On the deformed areal spaces

By OM P. SINGH (Agra)

§ 1. Introduction. The deformation theories of metric spaces have been studied by SHOUTEN, KAMPEN, DAVIES, DINES, YANO and others[1]), and of generalized metric spaces were developed by KNEBELMAN [1][2]) in Finsler space and by DAVIES [2] in a Cartan Space, but the works of these two later Geometers mainly concern with the spaces admitting motions. However, in a most general case, a coherent and very lucid study of the deformation of Riemannian space was made by SUGURI [3]. Later on, generalizing the idea, MISRA [4] studied the properties of the deformed Finsler space. In the present paper, we shall discuss the deformation of a general Areal space, inaugurated by A. KAWAGUCHI [5] and studied by himself and many others [6, 7]. Mainly, the problem in hand has been treated in such a manner that our course of investigations would be motivated with a new avenue of approach to develop the basic aspects of the theory by stand point of view in the general interests. On the other hand, the treatment of the problem is based on the idea that our results are the natural consequence of the generalisation of the deformation theory of Riemannian space to the areal space of the general type so as to include the results of the deformations of the areal space of submetric and metric class and particularly those of Riemannian and Finsler one as special cases.

AREAL SPACE AND SOME FUNDAMENTAL NOTATIONS. Let us consider an n-dimensional space, in which the area of a domain on an m-dimensional subspace given by the parameter form[3]):

$$x^i = x^i(u^\alpha); \qquad i=1, 2, \dots, n; \ \alpha=1, 2, \dots, m,$$

is defined by an m-ple integral

$$S = \underbrace{\int \cdots \int}_{m} F(x^i, \partial x^i/\partial u^\alpha)\, du^1\, du^2 \dots du^m$$

extended over a region $\mathscr{R}$ of the subspace, where $F(x, p)$; $p=p^i_\alpha \equiv \dfrac{\partial x^i}{\partial u^\alpha}$ being given by the *a priori* with the measure of its m-dimensional plane element (x, p) which

[1]) J. A. SHOUTEN: Ricci-Calculus (1954), p. 335.

[2]) Numbers in brackets refer to the references at the end of the paper.

[3]) The Latin indices $h, i, j, \dots$ run from 1 to n and the Greek indices $\alpha, \beta, \gamma, \dots$ from 1 to m, throughout this papaer.

is a positive analytical function of x's and $\partial x/\partial u$'s, and satisfies the following conditions:[4])

a) $F(x,p) > 0$ for linearly independent m vectors p^i_α,

b) $F(x, \lambda^\alpha_{\alpha'} p) = \|\lambda^\alpha_{\alpha'}\| F(x,p)$ for $\|\lambda^\alpha_{\alpha'}\| \geqq 0$,

c) $F;^\alpha_i p^i_\beta = \delta^\alpha_\beta F$, where $F;^\alpha_i \equiv \dfrac{\partial F}{\partial p^i_\alpha}$.

Such a space is called an areal space denoted by $A^{(m)}_n$, and $F(x,p)$ is named as the fundamental function of this space.

In such an areal space $A^{(m)}_n$, let $g_{i[m],j[m]}$ be the fundamental metric m-tensor which is expressed as

$$g_{i[m],j[m]} \equiv g_{i_1,i_2\dots i_m,j_1,j_2\dots j_m}.$$

The contravariant components $g^{i[m],j[m]}$ of the metric m-tensor $g_{i[m],j[m]}$ are determined uniquely by the relation

$$g_{i[m],j[m]}g^{k[m],j[m]} = m!\,\delta^{k[m]}_{i[m]}.$$

Putting

$$g_{ij}{}^{hk} \equiv g_{ii_2\dots i_m,jj_2\dots j_m}g^{hi_2\dots i_m,kj_2\dots j_m},$$

if we consider the quantity

$$\Phi_{ij}{}^{hk} \equiv \binom{n-2}{m-1}^{-1}\left\{((m-1)!)^{-2}g_{ij}{}^{hk} - \binom{n-2}{m-2}\delta^h_i\delta^k_j\right\},$$

where $\binom{n-2}{m-1}$ and $\binom{n-2}{m-2}$ represent binomial coefficients, then we can take the quantity $\Lambda_{ij}{}^{hk}$ which is the symmetric part of $\Phi_{ij}{}^{hk}$ and is given by

$$\Lambda_{ij}{}^{hk} \equiv \Phi_{(ij)}{}^{hk} \equiv \binom{n-2}{m-1}^{-1}\left\{((m-1)!)^{-2}g_{(ij)}{}^{hk} - \binom{n-2}{m-2}\delta^h_i\delta^k_j\right\}.$$

Now, let us put $\Lambda_{ij}{}^{\alpha\beta} \equiv \Lambda_{ij}{}^{hk}p^\alpha_h p^\beta_k$, where $p^\alpha_h = F^{-1}.\ \partial F/\partial p^h_\alpha$, and consider the quantity $\Lambda^{ij}{}_{\alpha\beta}$ which is determined uniquely by the relation

$$\Lambda_{ij}{}^{\alpha\beta}\Lambda^{hj}{}_{\gamma\beta} = \delta^h_i\delta^\alpha_\gamma,$$

under the assumption that the mn-rowed determinent $|\Lambda_{ij}{}^{\alpha\beta}|$ constructed of $\Lambda_{ij}{}^{\alpha\beta}$ does not vanish.

The parameter of the line connection coefficient $\Gamma^{*h}_{ij}(x,p)$ in $A^{(m)}_n$ is defined by

$$\Gamma^{*h}_{ij} = \gamma^h_{ij} - \{C^{\times h\gamma}_{i,k}B^k_{\gamma j} + C^{\times h\gamma}_{j,k}B^k_{\gamma i} - C^{\times h\gamma\gamma}_{ij,k}B^k_{\gamma\gamma}\},$$

where

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[4]) E. T. Davies, Areal spaces, *Annali di Mat.* (IV), **55** (1961), 63—76.

$\gamma^h_{ij}(x, p)$ is the Christoffel symbol of the space under consideration and is given by

$$\gamma^h_{ij} = \frac{1}{2m} \Lambda^{*h\gamma}{}_{\alpha\beta} \left\{ \frac{\partial \Lambda^*{}_{rj}{}^{\alpha\beta}}{\partial x^i} + \frac{\partial \Lambda^*{}_{ir}{}^{\alpha\beta}}{\partial x^j} - \frac{\partial \Lambda^*{}_{ij}{}^{\alpha\beta}}{\partial x^r} \right\},$$

$$\Lambda^*{}_{ij}{}^{\alpha\beta} = F^{2/m} \Lambda_{ij}{}^{\alpha\beta}, \quad \Lambda^{*ij}{}_{\alpha\beta} = F^{-2/m} \Lambda^{ij}{}_{\alpha\beta},$$

$$C^{\times}{}^{hr\gamma}_{ij,k} = \frac{1}{2m} \Lambda^{*hr}{}_{\alpha\beta} \frac{\partial \Lambda^*{}_{ij}{}^{\alpha\beta}}{\partial p^k_\gamma}, \quad C^{\times}{}^{i\gamma}_{j,k} = C^{\times}{}^{hi\gamma}_{hj,k} = \frac{1}{2m} \Lambda^{*hi}{}_{\alpha\beta} \frac{\partial \Lambda^*{}_{hj}{}^{\alpha\beta}}{\partial p^k_\gamma},$$

and $B^k_{\gamma j}$ is the base connection coefficient of the space $A^{(m)}_n$. Also, we have the quantity $C^{i}_{j,}{}^{\gamma}_{k}$ in $A^{(m)}_n$ given by

$$C^i_{j,}{}^\gamma_k = C^{\times}{}^{i\gamma}_{j,k} - C^{\times}{}^{i}_{h,}{}^{k}_{\gamma} p^h_\alpha p^\alpha_j.$$

Except this, in what follows, we shall frequently use the similar notations and terminologies as those employed by A. Kawaguchi and his collaboraters without explanations.

§ 2. The deformed areal space. Let us consider an n-dimensional areal space $A^{(m)}_n$ of the most general type, due to A. Kawaguchi and Y. Katsurada [8], which is not necessarily to be of the submetric class, say that is, not always referred to the existence of the matric tensor g_{ij} of covariant order two. The fundamental metric function of this space is $F(x, p)$, $p = p^i_\alpha \equiv \partial x^i / \partial u^\alpha$, which is homogeneous of degree one in every line element contained in the element (x, p) and is not only to be integrable but also of differentiability class at least three with respect to its argument p.

