phi|}\right\} \geqq 1-\max_{\varkappa}\lambda r\left\{\frac{(1-r^2)\varkappa+r(1-\varkappa^2)}{(1-r^2)(1-\lambda r\varkappa)}\right\},$$

$$r=|z|,\ \varkappa=|\Phi|,\ 0\leqq\varkappa\leqq 1,$$

where we have made use of the well-known fact that ([2])

$$|\Phi'| \leqq \frac{1-|\Phi|^2}{1-r^2}.$$

To maximize the second term on the right of (6), let us put

$$\Psi = \frac{(1-r^2)\varkappa+r(1-\varkappa^2)}{1-\lambda r\varkappa}$$

which yields on differentiation

$$\frac{d\Psi}{d\varkappa} = \frac{\lambda r^2\varkappa^2-2r\varkappa+1-r^2+\lambda r^2}{(1-\lambda r\varkappa)^2}$$

so that the absolute maximum of Ψ occurs for

$$\varkappa = \varkappa_0 = \frac{1-\sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda r}. \tag{7}$$

It is easy to verify that $\frac{d^2\Psi}{d\varkappa^2}<0$ so that we have indeed a maximum at (7). Also, $\varkappa_0$ is clearly positive but not always less than or equal to one. So we have to consider the case when $\varkappa_0\notin[0,1]$. In this case Ψ is monotonic increasing so that the maximum value of Ψ occurs at $\varkappa=1$.

Taking $\varkappa=1$, we obtain from (6)

$$\operatorname{Re}\left\{1+\frac{zh_\lambda''(z)}{h_\lambda'(z)}\right\} \geqq 1-\frac{\lambda r}{1-\lambda r} = \frac{1-2\lambda r}{1-\lambda r}$$

so that $h_\lambda(z)$ is convex in $|z|<r_\lambda$ where r_λ is the smallest positive root of the equation

$$1-2\lambda r = 0.$$

This gives

$$r\lambda = \frac{1}{2\lambda}. \tag{8}$$

Since $r_\lambda \leqq 1$, we must have $\lambda \geqq \frac{1}{2}$ so that formula (8) is not valid for $\lambda < \frac{1}{2}$.

Taking $\varkappa = \varkappa_0 = \dfrac{1-\sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda r}$, we obtain from (6)

$$\operatorname{Re}\left\{1+\frac{zh_\lambda''(z)}{h_\lambda'(z)}\right\} \geqq \frac{2\left(\dfrac{1-\sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda}\right)(1+\lambda r^2-\lambda)-2\lambda r^2}{(1-r^2)(1-\lambda r\varkappa)}.$$

Therefore $h_\lambda(z)$ is convex in $|z| < r_\lambda$ where r_λ is the smallest positive root of the equation

$$\left(1-\sqrt{(1-\lambda)(1+\lambda r^2)}\right)(1+\lambda r^2-\lambda)-\lambda^2 r^2 = 0 \tag{9}$$

that is,

$$\lambda r^4 + r^2(1-\lambda)-(1-\lambda) = 0.$$

This gives

$$r_\lambda = \left(\frac{\sqrt{(1-\lambda)(1+3\lambda)}-(1-\lambda)}{2\lambda}\right)^{1/2} = b \quad \text{(say)}. \tag{10}$$

The transition from formula (10) to formula (8) occurs for that value of λ for which χ_0 given by (7)$=1$. This gives

$$\frac{1-\sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda r} = 1 \tag{11}$$

Solving (11) with the help of (8), that is, taking $r = \dfrac{1}{2\lambda}$, we obtain $\lambda = \dfrac{\sqrt{5}+1}{4}$.

So we have to use (10) for $0 < \lambda \leqq \dfrac{\sqrt{5}+1}{4}$ and (8) for $\dfrac{\sqrt{5}+1}{4} \leqq \lambda < 1$.

We now determine the form of the extremal functions. In case (8) the extremal function is easily seen to be

$$h_\lambda(z) = (1-\lambda)z+\lambda\left(z+\frac{1}{2}z^2\right) = z+\frac{\lambda}{2}z^2-\frac{z-\beta}{1-z\beta} \tag{12}$$

In case (10) we obtain the extremal function thus: We take $\Phi(z) = -\dfrac{z-\beta}{1-z\beta}$ in (5) and

$$\beta = \frac{1-\lambda-\lambda b^2}{(1-\lambda)b}, \tag{13}$$

where b is given by (10). Then $h_\lambda(z)$ is given by

$$h_\lambda'(z) = 1-\frac{\lambda z(z-\beta)}{1-z\beta}. \tag{14}$$

From (14) we obtain

$$1+\frac{zh_\lambda''(z)}{h_\lambda'(z)}=\frac{\beta^2z^2(1-\lambda)+2\beta z(\lambda-1+\lambda z^2)+1-\lambda z^2-2\lambda z^2}{(1-\beta z)(1-\beta z+\lambda\beta z-\lambda z^2)}. \tag{15}$$

The numerator of the right side of (15) vanishes for $z=b$ and β given by (13) so that $h_\lambda(z)$ is not convex in a circle larger than $|z|<b$.

We have yet to prove that $|\beta|\leqq 1$ in order that Φ be bounded in E. That is, we must show that

$$-1\leqq\frac{1-\lambda-\lambda b^2}{(1-\lambda)b}\leqq 1. \tag{16}$$

From (10) we deduce that $\lambda=\dfrac{1-b^2}{1-b^2+b^4}$ so that the right inequality in (16) will hold if

$$\frac{1-b^2}{1-b^2+b^4}\geqq\frac{1-b}{1-b+b^2}$$

that is, if $1+b\geqq b^2$, so the right inequality holds. The left inequality will hold if we show that

$$\frac{1-b^2}{1-b^2+b^4}\leqq\frac{1+b}{1+b+b^2}$$

that is, if

$$b\geqq\frac{\sqrt{5}-1}{2}. \tag{17}$$

Since for $0<\lambda\leqq\dfrac{\sqrt{5}+1}{4}$, $b\geqq\dfrac{\sqrt{5}-1}{2}$, (17) always holds. This completes the proof of the theorem.

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