In the space $A^{(m)}_n$, for a contravariant vector $X^i(x, p)$, there exist two types of covariant derivative [10, 11] with respect to x^j and p^j_α respectively

$$X^i{}_{|j} = \partial_j X^i - \partial^\alpha_k X^i B^k_{\alpha j} + \Gamma^{*i}{}_{hj} X^h, \tag{2.1}$$

$$X^i|^\alpha_j = \partial^\alpha_j X^i + C^i_{h,}{}^\alpha_j X^h, \tag{2.2}$$

where

$$B^k_{\alpha j} = \Gamma^{*k}{}_{hj} p^h_\alpha, \quad \partial_j \equiv \frac{\partial}{\partial x^j}, \quad \partial^\alpha_j \equiv \frac{\partial}{\partial p^j_\alpha}.$$

Following Gama [10], these two types of covariant differentiation give rise to the curvature tensors R^i_{hkl}, K^i_{hkl}, $p^i_{hk,}{}^\lambda_r$ and $S^i_{h,}{}^\lambda_r{}^\mu_s$.

Now, we consider an infinitesimal transformation defined by

$$\bar{x}^i = x^i + \xi^i(x)\, d\tau, \tag{2.3}$$

where $\xi^i(x)$ is a contravariant vector field of class C^2, defined over a region $\mathscr{R}$ of $A^{(m)}_n$ depending only on the position x (or to say ξ^i is independent of the directions) and $d\tau$ is an infinitesimal constant.

The Lie-derivatives of a tensor field T^i_j and of the connection coefficient Γ^{*i}_{jk} with respect to the transformation (2. 3) are given by

$$\text{(2. 4)} \qquad \mathscr{L}T^i_j \equiv \frac{\overset{v}{d}T^i_j - \overset{m}{d}T^i_j}{d\tau} = T^i_{j|k}\xi^k + (\partial^\alpha_k T^i_j)\xi^k_{|h} p^h_\alpha - T^k_j \xi^i_{|k} + T^i_k \xi^k_{|j},$$

and

$$\text{(2. 5)} \qquad \mathscr{L}\Gamma^{*i}_{jk} \equiv \frac{\overset{v}{d}\Gamma^{*i}_{jk} - \overset{m}{d}\Gamma^{*i}_{jk}}{d\tau} = \xi^i_{|j|k} + (\partial^\alpha_l \Gamma^{*i}_{jk})\xi^l_{|h} p^h_\alpha + R^i_{jkl}\xi^l,$$

where a small vertical bar denotes the covariant differentiation with respect to connection coefficients Γ^{*i}_{jk}. Thus, if we interpret the deformation of a general geometric object $\Omega(x, p)$ in $A^{(m)}_n$ by the reasoning as given below:

$$\text{(2. 6)} \qquad \bar{\bar{\Omega}} = \Omega + (\overset{v}{d}\Omega - \overset{m}{d}\Omega),$$

where $\overset{v}{d}\Omega$ is the variation of $\Omega(x, b)$ arising from (2. 3) and $\overset{m}{d}\Omega$ is the difference in the displaced quantity $\Omega(\bar{x}, \bar{p})$ of $\Omega(x, p)$ from (x, p) to $(\bar{x}, \bar{p})$ (under the coordinate transformation (2. 3), if regarded) and the quantity $\Omega(x, p)$, then we can have the

Definition 1. The quantity $\bar{\bar{\Omega}}(x, p)$ is the deformed quantity of $\Omega(x, p)$ under the transformation (2. 3) and is given by (2. 6).

Hence, by reason of (2. 6), the deformed tensor field $\bar{\bar{T}}^i_j$ of T^i_j and the deformed connection coefficient $\bar{\bar{\Gamma}}^{*i}_{jk}$ of Γ^{*i}_{jk} may be defined as follows:

$$\text{(2. 7)} \qquad \bar{\bar{T}}^i_j = T^i_j + \mathscr{L}T^i_j\, d\tau,$$

$$\text{(2. 8)} \qquad \bar{\bar{\Gamma}}^{*i}_{jk} = \Gamma^{*i}_{jk} + \mathscr{L}\Gamma^{*i}_{jk}\, d\tau.$$

The transformation (2. 3) carries the point x of a surface $V_m: x^i = x^i(u^\alpha)$ to the neighbouring point $\bar{x}^i$ of a surface $\bar{V}_m: \bar{x}^i = \bar{x}^i(u^\alpha)$, such that u^α being always fixed and $\xi^i(x) = 0$ on the boundary point of both the surfaces V_m and $\bar{V}_m$. If we differentiate (2. 3) with respect to u^α, then

$$\bar{p}^i_\alpha = p^i_\alpha + (\partial_j \xi^i) p^j_\alpha\, d\tau, \quad \bar{p}^i_\alpha \equiv \partial \bar{x}^i / \partial u^\alpha,$$

so we notice immediately that, under the transformation (2. 3), the corresponding variation in the m-ple line element p^i_α may be expressed as

$$\overset{v}{d}p^i_\alpha = \bar{p}^i_\alpha - p^i_\alpha = (\partial_j \xi^i) p^j_\alpha\, d\tau,$$

but this is also the displaced value $\overset{m}{d}p^i_\alpha$ of p^i_α as is quite obvious when we regard (2. 3) as an infinitesimal coordinate transformation. Hence the Lie-derivative $\mathscr{L}p^i_\alpha$ of p^i_α as defined by $\mathscr{L}p^i_\alpha = \dfrac{\overset{v}{d}p^i_\alpha - \overset{m}{d}p^i_\alpha}{d\tau}$ vanishes, i.e.

$$\text{(2. 9)} \qquad \mathscr{L}p^i_\alpha = 0.$$

Thus, preserving the above statement, we can enunciate the

Theorem 1. *The m-ple line element p^i_α does not deform under the transformation* (2. 3).

Next, we see that, under the transformation (2. 3), the fundamental metric function $F(x, p)$ varies as

$$\bar{\bar{F}} = F + (\partial_i F)\xi^i d\tau + (\partial^\alpha_i F)(\partial_j \xi^i) p^j_\alpha d\tau. \tag{2.10}$$

Since the vector ξ^i is only a point function, so the relation (2. 10) can be written as

$$\bar{\bar{F}} = F + [F_{|i}\xi^i + (\partial^\alpha_i F)\xi^i_{|j} p^j_\alpha] d\tau,$$

but $F_{|i} = \partial_i F - (\partial^\alpha_j F) B^j_{\alpha i} = 0$. Therefore

$$\bar{\bar{F}} = F + (\partial^\alpha_i F)\xi^i_{|j} p^j_\alpha d\tau. \tag{2.11}$$

Again, if we put $\mathscr{L}F = (\partial^\alpha_i F)\xi^i_{|j} p^j_\alpha$, the relation (2. 11) is finally rewritten into the form

$$\bar{\bar{F}} = F + \mathscr{L}F d\tau. \tag{2.12}$$

THE DEFORMATION OF $\gamma^h_{ij}(x, p)$. The Lie-derivative of the Christoffel symbols $\gamma^h_{ij}(x, p)$ of the space under consideration with respect to $\Lambda^*{}_{ij}{}^{\alpha\beta}$ can not be determined by making use of the formula (2. 5), because the transformation law of the symbols γ^h_{ij} does not resemble with that of $\Gamma^*{}^h_{ij}$. Therefore, first we write down the transformation law of the symbols γ^h_{ij}:

$$\bar{\gamma}^h_{ij} = \frac{\partial \bar{x}^h}{\partial x^d}\frac{\partial x^b}{\partial \bar{x}^i}\frac{\partial x^c}{\partial \bar{x}^j}\gamma^d_{bc} + \frac{\partial \bar{x}^h}{\partial x^f}\frac{\partial x^b}{\partial \bar{x}^j} C^\times{}^{af\gamma}_{ab,d}\frac{\partial^2 x^d}{\partial \bar{x}^l \partial \bar{x}^i} p^l_\gamma + \frac{\partial \bar{x}^h}{\partial x^f}\frac{\partial x^a}{\partial \bar{x}^i} C^\times{}^{ef\gamma}_{ae,d}\frac{\partial^2 x^d}{\partial \bar{x}^l \partial \bar{x}^j} p^l_\gamma -$$

$$- \frac{\partial \bar{x}^k}{\partial x^e}\frac{\partial \bar{x}^h}{\partial x^f}\frac{\partial x^a}{\partial \bar{x}^i}\frac{\partial x^b}{\partial \bar{x}^j} C^\times{}^{ef\gamma}_{ab,d}\frac{\partial^2 x^d}{\partial \bar{x}^l \partial \bar{x}^k} p^l_\gamma + \frac{\partial \bar{x}^h}{\partial x^d}\frac{\partial^2 x^d}{\partial \bar{x}^i \partial \bar{x}^j},$$

and then proceed as usually in the same manner as to calculate $\mathscr{L}\Gamma^*{}^h_{ij}$. In this way the Lie-derivative of γ^h_{ij} is obtained in the form

$$\begin{aligned} \mathscr{L}\gamma^h_{ij} &= \partial_i\partial_j\xi^h + 2\gamma^h_{k(i}\partial_{j)}\xi^k - \gamma^k_{ij}\partial_k\xi^h + (\partial_k\gamma^h_{ij})\xi^k + \\ &+ \{(\partial^\gamma_k\gamma^h_{ij})\partial_l\xi^k + 2C^\times{}^{a\,h\,\gamma}_{a(j,d}\partial_{i)}\partial_l\xi^d - C^\times{}^{kh\,\gamma}_{ij,d}\partial_l\partial_k\xi^d\} p^l_\gamma. \end{aligned} \tag{2.13}$$

Using the formula (2. 4) and by reason of $\Lambda^*{}_{ij}{}^{\alpha\beta}{}_{|k} = 0$ and $\Lambda^{*ij}{}_{\alpha\beta|k} = 0$, the Lie-derivative of the four-index metric tensors are given by

$$\mathscr{L}\Lambda^*{}_{ij}{}^{\alpha\beta} = (\partial^\alpha_h \Lambda^*{}_{ij}{}^{\alpha\beta})\xi^h_{|l} p^l_\alpha + 2\Lambda^*{}_{l(j}{}^{\alpha\beta}\xi^l_{|i)}, \tag{2.14}$$

$$\mathscr{L}\Lambda^{*ij}{}_{\alpha\beta} = (\partial^\alpha_h \Lambda^{*ij}{}_{\alpha\beta})\xi^h_{|l} p^l_\alpha - 2\Lambda^{*l(i}{}_{\alpha\beta}\xi^{j)}_{|l}, \tag{2.15}$$

where the quantity $\Lambda^{*ij}{}_{\alpha\beta}$ is the contravariant counterpart of $\Lambda^*{}_{ij}{}^{\alpha\beta}$ and is determined uniquely by the relation

$$\Lambda^*{}_{ij}{}^{\alpha\beta}\Lambda^{*hj}{}_{\gamma\beta} = \delta^h_i \delta^\alpha_\gamma.$$

Hence in our present discussion, for the 4-index metric tensors, we have

$$\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta} = \Lambda^{*}{}_{ij}{}^{\alpha\beta} + \mathscr{L}\Lambda^{*}{}_{ij}{}^{\alpha\beta}\, d\tau, \tag{2.16}$$

$$\overline{\overline{\Lambda}}{}^{*ij}{}_{\alpha\beta} = \Lambda^{*ij}{}_{\alpha\beta} + \mathscr{L}\Lambda^{*ij}{}_{\alpha\beta}\, d\tau. \tag{2.17}$$

Now, we define the Christoffel symbols $\overline{\overline{\gamma}}{}^{h}_{ij}(x, p)$ with respect to $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$ as follows:

$$\overline{\overline{\gamma}}{}^{h}_{ij} = \frac{1}{2m}\, \overline{\overline{\Lambda}}{}^{*hr}{}_{\alpha\beta} \left\{ \frac{\partial \overline{\overline{\Lambda}}{}^{*}{}_{rj}{}^{\alpha\beta}}{\partial x^{i}} + \frac{\partial \overline{\overline{\Lambda}}{}^{*}{}_{ir}{}^{\alpha\beta}}{\partial x^{j}} - \frac{\partial \overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}}{\partial x^{r}} \right\}. \tag{2.18}$$

If we introduce the relations (2. 16) and (2. 17) in (2. 18), then on simplification, it follows that

$$\overline{\overline{\gamma}}{}^{h}_{ij} = \gamma^{h}_{ij} + \mathscr{L}\gamma^{h}_{ij}\, d\tau. \tag{2.19}$$

Consequently, we can state the

Theorem 2. *The Christoffel symbols $\overline{\overline{\gamma}}{}^{h}_{ij}$ calculated with respect to $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$ are the deformed counterparts of the symbols γ^{h}_{ij} of the space $A_n^{(m)}$, under the transformation* (2. 3).

Now, analogous to the quantity $B^{h}_{\gamma i} = \Gamma^{*h}_{ij} p^{j}_{\gamma}$, we consider the quantity $\overline{\overline{B}}{}^{h}_{\gamma i}$ by

$$\overline{\overline{B}}{}^{h}_{\gamma i} = \overline{\overline{\Gamma}}{}^{*h}_{ji} p^{j}_{\gamma} \quad (\text{because } \overline{\overline{p}}{}^{j}_{\gamma} = p^{j}_{\gamma}).$$

The Lie-derivatives of the base connection coefficients $B^{h}_{\gamma i}$ are determined with the same proceedure as to those needed in the calculation of $\mathscr{L}\gamma^{h}_{ij}$. In this way, we can obtain

$$\mathscr{L}B^{h}_{\gamma i} = \frac{\overset{v}{d}B^{h}_{\gamma i} - \overset{m}{d}B^{h}_{\gamma i}}{d\tau} = (\partial_i \partial_j \xi^h) p^{j}_{\gamma} + (\partial_j B^{h}_{\gamma i})\xi^{j} + (\partial^{\alpha}_{l} B^{h}_{\gamma i})(\partial_j \xi^{l}) p^{j}_{\alpha} - B^{j}_{\gamma i}\partial_j \xi^h + B^{h}_{\gamma j}\partial_i \xi^{j},$$

which on making the use of covariant derivatives of ξ^h with respect to x^i, may be written as

$$\mathscr{L}B^{h}_{\gamma i} = \xi^{h}_{|j|i} p^{j}_{\gamma} + R^{h}_{\gamma il}\xi^{l} + (\partial^{\alpha}_{l} B^{h}_{\gamma i})\xi^{l}_{|k} p^{k}_{\alpha} - \Gamma^{*h}_{ji}\xi^{j}_{|k} p^{k}_{\gamma},$$

where $R^{h}_{\gamma il} = R^{h}_{jil} p^{j}_{\gamma}$.

Now, following (2. 6), we can write

$$\overline{\overline{B}}{}^{h}_{\gamma i} = B^{h}_{\gamma i} + \mathscr{L}B^{h}_{\gamma i}\, d\tau. \tag{2.20}$$

Thus, in view point of the definition 1, and from (2. 12), (2. 16), (2. 17) and (2. 20), we can have the

Definition 2. The quantities $\overline{\overline{F}}(x, p)$, $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$, $\overline{\overline{\Lambda}}{}^{*ij}{}_{\alpha\beta}$ and $\overline{\overline{B}}{}^{h}_{\gamma i}$ may be defined as the deformed counterparts of the quantities the fundamental metric function $F(x, p)$, the four index metric tensors $\Lambda^{*}{}_{ij}{}^{\alpha\beta}$, $\Lambda^{*ij}{}_{\alpha\beta}$ and the base connection coefficient $B^{h}_{\gamma i}$ of the space $A_n^{(m)}$, under the infinitesimal transformation (2. 3), and will be known as the fundamental metric function, the four index metric tensors and the base connection coefficient of the space $\overline{\overline{A}}{}_n^{(m)}$ respectively.

Let us construct the quantities $\overline{\overline{C}}{}^{\times}{}^{hr\gamma}_{ij,k}$ with respect to $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$ as follows:

$$\overline{\overline{C}}{}^{\times}{}^{hr}_{ij,}{}^{\gamma}_{k} = \frac{1}{2m}\overline{\overline{\Lambda}}{}^{*hr}{}_{\alpha\beta}\frac{\delta\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}}{\partial p^k_\gamma}.$$

Making use of (2. 16), (2. 17) and the result $\mathscr{L}(\partial^\alpha_j X^i)=\partial^\alpha_j(\mathscr{L}X^i)$[5]) in the above relation and on simplifying this neglecting the higher powers of $d\tau$, it can easily be verified that

$$\overline{\overline{C}}{}^{\times}{}^{hr}_{ij,}{}^{\gamma}_{k} = C^{\times}{}^{hr}_{ij,}{}^{\gamma}_{k}+\mathscr{L}C^{\times}{}^{hr}_{ij,}{}^{\gamma}_{k}\,d\tau. \tag{2.21}$$

From this, it is easy to see that

$$\overline{\overline{C}}{}^{\times}{}^{i\alpha}_{j,k} = C^{\times}{}^{i\alpha}_{j,k}+\mathscr{L}C^{\times}{}^{i\alpha}_{j,k}\,d\tau, \tag{2.22}$$

where

$$\overline{\overline{C}}{}^{\times}{}^{i}_{j,}{}^{\alpha}_{k} = \overline{\overline{C}}{}^{\times}{}^{ih}_{jh,}{}^{\alpha}_{k} = \frac{1}{2m}\overline{\overline{\Lambda}}{}^{*ih}{}_{\gamma\delta}\frac{\partial\overline{\overline{\Lambda}}{}^{*}{}_{jh}{}^{\gamma\delta}}{\partial p^k_\alpha}.$$

Thus, obviously the quantities $\overline{\overline{C}}{}^{\times}{}^{hr}_{ij,}{}^{\gamma}_{k}$ and $\overline{\overline{C}}{}^{\times}{}^{i}_{j,}{}^{\alpha}_{k}$ defined with respect to $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$ are the deformed quantities of $C^{\times}{}^{hr}_{ij,}{}^{\gamma}_{k}$ and $C^{\times}{}^{i}_{j,}{}^{\alpha}_{k}$ respectively.

Finally, let us define the connection coefficients $\overline{\overline{\Gamma}}{}^{*h}_{ij}$ with respect to $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$ as follows:

$$\overline{\overline{\Gamma}}{}^{*h}_{ij} = \overline{\overline{\gamma}}{}^{h}_{ij}-(\overline{\overline{C}}{}^{\times}{}^{kh}_{kj,}{}^{\gamma}_{r}\overline{\overline{B}}{}^{r}_{\gamma i}+\overline{\overline{C}}{}^{\times}{}^{kh}_{ki,}{}^{\gamma}_{r}\overline{\overline{B}}{}^{r}_{\gamma j}-\overline{\overline{C}}{}^{\times}{}^{kh}_{ij,}{}^{\gamma}_{r}\overline{\overline{B}}{}^{r}_{\gamma k}). \tag{2.23}$$

Substitution of (2. 19), (2. 20) and (2. 21) in (2. 23) yields the result

$$\overline{\overline{\Gamma}}{}^{*h}_{ij} = \Gamma^{*h}_{ij}+\mathscr{L}\Gamma^{*h}_{ij}\,d\tau. \tag{2.24}$$

Hence we have the

Theorem 3. *The connection coefficients $\overline{\overline{\Gamma}}{}^{*h}_{ij}$ with respect to the four index metric tensor $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$ are nothing but only the deformed counterparts of the connection coefficients Γ^{*h}_{ij} with respect to $\Lambda^{*}{}_{ij}{}^{\alpha\beta}$ of the space $A^{(m)}_n$, under the transformation* (2. 3).

Of course from the above discussion, by cause of no doubt, we can give the following pertinent conclusion:

Definition 3. The areal space $\overline{\overline{A}}{}^{(m)}_n$ with the entities the fundamental metric function $\overline{\overline{F}}(x, p)$, the 4-index metric tensors $\overline{\overline{\Lambda}}{}^{*}{}_{ij}{}^{\alpha\beta}$, $\overline{\overline{\Lambda}}{}^{*ij}{}_{\alpha\beta}$, the parameters of the line metric connection coefficients $\overline{\overline{\Gamma}}{}^{*h}_{ij}$ and the base connection coefficients $\overline{\overline{B}}{}^{h}_{\gamma i}$ is called the deformed space of $A^{(m)}_n$ with respect to the infinitesimal transformation (2. 3).

Let us take the contravariant components of the m-vector p defined by

$$p^{i[m]} = m!\,p^{i_1}_{[1}p^{i_2}_{2}\cdots p^{i_m}_{m]},$$

[5]) T. Igarashi [13], p. 206.

so that $l^{i[m]}$ are the contravariant components of the unit m-vector which are given by

$$l^{i[m]} = F^{-1} p^{i[m]}.$$

Applying the formula (2. 4) for the Lie-derivative of the unit m-vector $l^{i[m]}$ along p^i_α, we can have

$$\mathscr{L} l^{i[m]} = -\frac{1}{F} l^{i[m]} (\partial^\alpha_j F) \xi^j_{|h} p^h_\alpha + \frac{1}{F} (\partial^\alpha_j p^{i[m]}) \xi^j_{|h} p^h_\alpha - l^{j[m]} \xi^i_{|j}.$$

Hence, under the transformation (2. 3), the unit m-vector $l^{i[m]}$ deforms into

$$(2.25) \quad \overline{\overline{l}}^{i[m]} = l^{i[m]} + \left[-\frac{1}{F} l^{i[m]} (\partial^\alpha_j F) \xi^j_{|h} p^h_\alpha + \frac{1}{F} (\partial^\alpha_j p^{i[m]}) \xi^j_{|h} p^h_\alpha - l^{j[m]} \xi^i_{|j} \right] d\tau.$$

Multiplying (2. 25) with (2. 12) and on simplifying by neglecting the higher powers of $d\tau$, we get at once

$$(2.26) \qquad \overline{\overline{F}}\, \overline{\overline{l}}^{i[m]} = p^{i[m]} + \mathscr{L} p^{i[m]} d\tau,$$

where we have put $\mathscr{L} p^{i[m]} = (\partial^\alpha_j p^{i[m]}) \xi^j_{|h} p^h_\alpha - p^{j[m]} \xi^i_{|j}$, so that if we notice that the m-vector $\overline{\overline{p}}^{i[m]}$ is the deformed m-vector of $p^{i[m]}$ and is defined by

$$\overline{\overline{p}}^{i[m]} = p^{i[m]} + \mathscr{L} p^{i[m]} d\tau,$$

then, the relation (2. 26) may now be put into the form

$$\overline{\overline{l}}^{i[m]} = \overline{\overline{p}}^{i[m]} / \overline{\overline{F}}.$$

Hence, we have the

Theorem 4. *The deformed m-vector* $\overline{\overline{l}}^{i[m]}$ *of the unit m-vector* $l^{i[m]}$ *along* p^i_α *in* $A^{(m)}_n$, *under the transformation* (2. 3), *is also the unit m-vector along the same direction in* $\overline{\overline{A}}^{(m)}_n$.

Now, we shall examine the deformation of the quantity p^α_i in $A^{(m)}_n$ under the transformation (2. 3). For the Lie-derivative of p^α_i, we approach with the fundamental procedure and after some simple calculation, we get our required result:

$$\mathscr{L} p^\alpha_i = (\partial_j p^\alpha_i) \xi^j + (\partial^\beta_j p^\alpha_i) \partial_k \xi^j p^k_\beta + p^\alpha_j (\partial_i \xi^j).$$

Hence, on writing the quantity $\overline{\overline{p}}^\alpha_i$ by the relation

$$(2.27) \qquad \overline{\overline{p}}^\alpha_i = p^\alpha_i + \mathscr{L} p^\alpha_i d\tau,$$

we can give the

Definition 6. The quantity $\overline{\overline{p}}^\alpha_i \overset{\text{def}}{=\!=} \dfrac{\partial \log \overline{\overline{F}}}{\partial p^i_\alpha}$ in $\overline{\overline{A}}^{(m)}_n$ is nothing but the deformed counterpart of the quantity p^α_i, under the transformation (2. 3).

At last to cover this section, we consider the connection coefficients $\overline{\overline{C}}^{i}_{j,k}{}^{\gamma}$ in the space $\overline{\overline{A}}^{(m)}_n$ in analogy with the connection coefficients $C^{i\gamma}_{j,k}$ of $A^{(m)}_n$, which are defined by

$$(2.28) \qquad \overline{\overline{C}}^{i}_{j,k}{}^{\gamma} = \overline{\overline{C}}{}^{\times i}_{h,k}{}^{\gamma} \overline{\overline{q}}^h_j,$$

where

$$\overline{\overline{q}}{}^h_j = \overline{\overline{\delta}}{}^h_j - \overline{\overline{p}}{}^h_\alpha \overline{\overline{p}}{}^\alpha_j.$$

If we substitute (2. 22) and (2. 27) in (2. 28) and simplify the result, then, in consequence of the theorem 1, it gives us

$$\overline{\overline{C}}{}^i_{j,}{}^\gamma_k = C^i_{j,}{}^\gamma_k + \mathscr{L} C^i_{j,}{}^\gamma_k \, d\tau, \tag{2.29}$$

showing that $\overline{\overline{C}}{}^i_{j,}{}^k_\gamma$ is also the deformed quantity of $C^i_{j,}{}^\gamma_k$, which will serve a very purposeful meaning in the discussion of the theory of covariant differentiation in $\overline{\overline{A}}{}^{(m)}_n$, dealt in the next section.

§ 3. The theory of Covariant derivation and the curvature tensors in the deformed space $\overline{\overline{A}}{}^{(m)}_n$.

In the present section, for the sake of brevity we shall consider an areal space $A^{(m)}_n$ of the submetric class and its corresponding deformed space $\overline{\overline{A}}{}^{(m)}_n$. Following Gama [10], let us introduce the process of covariant differentiation in the deformed space $\overline{\overline{A}}{}^{(m)}_n$. We define the covariant derivatives of a mixed tensor $\overline{\overline{T}}{}^i_j$ for the connection parameters $\overline{\overline{\Gamma}}{}^{*i}_{jk}$ and $\overline{\overline{C}}{}^i_{j,}{}^\gamma_k$ in the deformed space $\overline{\overline{A}}{}^{(m)}_n$ by

$$\overline{\overline{T}}{}^i_{j\|k} = \partial_k \overline{\overline{T}}{}^i_j - (\partial^\alpha_h \overline{\overline{T}}{}^i_j)\overline{\overline{B}}{}^h_{\alpha k} + \overline{\overline{\Gamma}}{}^{*i}_{hk}\overline{\overline{T}}{}^h_j - \overline{\overline{\Gamma}}{}^{*h}_{jk}\overline{\overline{T}}{}^i_h, \tag{3.1}$$

$$\overline{\overline{T}}{}^i_j\|^\alpha_k = \partial^\alpha_k \overline{\overline{T}}{}^i_j + \overline{\overline{C}}{}^i_{h,}{}^\alpha_k \overline{\overline{T}}{}^h_j - \overline{\overline{C}}{}^h_{j,}{}^\alpha_k \overline{\overline{T}}{}^i_h, \tag{3.2}$$

respectively.

Since it is well known that an areal space of the submetric class is a special case of an areal space of general type and consequently, the theory of an areal space of submetric class is easily derivable from the general theory of areal space in particular, so there is no harm in making the use of the results of preceding section.

We substitute for $\overline{\overline{T}}{}^i_j$, $\overline{\overline{B}}{}^h_{\alpha k}$ and $\overline{\overline{\Gamma}}{}^{*i}_{hk}$ from (2. 4), (2. 20) and (2. 24) respectively in (3. 1) and carry out the operations indicated therein. Thus, we obtain

$$\begin{aligned}\overline{\overline{T}}{}^i_{j\|k} = T^i_{j|k} + [\{T^i_{j|h|k} + T^l_i R^l_{jhk} - T^l_j R^i_{lhk} + (\partial^\alpha_l T^i_j) R^l_{mhk} p^m_\alpha\}\xi^h + \\ + T^i_{j|h}\xi^h_{|k} + T^i_{h|k}\xi^h_{|j} - T^h_{j|k}\xi^i_{|h} + (\partial^\alpha_h T^i_{j|k})\xi^h_{|l} p^l_\alpha] d\tau.\end{aligned} \tag{3.3}$$

Employing the expression for the Lie-derivative of the tensor $T^i_{j|k}$ in (3. 3), we get

$$\overline{\overline{T}}{}^i_{j\|k} = T^i_{j|k} + \mathscr{L} T^i_{j|k} \, d\tau, \tag{3.4}$$

where we have also used the commutation formula

$$T^i_{j|h|k} - T^i_{j|k|h} = T^l_j R^i_{lhk} - T^i_l R^l_{jhk} - (\partial^\alpha_l T^i_j) R^l_{mhk} p^m_\alpha.$$

Next, substituting the expressions for $\overline{\overline{T}}{}^i_j$ and $\overline{\overline{C}}{}^i_{h,}{}^\alpha_k$ in (3. 2) and on simplifying the concerned operations, we get

$$\overline{\overline{T}}{}^i_j\|^\alpha_k = T^i_\alpha|^\alpha_k + \mathscr{L} T^i_j|^\alpha_k \, d\tau. \tag{3.5}$$

Hence we have the

Theorem 5. *The tensors* $\bar{\bar{T}}{}^i_{j\|k}(x, p)$ *and* $\bar{\bar{T}}{}^i_j\|^\alpha_k(x, p)$ *are the deformed counterparts of the covariant derivatives* $T^i_{j|k}(x, p)$ *and* $T^i_j|^\alpha_k(x, p)$ *of the tensor* $T^i_j(x, p)$ *respectively, under the transformation* (2. 3).

Further, we note that the processes of covariant differentiation in the deformed space $\bar{\bar{A}}{}_n^{(m)}$ as introduced above will give rise to the following curvature tensors:

$$\bar{\bar{R}}{}^h_{ijk} = \partial_k \bar{\bar{\Gamma}}{}^{*h}_{ij} - \partial_j \bar{\bar{\Gamma}}{}^{*h}_{ik} + \partial^\alpha_l \bar{\bar{\Gamma}}{}^{*h}_{ik} \bar{\bar{\Gamma}}{}^{*l}_{\alpha j} - \partial^\alpha_l \bar{\bar{\Gamma}}{}^{*h}_{ij} \bar{\bar{\Gamma}}{}^{*l}_{\alpha k} + \bar{\bar{\Gamma}}{}^{*h}_{lk} \bar{\bar{\Gamma}}{}^{*l}_{ij} - \bar{\bar{\Gamma}}{}^{*h}_{lj} \bar{\bar{\Gamma}}{}^{*l}_{ik}, \tag{3.6}$$

$$\bar{\bar{K}}{}^h_{ijk} = \bar{\bar{R}}{}^h_{ijk} + \bar{\bar{C}}{}^{h\ \alpha}_{i,l} \bar{\bar{R}}{}^l_{mjk} \bar{\bar{p}}{}^m_\alpha, \tag{3.7}$$

$$\bar{\bar{P}}{}^{h\alpha}_{ij,l} = \partial^\alpha_l \bar{\bar{\Gamma}}{}^{*h}_{ij} - \bar{\bar{C}}{}^{h\ \alpha}_{i,l\|j} + \bar{\bar{C}}{}^{h\ r}_{i,m} \bar{\bar{p}}{}^r_\lambda \partial^\alpha_l \bar{\bar{\Gamma}}{}^{*m}_{rj}, \tag{3.8}$$

$$\bar{\bar{S}}{}^{h\ \alpha\ \lambda}_{i,l,\gamma} = \delta^\lambda_\gamma \bar{\bar{C}}{}^{h\ \alpha}_{i,l} - \bar{\bar{C}}{}^{h\ \alpha}_{m,l} \bar{\bar{C}}{}^{m\lambda}_{i,\gamma} - \partial^\alpha_l \bar{\bar{C}}{}^{h\ \lambda}_{i,\gamma} + \bar{\bar{C}}{}^{h\ \lambda}_{m,\gamma} \bar{\bar{C}}{}^{m\ \alpha}_{i,\ l}. \tag{3.9}$$

Substituting for $\bar{\bar{\Gamma}}{}^{*h}_{ij}$ from (2. 24) in (3. 6) and on solving out the relation, we obtain

$$\bar{\bar{R}}{}^h_{ijk} = R^h_{ijk} + \mathscr{L} R^h_{ijk}\, d\tau, \tag{3.10}$$

where we have used the definition (4. 5) of Gama [10, III].

Also, substituting the appropriate expressions for the quantities $\bar{\bar{\Gamma}}{}^{*h}_{ij}$, $\bar{\bar{C}}{}^{h\ \alpha}_{i,l}$ and $\bar{\bar{R}}{}^h_{ijk}$ inserted on the right hand sides of (3. 7), (3. 8) and (3. 9) from (2. 24), (2. 29), (3. 10) respectively, and carrying out the calculations for the operations indicated therein with the use of relation (3. 4) and the definitions (4. 2), (4. 3) and (4. 4) of Gama [10, III], we see that the identities (3. 7), (3. 8) and (3. 9) reduce to

$$\bar{\bar{K}}{}^h_{ijk} = K^h_{ijk} + \mathscr{L} K^h_{ijk}\, d\tau,$$

$$\bar{\bar{P}}{}^{h\ \alpha}_{ij,l} = P^{h\ \alpha}_{ij,l} + \mathscr{L} P^{h\ \alpha}_{ij,l}\, d\tau,$$

and

$$\bar{\bar{S}}{}^{h\alpha\ \lambda}_{i,l,\gamma} = S^{h\alpha\ \lambda}_{i,l,\gamma} + \mathscr{L} S^{h\alpha\ \lambda}_{i,l,\gamma}\, d\tau$$

respectively.

In this manner, we have the

Theorem 6. *The curvature tensors of the deformed space* $\bar{\bar{A}}{}_n^{(m)}$ *are respectively the deformed tensors of those* **of** $A_n^{(m)}$.

Remarks. It is very interesting to note that an areal space of the submetric class is a special case of the areal space of general type, and the areal space of the metric class is a subclass of the submetric class. Therefore, in the case when an areal space $A_n^{(m)}$ is of the submetric class and the following relations ([8], p. 152, theorem 3. 4)

$$\Lambda_{ij}{}^{hk} = g_{ij} g^{hk}, \quad \Lambda_{ij}{}^{\alpha\beta} = g_{ij} g^{\alpha\beta}, \quad \Lambda^{ij}{}_{\alpha\beta} = g^{ij} g_{\alpha\beta},$$

and

$$F^2 = \det |g_{\alpha\beta}|,$$

where g_{ij} and g^{ij} are the fundamental metric tensors of order two and $g_{\alpha\beta}=g_{ij}p^i_\alpha p^j_\beta$, hold good, then we have

$$\Lambda^{*hj}{}_{\alpha\beta}\frac{\partial\Lambda^{*}{}_{ij}{}^{\alpha\beta}}{\partial x^k}=g^{hj}\frac{\partial g_{ij}}{\partial x^k},\quad C^{\times}{}^{hr\gamma}_{ij,k}=g^{hr}\frac{\partial g_{ij}}{\partial p^k_\gamma},$$

and γ^h_{ij} is the Christoffel symbol in the usual sense.

Next, we remember that 'in virtue of the theorem of Iwamoto[6]), an areal space of the metric class belongs to the submetric class, and for a space of metric class, the above relations also hold good always ([8], p. 153).

Consequently, we deduce that the theory of an areal space of the submetric class and of metric class can directly be derived from that of the theory of an areal space of general type successively. From this we may have the

Conclusion 1. All the results of the theory of deformed areal space of the submetric and of metric class exist in the theory of deformed areal space of general type and thereby can be deduced without any loss of generality.

As well known, an areal space of the metric class is always one of the Riemannian, Finsler or cartan spaces, 'the theorem is due to Tandai'.[7]) By virtue of this theorem for the cases $m=1$ and $m=n-1$, our space $A_n^{(m)}$ becomes $A_n^{(1)}$ and $A_n^{(n-1)}$ as special cases, which are Finsler and Cartan spaces respectively. In these cases all the results of $A_n^{(m)}$ of metric class hold good almost coinciding with the results obtained by many researchers as the most natural approach to the problems in Finsler and Cartan spaces. Thus, in view point of this property of $A_n^{(m)}$ for the above particular cases, of course, we can argue that our results of the theory of deformed areal space also hold good in Finsler and Cartan spaces. Hence, we have the

Conclusion 2. All the results of the theory of deformed areal space consist of the results of a deformed Finsler space (the case $m=1$) as well as of a deformed Cartan space (the case $m=n-1$).

Finally, we notice that when an areal space $A_n^{(m)}$ is a Riemannian space in particular, $C^{\times}{}^{hr\gamma}_{ij,k}=0$, (A Kawaguchi [8], p. 143), then, no doubt, we can draw the following:

Conclusion 3. For the special case $C^{\times}{}^{hr\gamma}_{ij,k}=0$, all the results of the theory of deformed areal space $A_n^{(m)}$ hold good for the deformed Riemannian space.

Conclusively, if we make our results of the deformed areal space as specialized one for the cases $C^{\times}{}^{hr\gamma}_{ij,k}=0$ and for $m=1$, they almost coincide with those of Suguri [3] and Misra [4] for Riemannian and Finsler spaces respectively. From this discussion, it is worthwhile to state that our theory of deformed areal space stands as a most general one in itself consisting of the theory of deformed areal space of the submetric and metric class and of the deformed Riemannian, Finsler and Cartan spaces, and many results of theory of these deformed spaces can easily be obtained by making them specialized one rather than to achieve them by way of direct approach as a natural generalization.

[6]) A. Kawaguchi, ..., [6], p. 15; H. Iwamoto, On geometries associated with multiple integrals, Mathematica Japonicae, 1 (1948), 74—91.

[7]) K. Tandai [9], p. 44, Theorem 3.2.

§ 4. After having discussed the theory of covariant differentiation in the deformed space $\overline{\overline{A}}_n^{(m)}$ of submetric class, in this section we consider a tensor T^i_{jkh} defined by

$$T^i_{jkh}(x, p) = R^i_{jkh}(x, p) - R(\delta^i_h g_{jk} - \delta^i_k g_{jh}), \qquad (R \neq 0), \tag{4.1}$$

where $\overline{\overline{R}}^i_{jkh}(x, p)$ is the curvature tensor of $A_n^{(m)}$ and R is a scalar called Riemannian curvature in $A_n^{(m)}$. Then, under the transformation (2. 3) the deformed tensor $\overline{\overline{T}}^i_{jkh}(x, p)$ of $T^i_{jkh}(x, p)$ is expressed by the relation

$$\overline{\overline{T}}^i_{jkh}(x, p) = \overline{\overline{R}}^i_{jkh}(x, p) - \overline{\overline{R}}(\delta^i_h \overline{\overline{g}}_{jk} - \delta^i_k \overline{\overline{g}}_{jh}), \tag{4.2}$$

where $\overline{\overline{R}}^i_{jkh}(x, p)$ and $\overline{\overline{g}}_{jk}$ are the deformed tensors of R^i_{jkh} and g_{jk}, under the transformation (2. 3) and are said to be the curvature tensor and fundamental metric tensor of order two of the deformed space $\overline{\overline{A}}_n^{(m)}$ respectively, and $\overline{\overline{R}}$ is the deformed scalar of R under the transformation (2. 3).

For the purpose of our present discussion, we see that, under the transformation (2. 3), the fundamental metric tensor $g_{ij}(x, p)$ deforms as

$$\overline{\overline{g}}_{ij} = g_{ij} + \mathscr{L} g_{ij}\, d\tau, \tag{4.3}$$

where

$$\mathscr{L} g_{ij} = \xi_{i|j} + \xi_{j|i} + (\partial^\alpha_k g_{ij})\, \xi^k_{|h} p^h_\alpha.$$

Now substituting for $\overline{\overline{R}}^i_{jkh}$ and $\overline{\overline{g}}_{ij}$ from (3. 10) and (4. 3) respectively in (4. 2), we get

$$\overline{\overline{T}}^i_{jkh} = T^i_{jkh} + \mathscr{L} T^i_{jkh}\, d\tau, \tag{4.4}$$

where we have assumed R as a constant.

If the areal space $A_n^{(m)}$ is of constant Riemannian curvature [12], R is a constant (as assumed before) and the tensor $T^i_{jkh}(x, p)$ vanishes identically. Therefore, we see that for $\overline{\overline{R}}(x, p) = R(x, p)$ and from (4. 4), $\overline{\overline{T}}^i_{jkh}(x, p) = 0$. Consequently, from (4. 2), we obtain

$$\overline{\overline{R}}^i_{jkh}(x, p) = \overline{\overline{R}}(\delta^i_h \overline{\overline{g}}_{jk} - \delta^i_k \overline{\overline{g}}_{jh}). \tag{4.5}$$

Thus, we conclude that $\overline{\overline{R}} = R$ will be a constant Riemannian curvature of the deformed areal space $\overline{\overline{A}}_n^{(m)}$, because the relation (4. 5) is identically satisfied. Hence, we have the

Theorem 7. *If $A_n^{(m)}$ is an areal space of the submetric class of constant Riemannian curvature, the deformed space $\overline{\overline{A}}_n^{(m)}$ is also a one of the same constant Riemannian Curvature.*

On the other hand, KIKUCHI [12] has shown that an areal space $A_n^{(m)}$ of the submetric class is isotropic when and only when the areal space $A_n^{(m)}$ of the submetric class has a constant Riemannian curvature. Therefore, from the theorem 7, we establish the following:

Corollary: If an areal space $A_n^{(m)}$ of the submetric class is isotropic, the deformed space $\overline{\overline{A}}_n^{(m)}$ of the submetric class is also isotropic.

Of course, without any difficulty, we can also prove the following theorems:

Theorem 8. *If an areal space $A_n^{(m)}$ of the submetric class is of affinely connected, the deformed space $\overline{\overline{A}}_n^{(m)}$ is also of affinely connected.*

Theorem 9. *If an areal space $A_n^{(m)}$ of the submetric class is Minkowskian, the deformed space $\overline{\overline{A}}_n^{(m)}$ is also Minkowskian.*

Theorem 10. *If an areal space $A_n^{(m)}$ of the submetric class is a conformally flat space, the deformed space $\overline{\overline{A}}_n^{(m)}$ is also a conformally flat space having the same property as that of original one.*

§ 5. In this last section, we shall generalize some of the theorems of Suguri [3]. Let us consider $U^{\cdots}_{\cdots}(x, p)$ and $V^{\cdots}_{\cdots}(x, p)$ two tensor fields in an areal space $A_n^{(m)}$. Then, by virtue of (2. 4), their deformed tensor fields $\overline{\overline{U}}{}^{\cdots}_{\cdots}(x, p)$ and $\overline{\overline{V}}{}^{\cdots}_{\cdots}(x, p)$ respectively are given by

$$\overline{\overline{U}}{}^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} = U^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} + \mathscr{L} U^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\, d\tau,$$

$$\overline{\overline{V}}{}^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s} = V^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s} + \mathscr{L} V^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s}\, d\tau.$$

Multiplying these two relations side by side and on simplifying by neglecting the higher powers of $d\tau$, we get

$$(5.1) \qquad \overline{\overline{U}}{}^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \overline{\overline{V}}{}^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s} = U^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} V^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s} + \mathscr{L} U^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} V^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s}\, d\tau,$$

from which, it is quite obvious that the product $\overline{\overline{U}}{}^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \overline{\overline{V}}{}^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s}$ of two tensors is nothing but the deformed tensor of the product $U^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} V^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s}$ of two tensors.

Now we shall define a harmonic vector in an areal space. Let $\Omega_i(x, p)$ be a vector field well defined in a region $\mathscr{R}$ of an areal space $A_n^{(m)}$ of the submetric class. If the vector $\Omega_i(x, p)$ satisfies the following relations:

$$\Omega_{i|j} - \Omega_{j|i} = 0 \quad \text{and} \quad g^{ij}\Omega_{i|j} = 0,$$

then we will call this vector a harmonic vector in the space $A_n^{(m)}$ of the submetric class.

Let us consider a tensor field $T_{ij}(x, p)$ and a scalar field $\mathscr{G}(x, p)$ for any vector field $\Omega_i(x, p)$ in the space $A_n^{(m)}$ of the submetric class which are defined by

$$(5.2) \qquad \text{a)} \quad T_{ij}(x, p) = \Omega_{i|j} - \Omega_{j|i}; \qquad \text{b)} \quad \mathscr{G}(x, p) = g^{ij}\Omega_{i|j},$$

such that, if these are vanished identically, they leave $\Omega_i(x, p)$ as a harmonic vector field in the space under consideration.

Correspondingly, we are now in a position to construct the tensor field $\overline{\overline{T}}_{ij}(x, p)$ and a scalar field $\overline{\overline{\mathscr{G}}}(x, p)$ with regard to the deformed vector $\overline{\overline{\Omega}}_i(x, p)$, such that they are given by

$$(5.3) \qquad \text{a)} \quad \overline{\overline{T}}_{ij}(x, p) = \overline{\overline{\Omega}}_{i\|j} - \overline{\overline{\Omega}}_{j\|i}, \qquad \text{b)} \quad \overline{\overline{\mathscr{G}}}(x, p) = \overline{\overline{g}}{}^{ij}\overline{\overline{\Omega}}_{i\|j}.$$

Making use of (3. 4) and (4. 3) in the above relations, we get

$$(5.4) \quad \text{a)} \quad \overline{\overline{T}}_{ij}(x, p) = T_{ij} + \mathscr{L}T_{ij}\, d\tau, \qquad \text{b)} \quad \overline{\overline{\mathscr{G}}}(x, p) = \mathscr{G}(x, p) + \mathscr{L}\mathscr{G}(x, p)\, d\tau.$$

Thus, by cause of no doubt, we can state that the tensor $\overline{\overline{T}}_{ij}(x, p)$ and the scalar $\overline{\overline{\mathscr{G}}}(x, p)$ are nothing but the deformed counterparts of the tensor $T_{ij}(x, p)$ and the scalar $\mathscr{G}(x, p)$ respectively.

On account of the fact that, if $\Omega_i(x, p)$ is a harmonic vector, the tensor $T_{ij}(x, p)$ and the scalar $\mathscr{G}(x, p)$ vanish identically, we have

$$\overline{\overline{T}}_{ij}(x, p) = 0, \quad \text{and} \quad \overline{\overline{\mathscr{G}}}(x, p) = 0.$$

By this reason, from (5. 4), we get at once

$$\overline{\overline{\Omega}}_{i\|j} - \overline{\overline{\Omega}}_{j\|i} = 0 \quad \text{and} \quad \overline{\overline{g}}^{ij}\,\overline{\overline{\Omega}}_{i\|j} = 0. \tag{5. 5}$$

Consequently, we can state the

Theorem 11. *If $\Omega i(x, p)$ is any harmonic vector field in an areal space $A_n^{(m)}$ of the submetric class, then the deformed vector field $\overline{\overline{\Omega}}i(x, p)$ of the vector field $\Omega i(x, p)$, under the transformation* (2. 3), *is also harmonic in the deformed space $\overline{\overline{A}}_n^{(m)}$ of the submetric class.*

Furthermore, we shall endeavour ourselves to generalize the preceeding theorem for a harmonic tensor field in the space $A_n^{(m)}$ of the submetric class. For this specific purpose in hand, we take any tensor field $\Omega i_1 \dots i_p(x, p)$ in the considered space. Then, with respect to this field, we can consider two tensor field $Ti_1 \dots i_p j(x, p)$ and $\mathscr{G}i_1 \dots i_{p-1}(x, p)$ defined by

$$T_{i_1 \dots i_p j}(x, p) = \Omega_{i_1 \dots i_p | j} - \sum_{r=1}^{p} \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p | i_r}, \tag{5. 6}$$

$$\mathscr{G}_{i_1 \dots i_{p-1}}(x, p) = g^{i_p j}\,\Omega_{i_1 \dots i_{p-1} i_p | j} \tag{5. 7}$$

respectively.

Now, if we suppose that $\overline{\overline{\Omega}}i_1 \dots i_p(x, p)$ is the deformed tensor field of the considered tensor field $\Omega i_1 \dots i_p(x, p)$, then, analogous to the above two tensor fields we can construct two tensor fields $\overline{\overline{T}}i_1 \dots i_p j(x, p)$ and $\overline{\overline{\mathscr{G}}}i_1 \dots i_{p-1}(x, p)$ successively with regard to the deformed tensor field $\overline{\overline{\Omega}}i_1 \dots i_p(x, p)$, which will be defined by

$$\overline{\overline{T}}_{i_1 \dots i_p j}(x, p) = \overline{\overline{\Omega}}_{i_1 \dots i_p \| j} - \sum_{\gamma=1}^{p} \overline{\overline{\Omega}}_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p \| i_r}, \tag{5. 8}$$

$$\overline{\overline{\mathscr{G}}}_{i_1 \dots i_{p-1}}(x, p) = \overline{\overline{g}}^{i_p j}\,\overline{\overline{\mathscr{G}}}_{i_1 \dots i_{p-1} i_p \| j}, \tag{5. 9}$$

respectively.

On account of the relations (3. 4) and (4. 3), we approach in the same manner as before and we get finally

$$\overline{\overline{T}}_{i_1 \dots i_p j}(x, p) = T_{i_1 \dots i_p j}(x, p) + \mathscr{L} T_{i_1 \dots i_p j}(x, p)\, d\tau, \tag{5. 10}$$

$$\overline{\overline{\mathscr{G}}}_{i_1 \dots i_{p-1}}(x, p) = \mathscr{G}_{i_1 \dots i_{p-1}}(x, p) + \mathscr{L} \mathscr{G}_{i_1 \dots i_{p-1}}(x, p)\, d\tau. \tag{5. 11}$$

Evidently, the tensors $\overline{\overline{T}}_{i_1 \dots i_p j}(x, p)$ and $\overline{\overline{\mathscr{G}}}_{i_1 \dots i_{p-1}}(x, p)$ are the deformed tensors of $T_{i_1 \dots i_p j}(x,p)$ and $\mathscr{G}_{i_1 \dots i_{p-1}}(x,p)$ of the space $A_n^{(m)}$ respectively, under the transformation (2. 3).

Now, we assume that the tensor field $\Omega_{i_1 \dots i_p}(x,p)$ is harmonic in $A_n^{(m)}$, i.e.

$$(5.12) \qquad \begin{cases} \text{a) } \Omega_{i_1 \dots i_p}(x,p)\text{: is skew symmetric in all its indices,} \\ \text{b) } T_{i_1 \dots i_p j}(x,p) = 0, \\ \text{c) } \mathscr{G}_{i_1 \dots i_{p-1}}(x,p) = 0. \end{cases}$$

Employing these relations in (5. 10) and (5. 11), we see that the following relations hold good:

$$(5.13) \qquad \begin{cases} \text{a) } \bar{\bar{\Omega}}_{i_1 \dots i_p}(x,p)\text{: is Skew symmetric in all its indices,} \\ \text{b) } \bar{\bar{T}}_{i_1 \dots i_p j}(x,p) = 0, \\ \text{c) } \bar{\bar{\mathscr{G}}}_{i_1 \dots i_{p-1}}(x,p) = 0, \end{cases}$$

which shows that the deformed tensor field $\bar{\bar{\Omega}}_{i_1 \dots i_p}(x,p)$ is also harmonic in $\bar{\bar{A}}_n^{(m)}$. Hence we have the

Theorem 12. *If $\Omega_{i_1 \dots i_p}(x,p)$ is any harmonic tensor field in the space $A_n^{(m)}$ of the submetric class, the deformed tensor field $\bar{\bar{\Omega}}_{i_1 \cdots i_p}(x,p)$ of $\Omega_{i_1 \dots i_p}(x,p)$ is also harmonic in the deformed space $\bar{\bar{A}}_n^{(m)}$ of the submetric class.*

Of course, we know that, in case the tensor $\Omega_{i_1 \cdots i_p}(x,p)$ is a harmonic tensor, the tensor $T_{i_1 \cdots i_p} j(x,p)$ defined by (5. 6) vanishes identically so that, in the present case, the Lie operation on both sides of (5. 6) gives us

$$(5.14) \qquad \mathscr{L}\Omega_{i_1 \dots i_p | j} = \sum_{r=1}^{p} \mathscr{L}\Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p | i_r}.$$

Using the relations (2. 1), (2. 4) and (2. 5), we can easily derive the commutation formula

$$(5.15) \qquad \mathscr{L}(\Omega_{i_1 \dots i_p | j}) - (\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} =$$

$$= -(\partial_k^\alpha \Omega_{i_1 \dots i_p})\mathscr{L}B^k_{\alpha j} - \sum_{r=1}^{p} \Omega_{i_1 \dots i_{r-1} k i_{r+1} \dots i_p} \mathscr{L}\Gamma^{*k}_{i_r j},$$

which is nothing but the generalized identity of Igarashi (relation (3. 7), [11], p. 210). Using the identity (5. 15) in (5. 14), we have

$$(\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} - (\partial_k^\alpha \Omega_{i_1 \dots i_p})\mathscr{L}B^k_{\alpha j} =$$

$$= \sum_{r=1}^{p} \Big\{ (\mathscr{L}\Omega_{i_1 \dots i_{r-1} i j_{r+1} \dots i_p})_{|i_r} - (\partial_k^\alpha \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p})\mathscr{L}B^k_{\alpha i_r} -$$

$$- \sum_{s=1}^{r-1} \Omega_{i_1 \dots i_{s-1} k i_{s+1} \dots i_{r-1} j i_{r+1} \dots i_p} \mathscr{L}\Gamma^{*k}_{i_s i_r} - \Omega_{i_1 \dots i_{1-1} k i_{r+1} \dots i_p} \mathscr{L}\Gamma^{*k}_{j i_r} -$$

$$- \sum_{s=r+1}^{p} \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_{s-1} k i_{s+1} \dots i_p} \mathscr{L}\Gamma^{*k}_{i_s i_r} \Big\} - \sum_{r=1}^{p} \Omega_{i_1 \dots i_{r-1} k i_{r+1} \dots i_p} \mathscr{L}\Gamma^{*k}_{i_r j},$$

which on simplification, gives rise to

$$(5.16)\quad \begin{aligned} &(\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} - (\partial^\alpha_k \Omega_{i_1 \dots i_p})\mathscr{L}B^k_{\alpha j} = \\ &= \sum_{r=1}^{p} \{(\mathscr{L}\Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p})_{|i_r} - (\partial^\alpha_k \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p})\mathscr{L}B^k_{\alpha i_r}\} - \\ &- \sum_{r=2}^{p}\sum_{s=1}^{r-1} \Omega_{i_1 \dots i_{s-1} k i_{s+1} \dots i_{r-1} j i_{r+1} \dots i_p}\mathscr{L}\Gamma^{*k}_{i_s i_r} - \\ &- \sum_{r=1}^{p-1}\sum_{s=r+1}^{p} \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_{s-1} k i_{s+1} \dots i_p}\mathscr{L}\Gamma^{*k}_{i_s i}. \end{aligned}$$

Interchanging the order of summation and taking note of (5. 12) a), we can see that

$$\begin{aligned} &\sum_{r=2}^{p}\sum_{s-1}^{r-1} \Omega_{i_1 \dots i_{s-1} k i_{s+1} \dots i_{r-1} j i_{r+1} \dots i_p}\mathscr{L}\Gamma^{*k}_{i_s i_r} = \\ &= \sum_{s=1}^{p-1}\sum_{r=s+1}^{p} \Omega_{i_1 \dots i_{s-1} k i_{s+1} \dots i_{r-1} j i_{r+1} \dots i_p}\mathscr{L}\Gamma^{*k}_{i_s i_r} = \\ &= -\sum_{s=1}^{p-1}\sum_{r=s+1}^{p} \Omega_{i_1 \dots i_{s-1} j i_{s+1} \dots i_{r-1} k i_{r+1} \dots i_p}\mathscr{L}\Gamma^{*k}_{i_r i_s} = \\ &= -\sum_{r=1}^{p-1}\sum_{s=r+1}^{p} \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_{s-1} k i_{s+1} \dots i_p}\mathscr{L}\Gamma^{*k}_{i_s i_r}. \end{aligned}$$

By this reason of fact, (5. 16) reduces to

$$(5.17)\quad \begin{aligned} &(\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} - (\partial^\alpha_k \Omega_{i_1 \dots i_p})\mathscr{L}B^k_{\alpha j} = \\ &= \sum_{r=1}^{p} \{(\mathscr{L}\Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p})_{|j} - (\partial^\alpha_k \Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p})\mathscr{L}B^k_{\alpha i_r}\}. \end{aligned}$$

On the other hand by making use of (5. 7), (5. 12) c) and (5. 15), we can easily derive the relation

$$(5.18)\quad \begin{aligned} &(\mathscr{L}g^{i_p j})\Omega_{i_1 \dots i_p |j} + g^{i_p j}\{(\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} - (\partial^\alpha_k \Omega_{i_1 \dots i_p})\mathscr{L}B^k_{\alpha j} - \\ &- \sum_{r=1}^{p} \Omega_{i_1 \dots i_{r-1} k i_{r+1} \dots i_p}\mathscr{L}\Gamma^{*}{}_{i_r}{}^{k}{}_{j}\} = 0. \end{aligned}$$

Hence, in order that the Lie derivative $\mathscr{L}\Omega_{i_1 \dots i_p}(x, p)$ of the tensor $\Omega_{i_1 \dots i_p}(x, p)$ holds the relations

$$(5.19)\quad (\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} = \sum_{r=1}^{p} (\mathscr{L}\Omega_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p})_{|i_r},$$

$$(5.20)\quad g^{i_p j}(\mathscr{L}\Omega_{i_1 \dots i_p})_{|j} = 0,$$

it is necessary and sufficient that the relations

$$\mathscr{L}g^{ij} = 0, \tag{5.21}$$

$$\mathscr{L}B^{i}_{\alpha j} = 0, \tag{5.22}$$

$$\mathscr{L}\Gamma^{*i}_{jk} = 0, \tag{5.23}$$

are identically satisfied. But from (5. 19) and (5. 20), it is quite evident that the Lie derivative $\mathscr{L}\Omega i_1 \ldots i_p(x, p)$ of the tensor $\Omega i_1 \ldots i_p(x, p)$ is a harmonic tensor, and the relations (5. 21) and (5. 23) clearly shows us that the infinitesimal transformation (2. 3) is now an areal motion. Therefore we have following:

Theorem 13. *In order that the Lie derivative $\mathscr{L}\Omega i_1 \ldots i_p(x, p)$ of any harmonic tensor $\Omega i_1 \ldots i_p(x, p)$ with respect to the transformation* (2. 3) *in an areal space $A_n^{(m)}$ of the submetric class is also harmonic, it is necessary and sufficient that the infinitesimal transformation* (2. 3) *is now an areal motion in the same space.*

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